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Late-time evolution of massive Dirac fields in the Kerr background*

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The late-time tail of massive Dirac fields in Kerr spacetime is investigated by using the black hole Green function. It is shown that in the intermediate late times there are two kinds of new properties. The one is that the asymptotic behaviour of the massive Dirac fields is dominated by a decaying tail without any oscillation, which is different from the oscillatory decaying tails of the massive scalar field; the other is that the dumping exponent for the massive Dirac field depends not only on the multiple number of the wave mode and the mass of the Dirac particle but also on the rotating parameter of the black hole.

Keywords: black hole, late-time evolution, massive Dirac field

PACC: 0470, 0365, 0430, 9760

1. Introduction

The subject that the late-time evolution of various field perturbations outside a black hole plays a great role mainly involves two aspects of black hole physics: the no-hair theorem and the mass inflation scenario.[1,2] The external field of a Kerr black hole is characterized solely by the black hole mass and angular momentum.[3,4] In general, the evolution of field perturbation around a black hole consists roughly of three stages.[5] The first one is an initial wave burst coming directly from the source of perturbation. The second one involves the damped oscillations called the quasinormal modes.[6] And the last one is the propagation of a wave in such a curved background at very late times, named the power-law tail.[7,8] Physically, these inverse power-law tails are associated with the backscattering off the effective curvature potential at asymptotically far regions.

Leaver[9] demonstrated that the late-time tail can be associated mathematically with the existence of a branch cut in the Green's function for the wave propagation problem. Therefore, the decay rate of the various fields have been extensively studied.[10−22] We have studied the late-time tail behaviour of massive Dirac fields in the Schwarzschild black hole[18] and Reissner-Nordström background,[19] and found some new properties. In this work, we will extend our study to the late-time behaviour of the massive Dirac fields in the Kerr spacetime. We find that the decay rate of the massive Dirac perturbations is affected by the rotating parameter a and the quantum numbers l and m.

The rest of this paper is organized as follows. In Section 2 the decoupled massive Dirac equations in the Kerr spacetime are presented by using Newman–Penrose formalism. In Section 3 the black-hole Green function is introduced and the late-time evolution of the massive Dirac fields in the Kerr background is investigated. Section 4 is devoted to a summary.

2. Massive Dirac equations in the Kerr spacetime

For the Kerr spacetime, the null tetrad[23] can be taken as

$$\begin{aligned} l^\mu &= \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \\ n^\mu &= \frac{1}{2\rho^2}((r^2+a^2), -\Delta, 0, a), \\ m^\mu &= \frac{1}{\sqrt{2}\bar{\rho}}\left(\mathrm{i}a\sin\theta, 0, 1, \frac{\mathrm{i}}{\sin\theta}\right), \end{aligned} \tag{1}$$

with

$$\Delta = r^2 + a^2 - 2Mr, \quad \bar{\rho} = r + \mathrm{i}a\cos\theta, \tag{2}$$

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where M and a represent the mass and angular momentum per unit mass of the Kerr black hole respectively. After tedious calculation, we find that the radial decoupled equation of the Dirac fields[24] can be expressed as

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{1+s}\frac{\mathrm{d}R_s}{\mathrm{d}r}\right)+\frac{2\mathrm{i}s\mu\Delta}{\lambda-2\mathrm{i}s\mu r}\frac{\mathrm{d}R_s}{\mathrm{d}r}+P_sR_s=0,\quad(s=\pm\frac{1}{2}),\tag{3}$$

with

$$\begin{aligned}P_s=&\frac{K^2-2\mathrm{i}sK(r-M)}{\Delta}+4\mathrm{i}s\omega r+\left(s+\frac{1}{2}\right)\\&+\frac{\mathrm{i}\mu(s+1/2)(r-M)-\mu K}{\lambda-2\mathrm{i}s\mu r}-\mu^2r^2-\lambda^2,\end{aligned}\tag{4}$$

where μ is the mass of the Dirac particle and λ^2 is the separation constant that can be taken as

$$\lambda^2=\left(1+\frac{1}{2}\right)^2-2m\omega a\left(1+\frac{s^2}{l(l+1)}\right)+\frac{m\mu a(2l+1)}{2l(l+1)}+O(a^2),\tag{5}$$

for the slow rotating black hole (neglecting the $O(a^2\omega^2)$ and higher), and l and m are the angular quantum numbers.

