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# Impulsive control for synchronization of nonlinear Rössler chaotic systems*

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This paper reports that an impulsive control theory for synchronization of nonlinear Rössler chaotic systems is developed. A new framework for impulsive synchronization between such chaotic systems is presented, which makes the synchronization error system a linear impulsive control system. Therefore, it is easy to derive the impulsive synchronization law. The proposed impulsive control scheme is illustrated by nonlinear Rössler chaotic systems and the simulation results demonstrate the effectiveness of the method.

**Keywords:** chaos, impulsive synchronization, Rössler's system
**PACC:** 0545

## 1. Introduction

In recent years, control theory for chaos synchronization has attracted increasing attention. A chaotic system has complex dynamical behaviours that possess some special features, such as being extremely sensitive to tiny variations of initial conditions, having bounded trajectories in the phase space with a positive leading Lyapunov exponent, and so on. Chaos control and synchronization have been flurried research activities over a decade.[1−15]

Synchronization is an interesting topic and chaos synchronization that has attracted increasing attention in the areas of control theory and secure communication. Many control methods have been introduced for chaos synchronization, including impulsive control. However, there are very few results in impulsive control for the synchronization of general chaotic systems. Because impulsive control allows the stabilization and synchronization of chaotic systems using only small control impulses, it has been widely used to stabilize and synchronize chaotic systems,[5−7,16−19] even though the chaotic behaviour may follow unpredictable patterns. Recently, several impulsive synchronization schemes have been reported in the literature.[6,7,17] Yang *et al*[17] achieved the synchronization of two identical chaotic systems, i.e. the Chua circuit, using the state variable at the fixed instant time as the impulsive signal. The authors investigated the impulsive synchronization criteria for chaotic systems via a unidirectional linear error feedback approach in Ref.[6]. The stabilization and synchronization of Lorenz systems via impulsive signal are studied in Ref. [7]. However, for many-dimensional chaotic systems described by the ODE or DDE (ordinary or delayed differential equations), the less conservative synchronization criteria such as, e.g., Theorem 2 in Ref. [5], are generally difficult to be verified.

In this article, based on impulsive control theory and nonlinear technique, we propose a new impulsive control scheme for the synchronization of nonlinear Rössler systems including chaotic systems. A sufficient condition is derived for the proposed control scheme. This scheme leads to a linear impulsive error system. Therefore, it allows us to obtain the synchronization conditions easily. The proposed impulsive synchronization scheme is illustrated by a nonlinear Rössler chaotic system and the simulation results demonstrate the effectiveness of the proposed method.

## 2. Impulsive synchronization scheme

Consider the following chaotic system:

$$\dot{x} = Ax + bf(x) + \mu\,, \tag{1}$$

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where $x \in R^n$ denotes the state of the system, $\mu \in R^n$ the external output and $f : R^n \to R^n$ presents a continuous and nonlinear function. $A \in R^{n\times n}$ and $b \in R^n$ are constant.

**Remark 1** Several chaotic or hyperchaotic systems satisfy Eq.(1). For example, Rossler's system,[3] the MCK circuit,[4] the Duffing system and Chua's circuit all belong to the class defined by Eq.(1).

To synchronize the system Eq.(1), let $s(x) = f(x) + kx \in R$ be the transmitted signal, where $k = [k_1, k_2, \cdots, k_n] \in R^{1\times n}$ is the coupling strength. Construct the nonlinear coupled system:

$$\begin{aligned}\dot{y} &= Ay + bf(y) + \mu + b[s(x) - s(y)] \\ &= Ay + b[f(x) + k(x - y)] + \mu\,. \end{aligned} \tag{2}$$

Moreover, at the discrete time instants, $\tau_i, i = 1, 2, \cdots$, the state variable of Eq.(1) is transmitted to the response system Eq.(2) and leads to a 'jump' of the state of the response system. In this case, the model of response system is rewritten as

$$\begin{aligned}&\dot{y} = Ay + b[f(x) + k(x - y)] + \mu, t \neq \tau_i \\ &\Delta y = y(t^+) - y(t^-) = B(y - x), \\ &t = \tau_i, i = 1, 2, \cdots, \end{aligned} \tag{3}$$

where $B \in R^{n\times n}$ is impulse strength. $\tau_i, i = 1, 2, \cdots$, are constants satisfying

$$t_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots, \quad \lim_{i\to\infty} \tau_i \to \infty.$$

Let the synchronization error $e = x - y$. It follows from Eqs.(1) and (3) that the error system is

$$\begin{aligned}&\dot{e} = (A - bk)e, t \neq \tau_i, \\ &\Delta e = Be, t = \tau_{i,}\,. \end{aligned} \tag{4}$$

which is a linear impulsive system.

