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The instability conditions of a weakly interacting Fermi gas trapped in weak magnetic field

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In this paper the analytical expression of free energy expressed by small parameter r of a weakly interacting Fermi gas trapped in weak magnetic field is derived by using 'the maximum approximation' method and the ensemble theory. Based on the derived expression, the exact instability conditions of a weakly interacting Fermi gas trapped in weak magnetic field at both high and low temperatures are given. From the instability conditions we get the following two results. (1) At the whole low-temperature extent, whether the interactions are repulsive or attractive with $(\alpha n + 4\varepsilon_{\rm F}/3)$ (n and $\varepsilon_{\rm F}$ denote the particle-number density and the Fermi energy respectively, $\alpha = 4\pi a\hbar^2/m$, and a is s-wave scattering length) positive, there is a lower-limit magnetic field of instability; in addition, there is an upper-limit magnetic field for the system of attractive interactions with $(\alpha n + 4\varepsilon_{\rm F}/3)$ negative. (2) At the whole high-temperature extent, the system with repulsive interactions is always stable, but for the system with attractive interactions, the greater the scattering length of attractive interactions $|a|$ is, the stronger the magnetic field is and the larger the particle-number density is, the bigger the possibility of instability in the system will be.

Keywords: Fermi gas, interaction, instability conditions
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1. Introduction

The properties of a real quantum gas are affected by many kinds of factors, which include total particle number of system, dimensionality of space, energy dispersion and the form of external potential. In addition, the inter-particle interaction is an important factor which affects properties of the system. People have looked for many approximation methods for dealing with the thermodynamic properties of systems with interactions, such as the pseudopotential method, cluster expansions method, Thomas–Fermi method, variational principle, Green function, etc.; moreover, a great deal of investigation fruits have been obtained.[1−10]

Interactions between particles not only have important effects on the thermodynamic properties of the system, but also have a direct influence on the stability of the system. For example, condensation of Bose gases will disintegrate and collapse[11−13] due to the interactions between particles. Ref.[14] has shown that degenerate Fermi gases will collapse in certain conditions. Ref.[15] has also investigated the instability of a trapped Fermi gas with interactions between particles and has given the density condition of instability of the system.

An external magnetic field is a kind of familiar and important trap, so it is significant to discuss and analyse the instability of a weakly interacting Fermi gas trapped in a weak magnetic field. In the present paper, within 'the maximum approximation' method and the ensemble theory, we will study analytically the free energy expressed by small parameter r of a weakly interacting Fermi gas trapped in a weak magnetic field. Based on the derived expression, we will give the instability conditions of the system at both high and low temperatures. Finally, we will analyse the effects of magnetic field and inter-particle interactions on the instability of the system.

2. The analytical expression of free energy expressed by small parameter r

We consider an imperfect gas of spin-1/2 fermions, with volume V and particle number N, confined in an even weak magnetic field $B = B_Z$. It has been derived that, if s-wave considered only, the en-

ergy eigenvalue of the system can be written as[16]

$$E=\sum_p (n_p^+ + n_p^-)\frac{p^2}{2m}+\frac{\alpha}{V}N^+N^- -(N^+-N^-)\mu B, \quad (1)$$

where μ is $\mu_{\rm B}$, the magnetic moment of fermions, $\alpha = 4\pi a\hbar^2/m$ is the parameter of interactions, a is s-wave scattering length, $n_P^+(n_p^-)$ is the number of particles in the states of momentum $\boldsymbol{p}$ with spin-up (spin-down), $N^+(N^-)$ is the total number of particles with spin-up (spin-down).