Introducing a usual tortoise coordinate $\mathrm{d}r_\star=\dfrac{r^2+a^2}{\Delta}\mathrm{d}r$, Eq.(3) can have a solution in the form

$$R_s=\frac{\Delta^{-s/2}}{\sqrt{r^2+a^2}}(\lambda^2+\mu^2r^2)^{1/4}\mathrm{e}^{-\mathrm{i}s\arctan\left(\frac{\mu r}{\lambda}\right)}\Psi_s.\tag{6}$$

Then, we obtain a wave equation

$$\frac{\mathrm{d}^2\Psi_s}{\mathrm{d}{r_\star}^2}+\left[\frac{\mathrm{d}H_s}{\mathrm{d}r_\star}-H_s^2+\frac{\Delta}{(r^2+a^2)^2}P_s\right]\Psi_s=0,\tag{7}$$

where

$$H_s=-\frac{s}{2(r^2+a^2)}\frac{\mathrm{d}\Delta}{\mathrm{d}r}-\frac{\Delta}{(r^2+a^2)^{3/2}}\frac{d\sqrt{r^2+a^2}}{\mathrm{d}r}-\frac{s\Delta}{(r^2+a^2)}\frac{\mathrm{i}\mu}{\lambda-2\mathrm{i}s\mu r}.\tag{8}$$

We use Eq.(7) to study the late-time evolution of the massive Dirac fields.

3. Late-time tail of the massive Dirac fields

The time evolution of massive Dirac fields in Kerr spacetime is given by

$$\Psi_s(r_\star,t)=\int[G_s(r_\star,r'_\star;t)\partial_t\Psi_s(r'_\star,0)+\partial_tG_s(r_\star,r'_\star;t)\Psi_s(r'_\star,0)]\mathrm{d}r'_\star.\tag{9}$$

For $t>0$, the black hole (retarded) Green's function $G_s(r_\star,r'_\star;t)$ is determined by

$$\begin{aligned}&\left\{\frac{\partial^2}{\partial{r_\star}^2}-\frac{\partial^2}{\partial t^2}+\frac{\mathrm{d}H_s}{\mathrm{d}r_\star}-H_s^2+\left[\frac{-2\mathrm{i}ma}{\varrho^2}+\frac{2s(r-M)}{\varrho^2}-\frac{4\mathrm{s}r\Delta}{\varrho^4}-\frac{\mathrm{i}\mu}{\lambda+2\mathrm{i}s\mu r}\frac{\Delta}{\varrho^2}\right]\frac{\partial}{\partial t}\right.\\&+\frac{m^2a^2+2\mathrm{i}s(r-M)ma}{\varrho^4}-\frac{\Delta(\mu^2r^2+\lambda^2-(s+1/2))}{\varrho^4}\\&\left.+\frac{\Delta}{\varrho^4}\frac{\mathrm{i}\mu(r-M)(s+1/2)+\mu ma}{\lambda-2\mathrm{i}s\mu r}\right\}G_s(r_{\star,},r'_{\star,};t)=\delta(t)\delta(r_\star-r'_\star),\end{aligned}\tag{10}$$

with $\varrho = \sqrt{r^2 + a^2}$.

The causality condition gives us the initial condition $G_s(r_\star, r'_\star; t) = 0$ for $t \leq 0$. We calculate $G_s(r_\star, r'_\star; t)$ through the Fourier transform

$$G_s(r_\star, r'_\star; t) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} \tilde{G}_s(r_\star, r'_\star; \omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega, \quad (11)$$

and the corresponding inversion formula is given by

$$\tilde{G}_s(r_\star, r'_\star; \omega) = \int_{0^-}^{\infty} G_s(r_\star, r'_\star; t) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t, \quad (12)$$

where c is a positive constant. The Fourier transform is analytic in the upper half ω-plane and it satisfies the equation

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r_\star^2} + \frac{\mathrm{d}H_s}{\mathrm{d}r_\star} - H_s^2 + \frac{\Delta}{(r^2+a^2)^2} P_s \right\} \tilde{G}_s(r_\star, r'_\star; \omega) = \delta(r_\star - r'_\star). \quad (13)$$

