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Parameter-induced stochastic resonance with a periodic signal*

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In this paper conventional stochastic resonance (CSR) is realized by adding the noise intensity. This demonstrates that tuning the system parameters with fixed noise can make the noise play a constructive role and realize parameter-induced stochastic resonance (PSR). PSR can be interpreted as changing the intrinsic characteristic of the dynamical system to yield the cooperative effect between the stochastic-subjected nonlinear system and the external periodic force. This can be realized at any noise intensity, which greatly differs from CSR that is realized under the condition of the initial noise intensity not greater than the resonance level. Moreover, it is proved that PSR is different from the optimization of system parameters.

Keywords: tuning parameters, stochastic resonance, signal-to-noise ratio
PACC: 0540, 0250

1. Introduction

The concept of stochastic resonance (SR) was originally put forward in two seminal papers by Benzi and collaborators,[1,2] in which the response of a nonlinear system to a weak periodic input signal is amplified by adding an optimal level of the noise.[3−5] Recently, SR was also found in linear systems [6] and with aperiodic signals and different kinds of noise.[7−9] In Ref.[1], Benzi *et al* suggested that SR was based on the cooperative effects between the stochastic-subjected dynamical system and the external periodic forcing. According to this viewpoint, adding noise is tuning the stochastic system parameter in essence. To vary the stochastic-subjected dynamical system, besides adding noise, there is another method, which is directly adjusting system parameters. It seems to be the usual dynamical resonance by adjusting the inherent frequency of the system to the frequency of an external periodic force. Thus, one natural idea is to research into the cooperative effect in an adjustable system subjected to a given mixture of the noise and the external time-dependent force. The system parameter is then viewed as an important tunable element for the study of SR.

According to the notion of conventional stochastic resonance (CSR), its occurrence needs three ingredients: a double well potential, a periodic forcing and a noise. Based on these ingredients, the dynamical system can be modelled by the differential equation

$$\dot{x}(t) = ax(t) - bx^3(t) + A\cos(\omega t) + \Gamma(t), \qquad (1)$$

where system parameters a and b are positive, $A\cos(\omega t)$ is the input signal, and the Gaussian white noise $\Gamma(t)$ is characterized by

$$\langle \Gamma(t) \rangle = 0, \quad \langle \Gamma(t)\Gamma(t+\tau) \rangle = 2D\delta(\tau), \qquad (2)$$

in which D is the noise intensity. In this article, parameter-induced stochastic resonance (PSR) means that the resonance phenomenon between the stochastic-subjected system and the input signal is presented by tuning system parameters a and b with fixed noise intensity D. Some papers suggested that for a fixed noise it was possible to realize SR by tuning system parameters,[10,11] while many authors did not agree with this point of view. In previous papers, we discussed the optimal nonlinear systems (a and b are optimal for the output of the system) and considered that the resonance occurred when the system parameters were optimal.[12−14] Furthermore, in Ref.[14], it was proved that adding noise could be viewed as a

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specific case of tuning system parameters. However, we did not carefully investigate the role of the noise in the optimal systems. In other words, the optimization of system parameters and PSR were not definitely distinguished (if PSR really exists). Consequently, the four following problems arise. 1) As predicted in Refs.[10,11], does PSR really exist in system (1)? 2) If yes, then how to tune a and b to realize PSR? 3) What is the relationship between CSR and PSR? 4) Is PSR the same as the optimization of system parameters? In the following, we will discuss these problems in detail.

2. PSR and the relation with CSR

We employ the signal-to-noise ratio (SNR) as a measure of the output of system (1), as generally used in other papers about SR. This SNR was defined by a number of authors, and it is[3,4,15−17]

$$\mathrm{SNR} = \frac{\pi A^2 a r_{\mathrm{K}}}{bD^2} = \frac{A^2 a^2}{\sqrt{2} b D^2} \exp(-\Delta V/D), \tag{3}$$

where r_{K} is the Kramers rate and $\Delta V = a^2/(4b)$. Assume the parameter group (a, b, D) as (a_0, b_0, D_0), so the output SNR becomes

$$\mathrm{SNR} = \frac{A^2 a_0^2}{\sqrt{2} b_0 D_0^2} \exp(-\Delta V_0/D_0), \tag{4}$$

where $\Delta V_0 = a_0^2/(4b_0)$. When D_0 is adjustable and $D_0 \leq \Delta V_0/2$, CSR can be realized at $D_{\mathrm{CSR}} = \Delta V_0/2$ by adding the noise, and the corresponding SNR is

$$\mathrm{SNR}_{\mathrm{CSR}} = \frac{4\sqrt{2} A^2}{D_{\mathrm{CSR}} \mathrm{e}^2}. \tag{5}$$