The partition function of the system is

$$\begin{aligned} Q=&\sum_{\{n_p^+,n_p^-\}} \exp(-\beta E)\\ =&\sum_{N^+=0}^{N} \exp\Big[\beta\mu B(2N^+-N)-\frac{\alpha}{V}\beta N^+(N-N^+)\Big]\\ &\times\sum_{N^+}\exp\Big(-\beta\sum_p n_p^+\frac{p^2}{2m}\Big)\\ &\times\sum_{N^-}\exp\Big(-\beta\sum_p n_p^-\frac{p^2}{2m}\Big). \end{aligned} \quad (2)$$

Based on the method which deal with nonideal Bose gases proposed by Huang, Yang and Lee *et al*,[1−4] we introduce $A_0(\xi)$ and consider it as the free energy of a 'fiction system' of ξ spinless noninteracting fermions

$$A_0(\xi)=-\frac{1}{\beta}\ln\sum_{\{\xi_p\}}\exp\Big(-\beta\sum_p \xi_p\frac{p^2}{2m}\Big). \quad (3)$$

The free energy of the interacting system is then given by

$$\begin{aligned} F=&-\frac{1}{\beta}\ln Q\\ =&N\mu B-\frac{1}{\beta}\ln\sum_{N^+}\exp\Big\{-\beta\Big[A_0(N^+)+A_0(N-N^+)\\ &+\frac{\alpha}{V}N^+(N-N^+)-2\mu BN^+\Big]\Big\}. \end{aligned} \quad (4)$$

For a system including a large number of particles, we use 'the maximum approximation' method, i.e. the logarithm of the Sum in Eq.(4) may be replaced by the logarithm of the largest term,[16]

$$\begin{aligned} F=&F(\bar{N}^+)\\ =&\mu B(N-2\bar{N}^+)+A_0(\bar{N}^+)\\ &+A_0(N-\bar{N}^+)+\frac{\alpha}{V}\bar{N}^+(N-\bar{N}^+), \end{aligned} \quad (5)$$

$F(\bar{N}^+)$ is the maximum of $F(N^+)$, $\bar{N}^+$ can be decided by $\partial F(N^+)/\partial N^+|_{N^+=\bar{N}^+}=0$:

$$-2\mu B+\Big[\frac{\partial A_0(N^+)}{\partial N^+}-\frac{\partial A_0(N-N^+)}{\partial N^+}\Big]_{N^+=\bar{N}^+}+\frac{\alpha}{V}(N-2N^+)\Big|_{N^+=\bar{N}^+}=0, \quad (6)$$

where

$$\begin{aligned} F(N^+)=&\mu B(N-2N^+)+A_0(N^+)\\ &+A_0(N-N^+)+\frac{\alpha}{V}N^+(N-N^+). \end{aligned}$$

Using the chemical potential u_0 of the "fiction system" described in this paper, we can express Eq.(6) as

$$u_0(\bar{N}^+)-u_0(N-\bar{N}^+)=2\mu B-\frac{\alpha}{V}(N-2\bar{N}^+). \quad (7)$$

Let

$$r=(2\bar{N}^+/N)-1,$$

thus Eq.(7) can be written as

$$u_0\Big(\frac{1+r}{2}N\Big)-u_0\Big(\frac{1-r}{2}N\Big)=2\mu B+\alpha nr, \quad (8)$$

where n is the particle-number density.

When $B=0$, we have $r=0$. When B is very small, r is very small, too. Thus the left-hand side of Eq.(8) can be expanded with respect to r by Taylor series . If only the terms up to the first power in r are considered, one can derive from Eq.(8) that

$$\begin{aligned} r\frac{\partial u_0(xN)}{\partial x}\Big|_{x=1/2}&\approx 2\mu B+\alpha nr,\\ r&\approx\frac{2\mu B}{\partial u_0(xN)/\partial x|_{x=1/2}-\alpha n}. \end{aligned} \quad (9)$$

The free energy of the system can be expressed by small parameter r:

$$\begin{aligned} F\Big(\frac{1+r}{2}N\Big)=&-N\mu Br+A_0\Big(\frac{1+r}{2}N\Big)\\ &+A_0\Big(\frac{1-r}{2}N\Big)+\frac{\alpha}{4}nN. \end{aligned}$$

We expand A_0 by Taylor series at the point of $N/2$

and keep only the terms up to the lowest order in r. The free energy of the system then can be expressed as

$$F\Big(\frac{1+r}{2}N\Big) \approx -N\mu Br + 2A_0\Big(\frac{N}{2}\Big) + \frac{\alpha}{4}nN. \qquad (10)$$