We define auxiliary functions $\tilde{\Psi}_{\pm 1}(r_\star, \omega)$ and $\tilde{\Psi}_{\pm 2}(r_\star, \omega)$ that are (linearly independent) solutions to Eq.(7). Using the solutions $\tilde{\Psi}_{\pm 1}$ and $\tilde{\Psi}_{\pm 2}$, the black hole Green function can be constructed as

$$\tilde{G}_\pm(r_\star, r'_\star; \omega) = -\frac{1}{W_\pm(\omega)} \begin{cases} \tilde{\Psi}_{\pm 1}(r_\star, \omega) \tilde{\Psi}_{\pm 2}(r'_\star, \omega), & r_\star < r'_\star; \\ \tilde{\Psi}_{\pm 1}(r'_\star, \omega) \tilde{\Psi}_{\pm 2}(r_\star, \omega), & r_\star > r'_\star, \end{cases} \quad (14)$$

where $W_\pm(\omega) = W_\pm(\tilde{\Psi}_{\pm 1}, \tilde{\Psi}_{\pm 2}) = \tilde{\Psi}_\pm \tilde{\Psi}_{\pm 2,x} - \tilde{\Psi}_{\pm 2} \tilde{\Psi}_{\pm 1,x}$ is the Wronskian.

When both the observer and the initial data are situated far away from the black hole ($r \gg M$), we expand the wave equation (7) for the massive Dirac fields as a power series in M/r. Neglecting terms of order $O((\omega/r)^2)$ and higher, we obtain

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \omega^2 - \mu^2 + \frac{4M\omega^2 - 2M\mu^2}{r} - \frac{(4M^2 - a^2)\mu^2 + 2am\omega + \lambda^2 + \left(2\mathrm{i}sM + \dfrac{\lambda}{\mu}\right)\omega}{r^2} \right\} \xi = 0, \quad (15)$$

where $\xi = \sqrt{\dfrac{\Delta}{r^2+a^2}}\, \Psi_s$. Taking the transformation

$$\begin{aligned} & z = 2\sqrt{\mu^2 - \omega^2} r = 2\varpi r, \\ & \xi = \mathrm{e}^{-z/2} z^{1/2+b} \Phi, \\ & b^2 = \frac{1}{4} + \lambda^2 + (4M^2 - a^2)\mu^2 + 2am\omega + \left(\frac{\lambda}{\mu} + 2\mathrm{i}sM\right)\omega, \\ & k = \frac{M\mu^2}{\varpi} - 2M\varpi, \end{aligned} \quad (16)$$

Eq.(15) becomes the confluent hypergeometric equation[25]

$$z \frac{\mathrm{d}^2 \Phi}{\mathrm{d}z^2} + (1 + 2b - z) \frac{\mathrm{d}\Phi}{\mathrm{d}z} - \left(\frac{1}{2} + b - k\right) \Phi = 0. \quad (17)$$

The two basic solutions requisite for building the Green's function can be expressed as

$$\tilde{\Psi}_1 = A \mathrm{e}^{-\varpi r} (2\varpi r)^{1/2+b} M\left(\frac{1}{2} + b - k, 1 + 2b, 2\varpi r\right), \quad (18)$$

$$\tilde{\Psi}_2 = B \mathrm{e}^{-\varpi r} (2\varpi r)^{1/2+b} U\left(\frac{1}{2} + b - k, 1 + 2b, 2\varpi r\right), \quad (19)$$

where A and B are normalization constants. The functions $M(\tilde{k},\tilde{b},z)$ and $U(\tilde{k},\tilde{b},z)$ represent the two standard solutions to the confluent hypergeometric equation.[25] $U(\tilde{k},\tilde{b},z)$ is a many-valued function, so there is a cut in $\tilde{\Psi}_2$. Making use of Eq.(11), we find that the branch cut that contributes to the Green's function can be expressed as

$$\begin{aligned} G^{\rm C}(r_\star,r'_\star;t) &= \frac{1}{2\pi}\int_{-\mu}^{\mu}\left[\frac{\tilde{\Psi}_1(r'_\star,\omega{\rm e}^{{\rm i}\pi})\,\tilde{\Psi}_1(r_\star,\omega{\rm e}^{{\rm i}\pi})}{W(\omega{\rm e}^{{\rm i}\pi})}-\frac{\tilde{\Psi}_1(r'_\star,\omega)\,\tilde{\Psi}_1(r_\star,\omega)}{W(\omega)}\right]{\rm e}^{-{\rm i}\omega t}{\rm d}\omega \\ &\equiv \frac{1}{2\pi}\int_{-\mu}^{\mu}F(\varpi){\rm e}^{-{\rm i}\omega t}{\rm d}\omega. \end{aligned} \tag{20}$$

