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A new chaotic system and its circuit realization*

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Based on the Lü system, a new chaotic system is constructed, which can generate a Lorenz-like attractor, Chen-like attractor, Lü-like attractor and new attractor when its parameters are chosen appropriately. The detailed dynamical behaviours of this system are also investigated, including equilibria and stability, bifurcations, and Lyapunov exponent spectrum. Moreover, a novel analogue circuit diagram is designed for the verification of various attractors.

Keywords: chaos, bifurcation, Lyapunov exponent, circuit realization

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1. Introduction

The Lorenz system that can produce a well-known Lorenz chaotic attractor[1] has served as a prototype model for studying chaotic behaviour for quite a long time. Recently, there has been increasing interest in creating chaos particularly since Chen and Ueta[2] found a new chaotic system, called the Chen system, from an engineering feedback control approach, which is not topologically equivalent to the original system. In 2002, Lü and Chen[3] further constructed a new chaotic system, called the Lü system hereafter, which represents the transition between Lorenz system and the Chen system. Thereafter a very rich family of the so-called unified system or Lorenz system family was introduced as a connection of the Lorenz, Chen and Lü systems.[4,5] Over the last few years, there have been carried out some studies of these typical chaotic systems.[6−20]

In retrospect, Chen found a new system[2] via chaotification, and a linear feedback control term $u = -ax + (c+1)$ was added to the second equation of the Lorenz system. Very recently, a new chaotic system[7] is presented based on some chaotification analysis, and a cross-product term is added to the first equation of the Lorenz system. Inspired by the consideration of chaotification, we constructed a new chaotic system by adding a cross-product nonlinear term to the first equation of Lü system.[3] It turns out that the new system has five equilibria, therefore it is not topologically equivalent to the Lorenz system, or the Chen system, or the Lü system, or the Lorenz system family. Of particular interest is the fact that this chaotic system can display a Lorenz-like attractor, Chen-like attractor, Lü-like attractor and new attractor when its parameters are chosen appropriately. The chaotic behaviours of this new system are verified not only by bifurcation analysis through computer simulations but also by the implementation of an analogue electronic circuit. Several bifurcation routes to chaos from periodic and chaotic evolutions are observed.

2. The new chaotic system

We add a cross-product term to the first equation of the slightly modified Lü system, obtaining a new system as follows:

$$\begin{aligned} \dot{x} &= a(-x + y + yz), \\ \dot{y} &= y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{1}$$

where a and b are the positive real constant parameters. It is noticed that each equation of system (1) has

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a single cross-product nonlinear term, different from that of the Lorenz system, or the Chen or the Lü system, or even Lorenz system family.

For system (1) with variable parameters a and b, several simulations have been carried out. The findings are summarized as follows.

When $a = 2.4$ is fixed while b is varied, one can observe that the attractor of system (1) is similar to the Lorenz attractor for $b = 0.78$; when $a = 2.4$, and $b = 0.28$, it becomes similar to the Lü attractor; and when $a = 2.4$ and $b = 0.12$, a new attractor is obtained. Furthermore, when $a = 1.5$, $b = 0.2$, system (1) displays a chaotic attractor, seemingly similar to the Chen attractor. The simulation results of $x - z$ trajectories of these cases are shown in Figs.1(a)–(d).