Rescale the variables according to

$$\bar{t} = at, \qquad y = \sqrt{\frac{a}{D}} x,$$
$$\bar{b} = \frac{bD}{a^2}, \quad \bar{A} = \frac{A}{\sqrt{aD}}. \tag{6}$$

Equation (1) can be rewritten as

$$\dot{y}(\bar{t}) = y(\bar{t}) - \bar{b} y^3(\bar{t}) + \bar{A} \cos(\omega \bar{t}/a) + \xi(\bar{t}), \tag{7}$$

where

$$\langle \xi(\bar{t}) \rangle = 0, \quad \langle \xi(\bar{t}) \xi(\bar{t} + \tau) \rangle = 2\delta(\tau). \tag{8}$$

In terms of $\bar{t} = at$, the Kramers rate for system (7) is $\bar{r}_{\mathrm{K}} = r_{\mathrm{K}}/a$, so the SNR in Eq.(3) expressed by the new parameter scale is

$$\mathrm{SNR} = \frac{\pi \bar{A}^2 a \bar{r}_{\mathrm{K}}}{\bar{b}}. \tag{9}$$

Hence, according to Eq.(6), we can transform the increase of initial noise intensity $D = D_0$ for realizing CSR into the variation of parameters a and b so that SR is produced by tuning the system parameters with fixed noise intensity D_0. The parameters for realizing PSR can be obtained by

$$\bar{b} = \frac{b_0 D_{\mathrm{CSR}}}{a_0^2} = \frac{b_{\mathrm{PSR}} D_0}{a_{\mathrm{PSR}}^2} \tag{10}$$

and

$$\bar{A} = \frac{A}{\sqrt{a_0 D_{\mathrm{CSR}}}} = \frac{A}{\sqrt{a_{\mathrm{PSR}} D_0}}, \tag{11}$$

it means that tuning the noise intensity with fixed system parameters a_0 and b_0 can be viewed as tuning the system parameters with fixed noise intensity D_0. From Eqs.(10) and (11), one has

$$a_{\mathrm{PSR}} = \frac{a_0 D_{\mathrm{CSR}}}{D_0} = \frac{a_0^3}{8 b_0 D_0}, \tag{12}$$

$$b_{\mathrm{PSR}} = \frac{b_0 D_{\mathrm{CSR}}^3}{D_0^3} = \frac{a_0^6}{512 b_0^2 D_0^3}. \tag{13}$$

Substituting a_{PSR} and b_{PSR} into Eq.(3) with $D = D_0$ we get the output SNR

$$\mathrm{SNR}_{\mathrm{PSR}} = \frac{4\sqrt{2} A^2}{D_0 \mathrm{e}^2}. \tag{14}$$

Obviously, SNR_{PSR} has no relation with the initial parameters and it only depends on the given noise intensity D_0.

To investigate the role of the noise with the system parameters taken as a_{PSR} and b_{PSR}, putting Eqs.(12) and (13) into Eq.(3) we have

$$\mathrm{SNR} = \frac{4\sqrt{2} A^2 D_0}{D^2} \exp(-2D_0/D). \tag{15}$$

From Eq.(15), by a simple derivation we find that with varying D, the SNR is maximized exactly at $D = D_0$ and it equals to SNR_{PSR}.

Figure 1 depicts the output SNR versus the noise intensity D. The dotted lines denote that the noise cannot be reduced in Eq.(1). The figure clearly shows that at $D = D_0$ there is a peak in the curve with PSR, which indicates that the noise plays a constructive role when the system parameters are taken as a_{PSR} and b_{PSR}, i.e., the system produces a resonance phenomenon when $a = a_{\mathrm{PSR}}$, $b = b_{\mathrm{PSR}}$ and fixed noise intensity $D = D_0$. Accordingly, PSR is really realizable in Eq.(1). It could be interpreted as

changing the intrinsic characteristic of the dynamical system to yield the cooperative effect between the stochastic-subjected nonlinear system and the external periodic force. For CSR, it is realized by changing the stochastic force to produce the cooperative effect. In fact, from Eqs.(12) and (13), we can obtain $a_{\rm PSR}^2/(4b_{\rm PSR}) = 2D_0$, which is the exact relation $(a_0^2/(4b_0) = 2D_{\rm CSR})$ between the initial parameters and $D_{\rm CSR}$ as CSR is realized. Thus it is not surprising that at $D = D_0$ there is a peak in the curve with PSR. Here, we can directly tune the parameters to realize PSR via the relation $a_{\rm PSR}^2/(4b_{\rm PSR}) = 2D_0$. Nevertheless, as this relation cannot be obtained (it is general) in other nonlinear systems and inputs, to realize PSR, the above method of tuning a and b could be employed.