**Remark 2** From the analysis above, it follows that a necessary and sufficient condition for synchronizing chaos is that the origin of Eq.(4) is asymptotically stable. It is obvious that the origin is one of the equilibriums of Eq.(4). Also, the origin is the unique equilibrium of Eq.(4) because $\Delta e = Be (B \neq 0)$ implies $e(\tau_i^+) \neq e(\tau_i)$ unless $e(\tau_i^+) = e(\tau_i) = 0$.

For the asymptotical stability of Eq.(4) at the origin, we obtain the following results.

**Theorem 1** Suppose that

$$\begin{aligned}0 < \alpha &= \inf_{i\in N}(\tau_{i+1} - \tau_i) \\ &\leq \sup_{i\in N}(\tau_{i+1} - \tau_i) = \beta < \infty, \end{aligned} \tag{5}$$

and

$$\|(I + B)\exp((A - bk)(\tau_i - \tau_{i-1}))\| < 1, \quad i \in N. \tag{6}$$

Then the origin of Eq.(4) is globally asymptotically stable.

**Proof** Let $e(t) = e(t, e_0) = e(t, t_0, e_0)$ be the solution of Eq.(4) with the initial condition $e(t_0) = e_0$. then

$$\begin{aligned}e(t, e_0) &= e_0\exp((A - bk)(t - \tau_i)) \\ &\quad \times \prod_{j=1}^{i}(I + B)\exp((A - bk)(\tau_j - \tau_{j-1})). \end{aligned} \tag{7}$$

Where $\tau_0 = t_0$, $t \in (\tau_i, \tau_{i+1})$, and the notation $\boldsymbol{I}$ denotes the identity matrix.

From Eq.(7), we have

$$\begin{aligned}\|e(t, e_0)\| &\leq \|e_0\|\|\exp((A - bk)(t - \tau_i))\| \\ &\quad \times \prod_{j=1}^{i}\|(I + B)\exp((A - bk)(\tau_j - \tau_{j-1}))\|. \end{aligned} \tag{8}$$

Notice that condition Eq.(5) implies

$$\|\exp((A - bk)(t - \tau_i))\| < \infty.$$

Then, it follows from Eq.(6) that

$$\lim_{t\to\infty}\|e(t, e_0)\| = \lim_{i\to\infty}\|e(t, e_0)\| = 0,$$

for any $e_0 \in R^n$, which completes the proof.

From Theorem 1 and Remark 2 the following corollary is immediately obtained.

**Corollary 1** Suppose that there exist synchronization gain $k$ and $B$ such that Eqs.(5) and (6) hold. Then, Eqs.(1) and (3) are globally synchronous.

## 3. Synchronizing Rössler systems

In this section, in order to show the effectiveness of the proposed impulsive control scheme, the nonlinear Rössler systems[3] via nonlinear feedback approach are considered. A dimensionless form a Rössler systems is given by[3]

$$\begin{aligned}&\dot{x} = -y - z, \\ &\dot{y} = x + \alpha' y, \\ &\dot{z} = z(x - \beta') + \alpha', \end{aligned} \tag{9}$$

where $\alpha'$, $\beta' \in R^+$.

Let $f(x) = x \times z$, $\mu = \alpha'$, then Eq.(9) can be rewritten as

$$\dot{x} = Ax + bf(x) + \mu, \tag{10}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha' & 0 \\ 0 & 0 & -\beta' \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

According to Eq.(3), we construct the corresponding response system as follows:

$$\begin{aligned} &\dot{y} = Ay + b[f(x) + k(x-y)] + \mu, \quad t \neq \tau_i, \\ &\Delta y = y(t^+) - y(t^-) = B(y-x), \\ &t = \tau_i, \quad i = 1, 2, \cdots, \end{aligned}$$

thus, we have

$$\dot{e} = (A - bk)e, t \neq \tau_i, \quad \Delta e = Be, t = \tau_i. \tag{12}$$

From Theorem 1, the following asymptotical stability condition for Eq.(12) is immediately obtained.