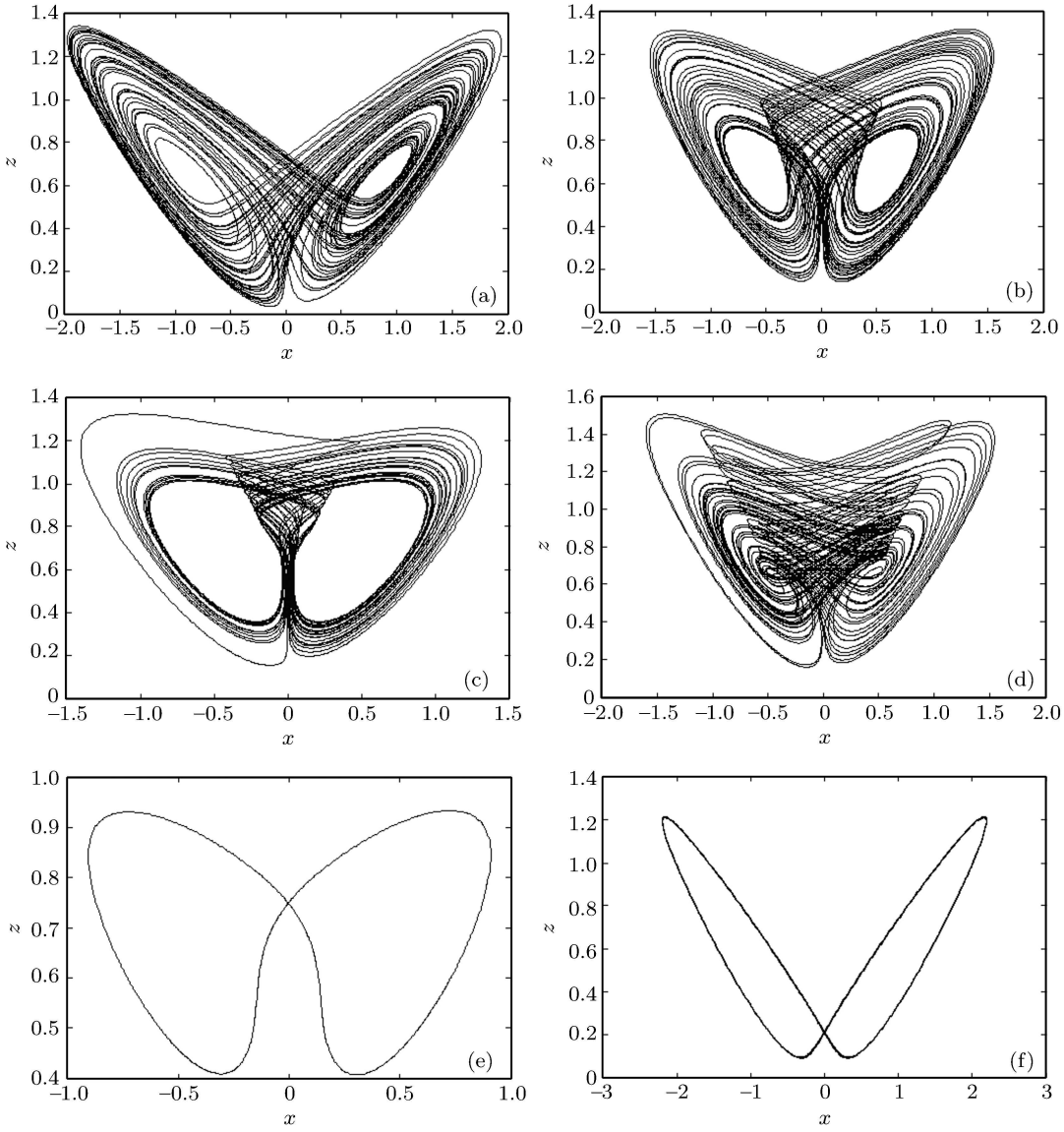


Fig.1. Projection on the phase plane x-z for the chaotic attractors of system (1): (a) Lorenz-like attractor, with $a = 2.4$ and $b = 0.78$; (b) Lü-like attractor, with $a = 2.4$ and $b = 0.28$; (c) new attractor, with $a = 2.4$ and $b = 0.12$; (d) Chen-like attractor, with $a = 1.5$ and $b = 0.2$; (e) periodic orbit, with $a = 2.4$ and $b = 0.2$; (f) periodic orbit, with $a = 2.4$ and $b = 1.5$.

Note that this new system (1) is not diffeomorphic with any of Lorenz, Chen and Lü systems, since their eigenvalues at the corresponding equilibria are not the same, nor in any sense equivalent. It is easy to see that there is no nonsingular coordinate transformation that can convert the new system to the others. Therefore, all the four mentioned systems are topologically not equivalent.

3. Basic dynamical analysis

The fact that Lorenz-like, Chen-like and Lü-like attractors can appear in system (1) shows that this system shares some basic properties with the Lorenz, Chen and Lü systems but it also possesses its own behaviours. Some basic dynamical behaviours of system (1) are analysed below.

3.1. Symmetry and dissipativity

It is easy to see that system (1) has a natural symmetry under the following transformation:

$$(x, y, z) \to (-x, -y, z),$$

i.e. reflection about the z-axis.

Consider a volume in a certain domain D_0 of the state space. For system (1), one has

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(a + b - 1). \tag{2}$$

This means that the dynamical system (1) with a set of parameters satisfying

$$a + b - 1 > 0 \tag{3}$$

is guaranteed to be dissipative, so the volume of any attractor of the system must be zero. The orbit flows into a certain bounded region as $t \to \infty$.