We can obtain the following relations by the use of Eqs.(13.1.32), (13.1.33), and (13.1.34) in Ref.[25]:

$$\tilde{\Psi}_1(2\varpi r) = A{\rm e}^{-\varpi r}(2\varpi r)^{\frac{1}{2}+b}M\left(\frac{1}{2}+b-k,1+2b,2\varpi r\right), \tag{21}$$

$$\begin{aligned} \tilde{\Psi}_2(2\varpi r) &= B\frac{\Gamma(-2b)}{\Gamma\left(\frac{1}{2}-b-k\right)}{\rm e}^{-\varpi r}(2\varpi r)^{\frac{1}{2}+b}M\left(\frac{1}{2}+b-k,1+2b,2\varpi r\right) \\ &\quad +B\frac{\Gamma(2b)}{\Gamma\left(\frac{1}{2}+b-k\right)}{\rm e}^{-\varpi r}(2\varpi r)^{\frac{1}{2}-b}M\left(\frac{1}{2}-b-k,1-2b,2\varpi r\right). \end{aligned} \tag{22}$$

Using Eq.(13.1.22) in Ref.[25], we have

$$W(\varpi{\rm e}^{{\rm i}\pi}) = -W(\varpi) = AB\frac{\Gamma(2b)}{\Gamma\left(\frac{1}{2}+b-k\right)}4b\varpi. \tag{23}$$

Then, we find that $F(\varpi)$ is given by

$$\begin{aligned} F(\varpi) = {} & \frac{{r_\star}^{\frac{1}{2}-b}{r'_\star}^{\frac{1}{2}+b}{\rm e}^{-\varpi(r_\star+r'_\star)}}{2b}\left[M\left(\frac{1}{2}+b-k,1+2b,2\varpi r'_\star\right)M\left(\frac{1}{2}-b-k,1-2b,2\varpi r_\star\right)\right. \\ & \left.-M\left(\frac{1}{2}+b+k,1+2b,2\varpi r'_\star\right)M\left(\frac{1}{2}-b+k,1-2b,2\varpi r_\star\right)\right]+\frac{\Gamma(-2b)\Gamma\left(\frac{1}{2}+b-k\right)}{\Gamma(2b)\Gamma\left(\frac{1}{2}-b-k\right)} \\ & \times\frac{{\rm e}^{-\varpi(r_\star+r'_\star)}(4\varpi^2 r_\star r'_\star)^{\frac{1}{2}+b}}{4b\varpi}\left[M\left(\frac{1}{2}+b-k,1+2b,2\varpi r'_\star\right)M\left(\frac{1}{2}+b-k,1+2b,2\varpi r_\star\right)\right. \\ & \left.+{\rm e}^{(1+2b){\rm i}\pi}M\left(\frac{1}{2}+b+k,1+2b,2\varpi r'_\star\right)M\left(\frac{1}{2}+b+k,1+2b,2\varpi r_\star\right)\right]. \end{aligned} \tag{24}$$

We focus our attention on the intermediate asymptotic behaviour of the massive Dirac fields. When the late-time behaviours of $G^{\rm C}$ at the time scale $M \ll r \ll t \ll \frac{M}{(M\mu)^2}$, which leads to $k \ll 1$, following Hod and Prian,[14,15] the effective contribution to the integral in Eq.(11) is limited to the range $\varpi = O(\sqrt{\mu/t})$. This is due to the rapidly oscillating term ${\rm e}^{-{\rm i}\omega t}$, which leads to a mutual cancellation between the positive and the negative parts of the integrand. There are both the field's parameter μ and the space-time parameter M in the factor k, which originates from the $1/r$ term and describes that the effect of backscattering off the spacetime curvature from the asymptotically far regions is negligible.