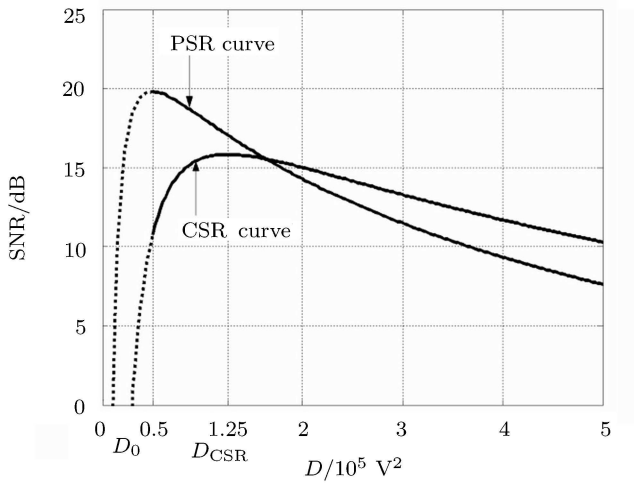


Fig.1. SNR as a function of noise intensity D with $a_0 = 10^4$, $b_0 = 100$ V^{-2}, $A = 2500$ V and $D_0 = 5\times10^4$ V^2. Here, V is the unit of voltage, parameter a is a coefficient and it is dimensionless in order to keep the dimensions of Eq.(1) accordant. The CSR curve: The behaviour of SNR(D_0) in Eq.(4). The PSR curve: the behaviour of SNR(D) in Eq.(15). Dashed lines denote that the noise intensity cannot be reduced. For $D_0 < D_{\rm CSR}$, CSR can be realized by adding the noise intensity.

From Eqs.(10) and (11), $a_{\rm PSR}$ and $b_{\rm PSR}$ have relation to $D_{\rm CSR}$ so PSR and CSR are closely related. It is well known that to realize CSR, the parameter condition (signal amplitude $A \le \sqrt{4a^3/(27b)}$, recently, some papers argued that this condition did not need to be strictly satisfied[18,19]) should be satisfied. However, for PSR, because the parameters are adjustable, this condition is not necessary to be satisfied any more. CSR can always be realized theoretically (the noise intensity could be reduced) and the values of $a = a_{\rm PSR}$ and $b = b_{\rm PSR}$ can always be obtained, i.e., for any values of D_0, PSR can be realized in practice by taking the system parameters as Eqs.(12) and (13), as illustrated in Fig.2.

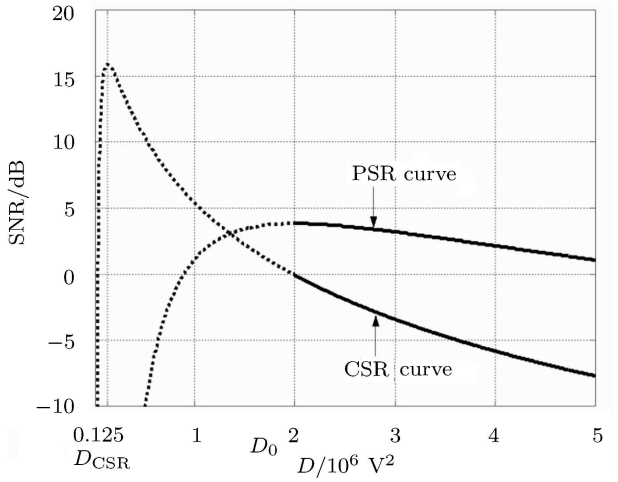


Fig.2. SNR as a function of noise intensity D with $D_0 = 2\times10^6$ V^2 and other parameters are the same as Fig.1. The CSR curve: the behaviour of SNR(D_0) in Eq.(4). The PSR curve: the behaviour of SNR(D) in Eq.(15). Dashed lines denote that the noise intensity cannot be reduced. For $D_0 > D_{\rm CSR}$, CSR cannot be realized by adding the noise intensity, while PSR still can be realized at fixed D_0 by taking system parameters as $a = a_{\rm PSR}$ and $b = b_{\rm PSR}$.

3. The comparison between PSR and the optimal system

In the following, the difference between PSR and the optimization of the system parameters is investigated. From Eq.(3), one has

$$\frac{\partial {\rm SNR}}{\partial a} = 0, \tag{16}$$

$$\frac{\partial {\rm SNR}}{\partial b} = 0. \tag{17}$$

The above two equations give

$$a_{\rm opt}^2 = 4bD, \tag{18}$$

$$b_{\rm opt} = a^2/(4D), \tag{19}$$

so Eqs.(16) and (17) are equivalent. According to the second-order conditions

$$\frac{\partial^2 {\rm SNR}}{\partial a^2}\Big|_{a=a_{\rm opt}} < 0, \tag{20}$$

$$\frac{\partial^2 {\rm SNR}}{\partial b^2}\Big|_{b=b_{\rm opt}} < 0, \tag{21}$$

we find that the SNR is maximized with $a = a_{\rm opt}$ and $b = b_{\rm opt}$. Putting Eq.(18) or Eq.(19) into Eq.(3)