3.2. Equilibria and bifurcation

If $b > 0$, the dynamical system (1) has five equilibria:

$$S_0(0, 0, 0), \quad S_1(\sqrt{b}, p_1\sqrt{b}, p_1), \quad S_2(-\sqrt{b}, -p_1\sqrt{b}, p_1)$$

$$S_3(\sqrt{b}, p_2\sqrt{b}, p_2), \quad S_4(-\sqrt{b}, -p_2\sqrt{b}, p_2),$$

where $p_1 = -0.5 + \sqrt{5}/2 > 0$ and $p_2 = -0.5 - \sqrt{5}/2 < 0$. If $b < 0$, then this system has a unique real equilibrium, $S_0(0, 0, 0)$. It is noticed that nonzero equilibria S_1 and S_2, S_3 and S_4 are symmetrical about the z-axis.

By considering a Jacobian matrix for one of these equilibria and calculating their eigenvalues, we can investigate the stability of each equilibrium based on the roots of the characteristic equation of the system.

Linearizing the system (1) about the equilibrium S_0 yields the characteristic equation

$$f(\lambda) = \lambda^3 + (a+b-1)\lambda^2 - (a - ab + b)\lambda - ab = 0, \tag{4}$$

and gives the corresponding eigenvalues as follows:

$$\lambda_1 = -a, \quad \lambda_2 = 1, \quad \text{and} \quad \lambda_3 = -b. \tag{5}$$

Consider the expressions (5), one can easily find $\lambda_1 < 0$, $\lambda_2 > 0$, and $\lambda_3 < 0$, implying that the equilibrium S_0 is unstable, and always has a saddle point in three-dimensional space.

Next, linearizing the system about the equilibria S_1 and S_2 gives the same characteristic equation as follows:

$$f(\lambda) = \lambda^3 + (a+b-1)\lambda^2 + p_1 ab\lambda + (4-2p_1)ab = 0. \tag{6}$$

Note that the coefficients of the characteristic polynomial (6) are all positive. Therefore, $f(\lambda)$ are positive for all $\lambda > 0$. Consequently, an instability ($R(\lambda) > 0$) appears only if there are two complex conjugate zeros of f. Let the two complex zeros be $\lambda_1 = \mathrm{i}\omega$ and $\lambda_2 = -\mathrm{i}\omega$ for some real ω. The sum of the three zeros of the characteristic polynomial (6) is

$$\lambda_1 + \lambda_2 + \lambda_3 = -(a + b - 1). \tag{7}$$

Hence, $\lambda_3 = -(a + b - 1)$, on the margin of stability, where $\lambda_{1,2} = \pm\mathrm{i}\omega$. On this margin, one has

$$f(\lambda_3) = -p_1 ab(a + b - 1) + (4 - 2p_1)ab = 0,$$

that is,

$$p_1(a + b - 1) = (4 - 2p_1).$$

Therefore,

$$b_{\mathrm{h}} = \frac{4 - 2p_1}{p_1} - a + 1. \tag{8}$$

The parameter b_{h} is the Hopf bifurcation value for both S_1 and S_2. When $b > b_{\mathrm{h}}$, S_1 and S_2 are both stable sinks. At $b = b_{\mathrm{h}}$, however, they change to two two-dimensional unstable saddles. When $0 < b < b_{\mathrm{h}}$, S_1 and S_2 are both unstable saddle-foci.

For equilibria S_3 and S_4, they also have the same characteristic equation:

$$f(\lambda) = \lambda^3 + (a+b-1)\lambda^2 + p_2 ab\lambda + (4-2p_2)ab = 0, \tag{9}$$

where $p_2 < 0$.

Theorem Hopf bifurcation does not appear at equilibria S_3 and S_4 of the system (1).

Proof For equilibria S_3 and S_4 they have the same characteristic equation (9). If Hopf bifurcation appears at equilibrium S_3 or S_4, then it may be assumed that two zeros are $\lambda = \pm\mathrm{i}\omega$ for some real ω, and the sum of the three zeros of the cubic polynomial f satisfies

$$\lambda_1 + \lambda_2 + \lambda_3 = -(a + b - 1).$$

Hence,

$$\lambda_3 = -(a + b - 1).$$

However,

$$f(\lambda_3) = -p_2(a + b - 1)ab + (4 - 2p_2)ab = 0,$$

that is,

$$(a + b - 1) = (4 - 2p_2)/p_2 < 0,$$

which is in contradiction with the condition $(a + b - 1) > 0$. Therefore, Hopf bifurcation does not appear at S_3 and S_4. The proof of the theorem ends here.