3. The instability conditions of the system at high and low temperatures

The equation of state of the system with respect to free energy can be written as

$$P = -\Big(\frac{\partial F}{\partial V}\Big)_{N,T} = -\Big[N\mu B\Big(\frac{\partial r}{\partial V}\Big)_{N,T} + 2\frac{\partial A_0(N/2)}{\partial V}\Big|_{N,T} - \frac{1}{4}\alpha n^2\Big], \qquad (11)$$

moreover

$$\Big(\frac{\partial P}{\partial V}\Big)_T = -\Big[N\mu B\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} + 2\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} + \frac{1}{2}\frac{\alpha n^2}{V}\Big]. \qquad (12)$$

So the instability condition of the system is

$$N\mu B\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} + 2\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} + \frac{1}{2}\frac{\alpha n^2}{V} < 0. \qquad (13)$$

At low-temperature limit, i.e. $T \to 0$, there are conclusions

$$u_0\Big(\frac{N}{2}\Big) = \Big(\frac{3N}{8\pi V}\Big)^{2/3}\frac{h^2}{2m} = \varepsilon_{\rm F} = kT_{\rm F},$$

$$\frac{\partial u_0(xN)}{\partial x}\Big|_{x=1/2} = \frac{4}{3}\varepsilon_{\rm F}, \quad r = \frac{3\mu B}{2\varepsilon_{\rm F}}\Big[1 + \frac{3\alpha n}{4\varepsilon_{\rm F}}\Big],$$

$$A_0(x) = \frac{h^2}{2m}\int\Big(\frac{3}{4\pi V}\Big)^{2/3}x^{2/3}{\rm d}x = \frac{3}{5}xu_0(x),$$

$$A_0\Big(\frac{N}{2}\Big) = \frac{3}{10}Nu_0\Big(\frac{N}{2}\Big) = \frac{3}{10}N\varepsilon_{\rm F}.$$

Where $\varepsilon_{\rm F}$ and $T_{\rm F}$ are Fermi energy and Fermi temperature respectively. We can also obtain the following equations by calculation:

$$\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} = -\frac{\mu B}{3\varepsilon_{\rm F}V^2}\Big[1 + \frac{3\alpha n}{4\varepsilon_{\rm F}}\Big],$$

$$\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} = \frac{1}{3}\frac{n}{V}\varepsilon_{\rm F}.$$

Therefore, the instability condition of the system is

$$\Big(\alpha n + \frac{4}{3}\varepsilon_{\rm F}\Big)\varepsilon_{\rm F}^2 < \frac{1}{2}\Big(\alpha n + \frac{4}{3}\varepsilon_{\rm F}\Big)\mu^2B^2. \qquad (14)$$

At low-temperature region, i.e. $T < T_{\rm F}$, we have

$$u_0\Big(\frac{N}{2}\Big) = \varepsilon_{\rm F}\Big[1 - \frac{\pi^2}{12}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big],$$

$$r = \frac{2\mu B}{(4/3)\varepsilon_{\rm F}[1 + (\pi^2/12)(T/T_{\rm F})^2] - \alpha n}$$

$$\approx \frac{3\mu B}{2\varepsilon_{\rm F}}\Big\{1 - \frac{\pi^2}{12}\Big(\frac{T}{T_{\rm F}}\Big)^2 + \frac{3\alpha n}{4\varepsilon_{\rm F}}\Big[1 - \frac{\pi^2}{6}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big]\Big\},$$

$$A_0(x) = \int\Big[\varepsilon_{\rm F}(x) - \frac{\pi^2}{12}\frac{(kT)^2}{\varepsilon_{\rm F}(x)}\Big]{\rm d}x$$

$$= \frac{3}{5}x\varepsilon_{\rm F}(x) - \frac{\pi^2}{4}x\frac{(kT)^2}{\varepsilon_{\rm F}(x)},$$

$$A_0\Big(\frac{N}{2}\Big) = \frac{3}{10}N\varepsilon_{\rm F} - \frac{\pi^2}{8}N\frac{(kT)^2}{\varepsilon_{\rm F}}.$$