**Corollary 2** Let the impulses be equidistant and separated by interval $\delta$. Suppose that there exist constants $\alpha, \beta \in R$ such that $k = \alpha \times [\,1\ 1\ 1\,]$, $B = \beta \times I$ and

$$\|\exp(A - bk)\||1 + \beta|\delta < 1, \tag{13}$$

then the origin of Eq.(12) is globally asymptotically stable.

In the following simulations, Eq.(9) is chaotic if $\alpha' = 0.2, \beta' = 5.7$. The trajectories of driving system are shown in Fig.1, which is the nonlinear Rössler's chaotic systems. We choose $\tau_i - \tau_{i-1} \equiv \delta (i = 1, 2, \cdots)$. Moreover, let $k = \alpha \times [\,1\ 1\ 1\,]$ and $B = \beta \times I$, where $I$ denotes the identity matrix. Note that the stable region of chaos synchronization is determined by Eq.(13).

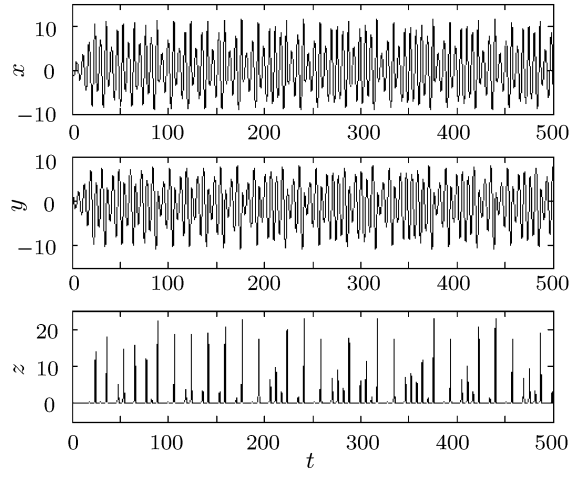



**Fig.1.** The nonlinear Rössler's system is chaotic.

The initial conditions for driving and driven systems are given by

$$(x_1(0), x_2(0), x_3(0)) = (0.25, 0.2, 0.2)$$

and

$$(\tilde{x}_1(0), \tilde{x}_2(0), \tilde{x}_3(0)) = (-0.05, -0.05, -0.2)$$

respectively. Figures 2 shows the stable regions of synchronization for the special parameters. The entire region below the curve corresponding to given parameter is the predicted stable region. Figure 2 shows the curve of impulse time interval $\delta$ with respect to control parameter $\beta$. Figures 3–5 show the time response curves of synchronization error systems. Figures 3, 4 and 5 show the stable results within the stable region for $\alpha = -4, \beta = -0.9, \delta = 0.01$, for $\alpha = -3$, $\beta = -0.5$, $\delta = 0.001$ and for $\alpha = -2$, $\beta = -0.5$, $\delta = 0.0001$ respectively. Figure 6 shows the unstable results outside the stable region for $\alpha = -2$, $\beta = -2.5$, $\delta = 0.01$.