In fact, there always exist one negative real eigenvalue and a pair of complex conjugate eigenvalues with positive real parts for equilibria S_3 and S_4 over the entire spectrum of the system parameters: that is, S_3 and S_4 are both always unstable saddle-foci.

4. Transition between various attractors

This section is devoted to investigating the transition between the Lorenz-like, Chen-like, and Lü-like attractors by manipulating the new chaotic system (1).

When the parameter $a = 2.4$ is fixed while parameter b is varied, the spectrum of Lyapunov exponents and the corresponding bifurcation diagram of state x with respect to b are obtained as shown in Fig.2.

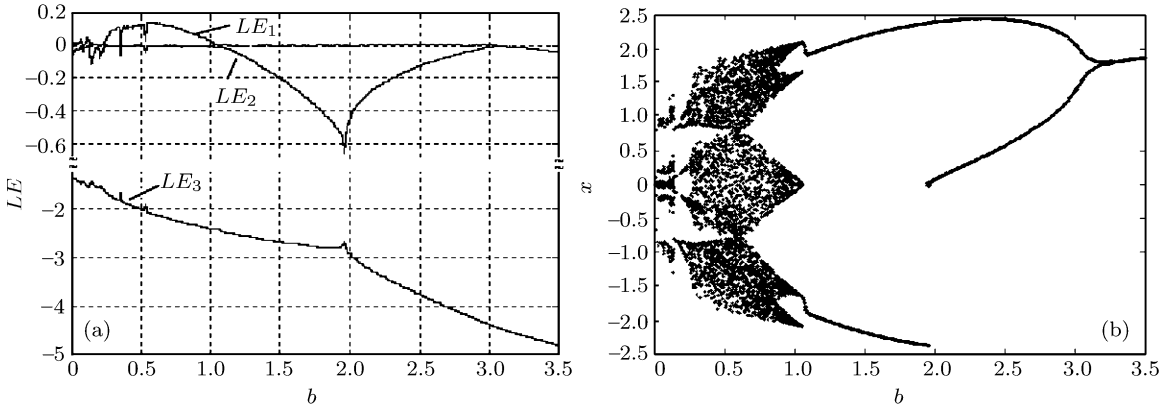


Fig.2. (a) Spectrum of Lyapunov exponents of system (1), where LE stands for Lyapunov exponent. (b) Bifurcation diagram of x. $a = 2.4$, $b \in [0, 3.5]$.

When $b \in [0, 0.139]$, system (1) is a chaotic system (corresponding to a new attractor shown in Fig.1(c)) with a single positive Lyapunov exponent; while for $b \in (0.139, 0.248]$, the maximum Lyapunov exponent LE_1 equals zero, implying that the system has a periodic orbit shown in Fig.1(e). In the interval $0.248 < b \leq 1.07$, there exists a tiny periodic window for $b \in [0.348, 0.363]$, and the window divides the interval into two chaotic regions $C_1 = (0.248, 0.348]$ and $C_2 = (0.363, 1.07]$, corresponding to a Lü-like attractor and a Lorenz-like attractor (Figs.1(b) and (a)), respectively. As b increases further in the region $b \in [1.073, 3.072]$ the system follows a period-2 window (Fig.1(f)) and then arrives at the Hopf bifurcation point $b_{\rm h} = 3.072$ shown in Fig.2(b). When $b > b_k$, the maximum Lyapunov exponent becomes negative and the bifurcation of x changes into a line, both of which imply that the orbit of the system converges to an equilibrium as shown in Figs.2.(a) and 2(b).

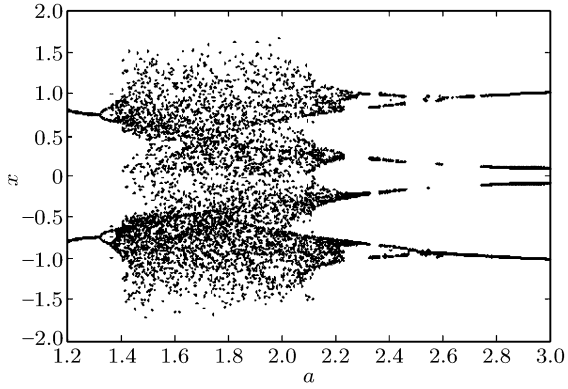


Fig.3. Bifurcation diagram of x, $b = 0.2$, $a \in [1.2, 3.0]$.