By calculation we have

$$\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} = -\frac{\mu B}{3\varepsilon_{\rm F}V^2}\Big\{1 + \frac{3}{2}\pi^2\Big(\frac{T}{T_{\rm F}}\Big)^2 + \frac{3\alpha n}{4\varepsilon_{\rm F}}\Big[1 + \frac{1}{6}\pi^2\Big(\frac{T}{T_{\rm F}}\Big)^2\Big]\Big\},$$

$$\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} = \frac{N\varepsilon_{\rm F}}{3V^2} + \frac{\pi^2}{36}\frac{(kT)^2}{\varepsilon_{\rm F}V^2}N,$$

where k is Boltzmann's constant. The instability condition of the system is

$$\Big\{\frac{4}{3}\varepsilon_{\rm F}\Big[1 + \frac{\pi^2}{12}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big] + \alpha n\Big\}\varepsilon_{\rm F}^2 < \mu^2B^2\frac{1}{2}\Big\{\frac{4}{3}\varepsilon_{\rm F}\Big[1 + \frac{3}{2}\pi^2\Big(\frac{T}{T_{\rm F}}\Big)^2\Big] + \alpha n\Big[1 + \frac{\pi^2}{6}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big]\Big\}. \qquad (15)$$

When $T > T_{\rm F}$, there are conclusions

$$u_0\Big(\frac{N}{2}\Big) = kT\ln\frac{n\lambda^3}{2} + \frac{kT}{2^{3/2}}\frac{n\lambda^3}{2},$$

$$r = \frac{\mu B}{kT}\Big[1 - \frac{n\lambda^3}{2^{5/2}}\Big]\Big[1 + \frac{\alpha n}{2kT}\Big]$$

$$\approx \frac{\mu B}{kT}\Big[1 + \frac{\alpha n}{2kT} - \frac{n\lambda^3}{2^{5/2}}\Big],$$

$$A_0\Big(\frac{N}{2}\Big) = \frac{NkT}{2}\Big[\ln\frac{n\lambda^3}{2} - 1 + \frac{n\lambda^3}{2^{7/2}}\Big],$$

where $\lambda = h/\sqrt{2\pi mkT}$ is the thermal wavelength. By further calculation, we have

$$\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} = \frac{\mu B}{kTV^2}\Big[\frac{\alpha n}{kT} - \frac{n\lambda^3}{2^{3/2}}\Big],$$

$$\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} = \frac{NkT}{2V^2}\Big[1 + \frac{n\lambda^3}{2^{5/2}}\Big].$$

So the instability condition of the system in this condition is

$$\Big(\frac{\mu B}{kT}\Big)^2 \alpha n - \frac{n\lambda^3\mu^2 B^2}{2^{3/2}kT} + kT\Big[1 + \frac{n\lambda^3}{2^{5/2}}\Big] + \frac{1}{2}\alpha n < 0. \quad (16)$$

At high temperatures, i.e. $T \gg T_{\rm F}$, there are conclusions

$$u_0\Big(\frac{N}{2}\Big) = kT\ln\frac{n\lambda^3}{2}, \quad \frac{\partial u_0(xN)}{\partial x}\Big|_{x=1/2} = 2kT,$$

$$r = \frac{2\mu B}{2kT - \alpha n} = \frac{\mu B}{kT}\Big[1 + \frac{\alpha n}{2kT}\Big],$$

$$A_0\Big(\frac{N}{2}\Big) = \int u_0\Big(\frac{N}{2}\Big)\mathrm{d}\Big(\frac{N}{2}\Big) = \frac{N}{2}kT\Big(\ln\frac{n\lambda^3}{2} - 1\Big).$$

By calculation, there are

$$\Big(\frac{\partial^2 r}{\partial V^2}\Big)_{N,T} = \frac{\mu B}{(kT)^2V^2}\alpha n,$$

$$\frac{\partial^2 A_0(N/2)}{\partial V^2}\Big|_{N,T} = \frac{1}{2}\frac{NkT}{V^2}.$$