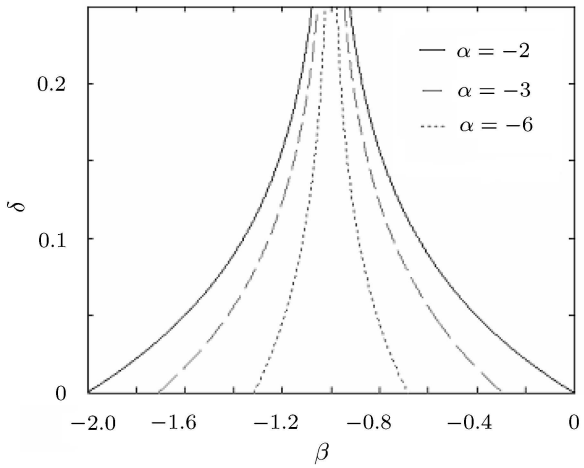



**Fig.2.** Estimate of the boundaries of stable region with the given $\alpha's$.

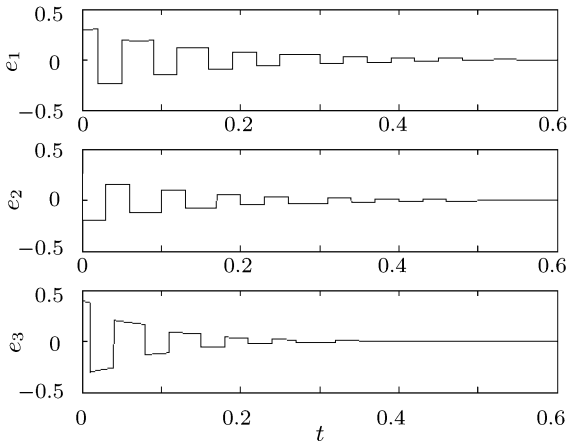



**Fig.3.** Time response of the synchronization error system with $\alpha = -4$, $\beta = -0.9$, $\delta = 0.01$.

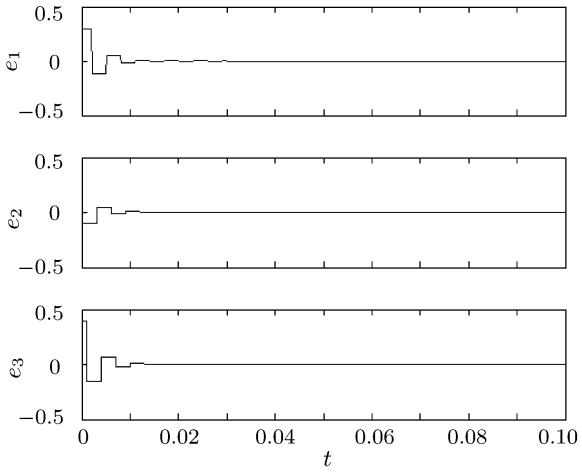



**Fig.4.** Time response of the synchronization error system with $\alpha = -3$, $\beta = -0.5$, $\delta = 0.001$.

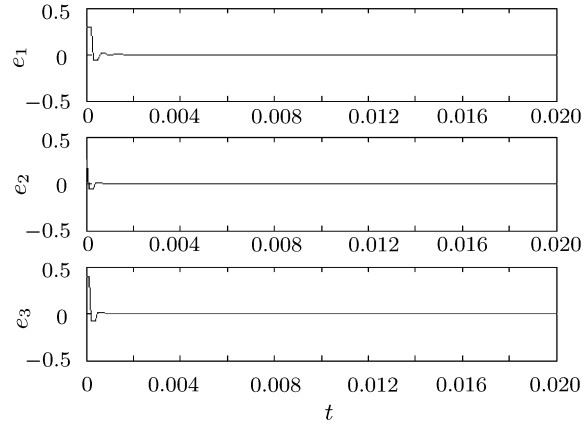



**Fig.5.** Time response of the synchronization error system with $\alpha = -2$, $\beta = -0.5$, $\delta = 0.0001$.

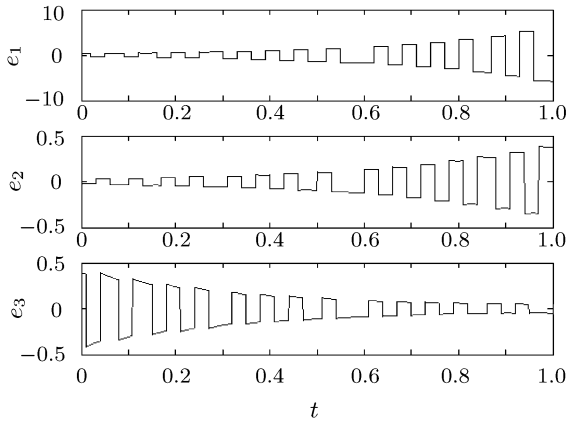



**Fig.6.** Time response of the synchronization error with $\alpha = -2$, $\beta = -2.5$, $\delta = 0.01$.

# 4. Conclusions

In this paper, a synchronization scheme has been proposed based on the nonlinear feedback approach. According to the derived synchronization conditions, we investigated the synchronization behaviour of the Rössler system and estimated the stable regions of synchronization for the given parameters. In comparison with the schemes reported in the literature, e.g. Ref.[3], our method does not require complex mathematical analysis. The effectiveness of the suggested method has been shown by computer simulation.

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