Starting with the periodic window ($b \in [0.139, 0.248]$), if $b = 0.2$ is fixed but a is varied, the periodic obit shown in Fig.1(e) can also evolve into the attractor shown in Fig.1(d), which is similar to the Chen attractor. It is very interesting to point out

that the Chen-like attractor, Lü-like attractor and the new attractor can be derived from the same periodic obit. The corresponding bifurcation diagram of x with respect to variable parameter a is shown in Fig.3.

5. Circuit design for chaotic system

In this section, an electronic circuit is designed to realize the new chaotic system (1). The designed circuitry realizing Eq.(1) is shown in Fig.4. It consists of three channels to conduct the integration of the three state variables x, y and z, respectively. Operational amplifiers and the associated circuitry perform the operations of addition, subtraction, and integration. Analogue multipliers are employed to perform the required multiplication operations. The voltage supplies of all active devices are of ± 14 V.

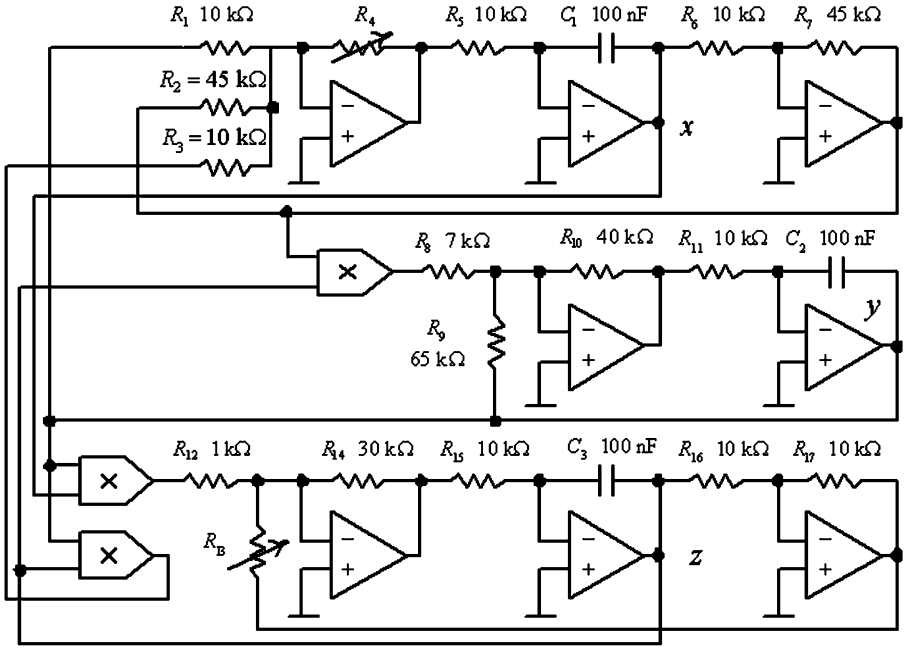
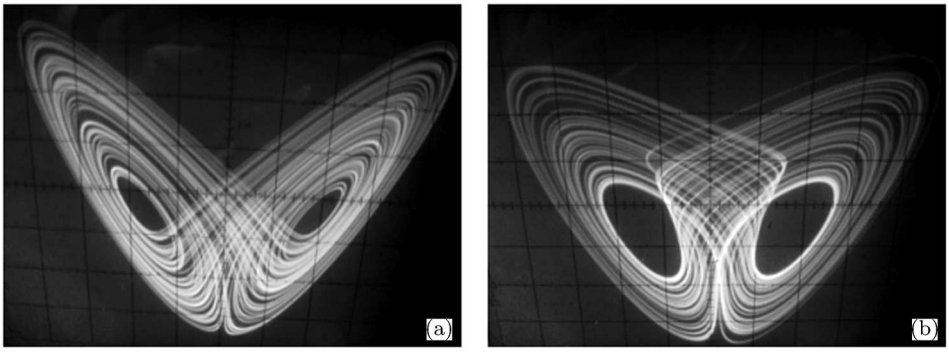


Fig.4. Circuit diagram for realizing the new chaotic system.