and taking the noise intensity as $D = D_0$, we get the optimal SNR

$$\mathrm{SNR}_{\mathrm{opt}} = \frac{2\sqrt{2}A^2}{D_0 \mathrm{e}}. \tag{22}$$

Compared the values of a_{PSR} and b_{PSR} with those of a_{opt} and b_{opt}, the parameters for realizing PSR are clearly different from the optimal parameters. To study the effect of the noise with intensity D_0 on the optimal system, putting $a_{\mathrm{opt}}^2 = 4b_0 D_0$ and $b = b_0$ into Eq.(3) we have

$$\mathrm{SNR} = \frac{2\sqrt{2}A^2 D_0}{D^2}\exp(-D_0/D), \tag{23}$$

which is depicted in Fig.3 as a function of D. The figure shows that although there is a peak in the curve, the corresponding noise intensity does not locate at D_0. Hence, in the optimal system, the noise level is not optimal and the system does not yield "resonance". The evaluation of a_{opt} and b_{opt} just makes the nonlinear system optimized and does not produce a resonance phenomenon.

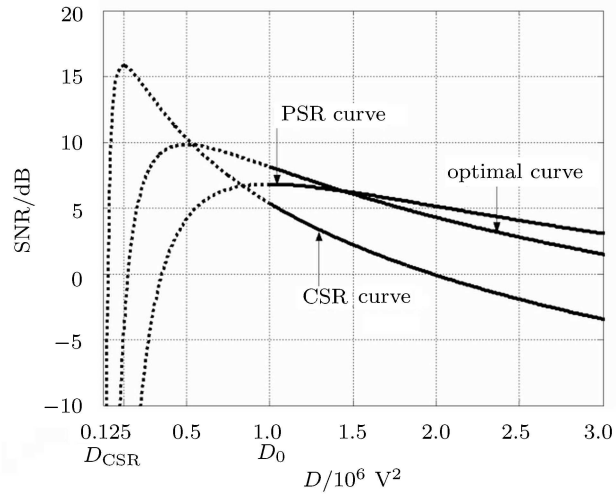


Fig.3. SNR as a function of noise intensity D with $D_0 = 1 \times 10^6$ V^2 and other parameters are the same as Fig.1. The CSR curve: the behaviour of SNR(D_0) in Eq.(4). The PSR curve: the behaviour of SNR(D) in Eq.(15). The optimal curve: the behaviour of SNR(D) in Eq.(23). Dashed lines denote that the noise intensity cannot be reduced. The peak of the optimal curve is not at $D = D_0$.

4. The SNR gain

In this section, the SNR gains of the above three cases (CSR, PSR and the optimization of the parameters) are discussed. Similar to the definition of the output SNR, define the input SNR as

$$\mathrm{SNR}_{\mathrm{in}} = \frac{S(\omega)}{\bar{S}_n} = \frac{\pi A^2}{4D_0}, \tag{24}$$

where $S(\omega)$ is the maximum of the signal power spectral density (PSD), and $\bar{S}_n$ is the average of the spectrum near but not at the signal frequency.[20] From Eqs.(5), (14), (22) and (24), we obtain $\mathrm{SNRGain}_{\mathrm{CSR}} \leq \mathrm{SNRGain}_{\mathrm{PSR}} < 1 < \mathrm{SNRGain}_{\mathrm{opt}}$. Therefore, from the viewpoint of practical application, the optimization of the parameters (some researchers named it as nonlinear bistable filter[21]) is the most valuable. It should be noted that despite $\mathrm{SNRGain}_{\mathrm{CSR}} < 1$ it has been proved that it is possible to get $\mathrm{SNRGain}_{\mathrm{CSR}} > 1$ in some nonlinear systems.[22,23]

5. Conclusions

With this work, we have theoretically demonstrated the existence of PSR in a nonlinear system and presented a method of tuning the system parameters for realizing PSR. When PSR is produced, the output SNR of the system shows a maximum with the noise at its initial intensity D_0. Hence, although the noise is fixed, by tuning the system parameters according to Eqs.(10) and (11), i.e., by changing the intrinsic characteristic of the nonlinear system, the stochastic-subjected system and the external periodic force can yield the cooperative effect that is produced by adding the noise for CSR. Furthermore, we have compared the parameters and the SNR gains of PSR with those of the optimal system, and find that they are different whether from the values of their parameters or from their output results. Finally, it has been proved that the SNR gain of the optimal system is maximal among the three cases although the system dose not show the phenomenon of SR, so it is more valuable to the practical application than the other two.

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