So the instability condition of the system in this condition is

$$\Big(\frac{\mu B}{kT}\Big)^2 \alpha n + kT + \frac{1}{2}\alpha n < 0. \quad (17)$$

4. The effects of magnetic field and inter-particle interactions on the instability

At the whole low-temperature extent, as far as the inter-particle interactions are concerned, whether they are repulsive force or attractive force with $(\alpha n + 4\varepsilon_{\rm F}/3)$ positive, there are the two following cases. (1) At the low-temperature limit, i.e. $T \to 0$, we can conclude from Eq.(14) that the system will display instability as long as $B > \sqrt{2}\varepsilon_{\rm F}/\mu = B_0$, where B_0 is the lower-limit magnetic field of instability. This lower-limit magnetic field is not strong as long as the particle-number density is relatively low. (2) When the temperature is relatively low but not close to 0 K, we can conclude from Eq.(15) that the system will display instability as long as

$$B > B_0' = B_0\left(\frac{\frac{4}{3}\varepsilon_{\rm F}\Big[1 + \frac{\pi^2}{12}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big] + \alpha n}{\frac{4}{3}\varepsilon_{\rm F}\Big[1 + \frac{3}{2}\pi^2\Big(\frac{T}{T_{\rm F}}\Big)^2\Big] + \alpha n\Big[1 + \frac{\pi^2}{6}\Big(\frac{T}{T_{\rm F}}\Big)^2\Big]}\right)^{1/2}.$$

Here the lower-limit magnetic field B_0' is relate to temperature and is weaker than B_0, and the lower-limit magnetic field will become weak when the temperature increases.

At the whole low-temperature extent, if the inter-particle interactions are attractive force with $(\alpha n + 4\varepsilon_{\rm F}/3)$ negative (this can be satisfied as long as the scattering length $|a|$ is relatively high under the condition of weak interactions), we have the following two cases. (1) At low-temperature limit, so the system will display instability as long as $B < \sqrt{2}\varepsilon_{\rm F}/\mu = B_0$, where B_0 is the upper-limit magnetic field of instability. (2) When the temperature is relatively low but not close to 0K, we can conclude from Eq.(15) that the system will display instability when the upper-limit magnetic field is different from B_0 slightly.

From Eq.(17), when the temperature is very high, i.e. $T \gg T_{\rm F}$, the system with repulsive interactions is always stable. If the interactions are attractive, the condition of instability becomes

$$kT < [(\mu B/kT)^2 + 1/2]|\alpha|n\,.$$

Thus in the premise of weak magnetic field and weak interactions, the greater the scattering length of attractive interactions $|a|$ is, the stronger the magnetic field is and the larger the particle-number density is, the bigger the possibility of instability in the system will be.

From Eq.(16), one can find that when temperature is relatively high, i.e. $T > T_{\rm F}$, the system with

repulsive interactions is stable. But when the interparticle interactions are attractive, the condition of instability becomes

$$kT\Big\{1+\frac{n\lambda^3}{2^{5/2}}\Big[1-2\Big(\frac{\mu B}{kT}\Big)^2\Big]\Big\}<\Big[\Big(\frac{\mu B}{kT}\Big)^2+\frac{1}{2}\Big]|\alpha|n.$$

Here, on condition that magnetic field, the scattering length $|a|$ of attractive interactions and the particle-number density n are all smaller than those under the condition $T \gg T_{\rm F}$, the system can display instability all the same.

5. Conclusions

We have discussed the weakly interacting Fermi gas trapped in weak magnetic field and have given the analytical expressions of the instability conditions at both high and low temperatures. It is shown that at the whole low-temperature extent, whether the interactions are repulsive or attractive with $(\alpha n + 4\varepsilon_{\rm F}/3)$ positive, there is a lower-limit magnetic field of instability; in addition, there is an upper-limit magnetic field for the system of attractive interactions with $(\alpha n+4\varepsilon_{\rm F}/3)$ negative. At the whole high-temperature extent, the system with repulsive interactions is always stable, but for the system with attractive interactions, the greater the scattering length of attractive interactions $|a|$ is, the stronger the magnetic field is and the larger the particle-number density is, the bigger the possibility of instability in the system will be.

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