The observations made in the experimental setup are shown in Figs.5(a)–(d). The adjusting of parameters a and b in system (1) can be implemented by adjusting the linear resistors R_4 and R_{13} in Fig.4, respectively. When $R_4 = 13\,\mathrm{k}\Omega$ (corresponding to $a = 2.4$) is fixed while R_{13} is varied, various attractors can be obtained. If $R_{13} = 120\,\mathrm{k}\Omega$, it is corresponding to $b = 0.78$, the Lorenz-like attractor as shown in Fig.5(a) can be observed from the circuit; if $R_{13} = 320\,\mathrm{k}\Omega$, it corresponds to $b = 0.28$, the circuit can generate the Lü-like attractor as shown in fig.5(b); and when $R_{13} = 900\,\mathrm{k}\Omega$, corresponding to $b = 0.12$, a new attractor appear as shown in Fig.4(c). Furthermore, when $R_{13} = 500\,\mathrm{k}\Omega$ (corresponding to $b = 0.2$) is fixed while R_4 varies from $R_4 = 13\,\mathrm{k}\Omega$ to $8\,\mathrm{k}\Omega$, the Chen-like attractor can be obtained as shown in Fig.5(d).



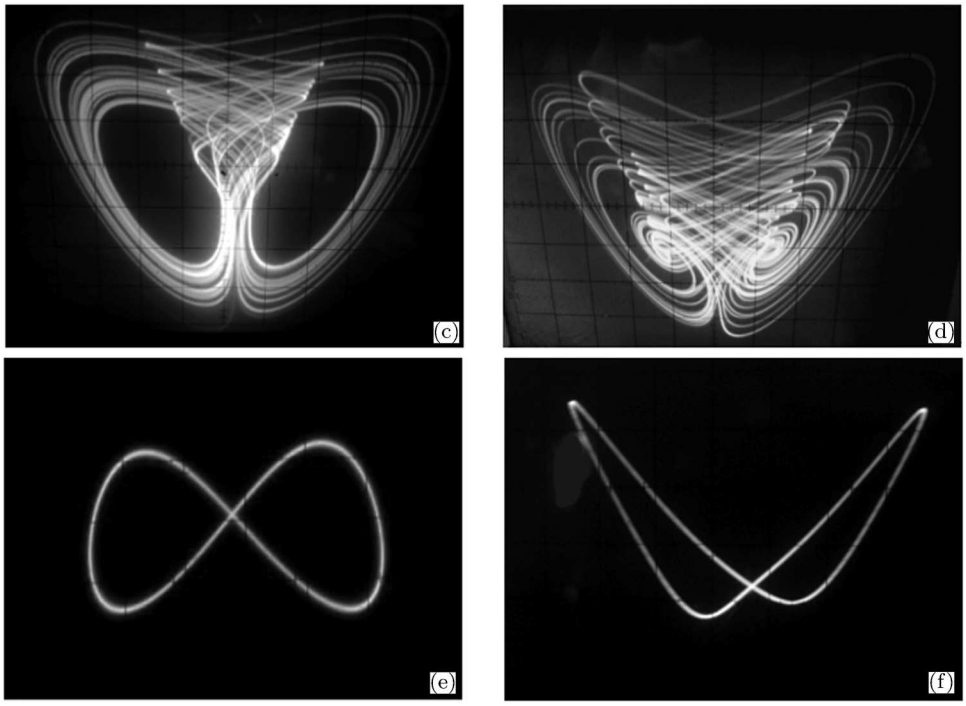


Fig.5. Projections on the x–z plane for the chaotic attractor of the system (1) measured in the experimental setup: (a) Lorenz-like attractor; (b) Lü-like attractor; (c) new attractor; (d) Chen-like attractor; (e) and (f) periodic orbits.

By comparing the results of Fig.1 and Fig.5, we can find good qualitative agreement between the numerical simulation and the experimental measurement. On the other hand, this design is simple, and is easier to achieve the new chaotic system than the implementation of Chen's attractor in Ref.[21] and a Lü's attractor in Ref.[6].

6. Conclusions

In this paper we have introduced a novel three-dimensional continuous autonomous system, in which each equation has a single cross-product nonlinear term. Some basic dynamical behaviours of the system are further explored by investigating its Lyapunov exponent spectrum and bifurcation diagrams. Furthermore, this new system has also been physically realized by using an analogue electronic circuit. Compared with the Lü system, the system has some apparent differences such as having five equilibria and more complex bifurcation behaviours.

The analysis of the new system reveals an application for digital chaotic communication to map different symbols onto different attractors that are produced from the proposed system with different parameter values. More detailed dynamical analysis about this new chaotic system will be reported later.

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