For the case $k \ll 1$, noting that $M(\tilde{k},\tilde{b},z) \simeq 1$ as $z \to 0$, we obtain

$$F(\varpi) \simeq -\frac{1+\mathrm{e}^{(1+2b)\mathrm{i}\pi}}{4\varpi b}\frac{\Gamma(-2b)\Gamma\left(\frac{1}{2}+b\right)}{\Gamma(2b)\Gamma\left(\frac{1}{2}-b\right)}(2\varpi)^{1+2b}(r_\star r'_\star)^{\frac{1}{2}+b}$$
$$= \frac{\pi}{\sin(\pi b)}\frac{1+\mathrm{e}^{1+2b}\mathrm{i}\pi}{2^{1+2b}b^2}\frac{\varpi^{2b}}{\Gamma(b)^2}(r_\star r'_\star)^{\frac{1}{2}+b}. \tag{25}$$

After tedious calculation, we have

$$G^{\mathrm{C}}(r_\star, r'_\star; t) = \frac{1}{4}\int_{-\mu}^{\mu}\frac{1}{\sin(\pi b)}\frac{1+\mathrm{e}^{(1+2b)\mathrm{i}\pi}}{2^{2b}b^2}\frac{(r_\star r'_\star)^{\frac{1}{2}+b}}{\Gamma(b)^2}(\mu^2-\omega^2)^b\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega. \tag{26}$$

As the above integral cannot be worked out analytically, we use numerical integral to give three figures to describe how the decay rate is affected by the rotating parameters a and m. Figures 1 and 2 show that the larger the parameter a is, the more slowly the perturbation decays for $m\lambda > 0$, but the larger the parameter a is, the more quickly it decays for $m\lambda < 0$. Figure 3 shows that when parameter m ranges from $\frac{3}{2}$ to $-\frac{3}{2}$, the decay rate increases for the $\lambda > 0$ case, but it decreases for $\lambda < 0$.

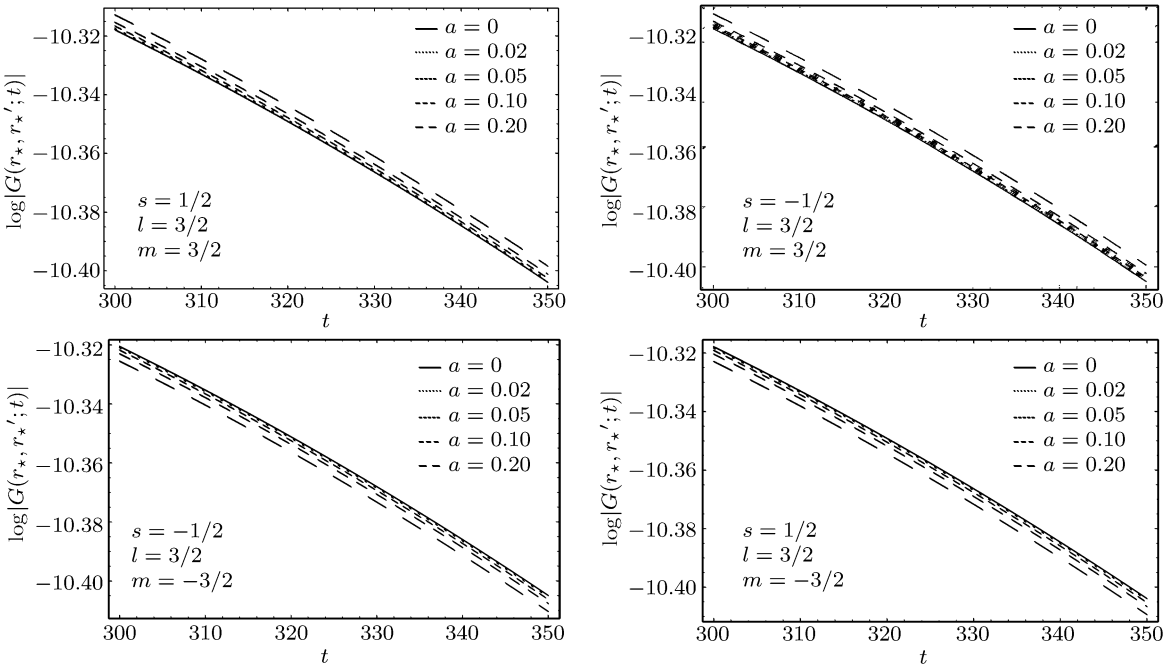


Fig.1. $\ln|G^C(r_\star, r'_\star; t)|$ versus t for different a and m with $\mu = 0.01$ and $\lambda > 0$. For $m > 0$, the larger the parameter a is, the more slowly the perturbation decays. But for $m < 0$, the larger the parameter a is, the more quickly the perturbation decays.

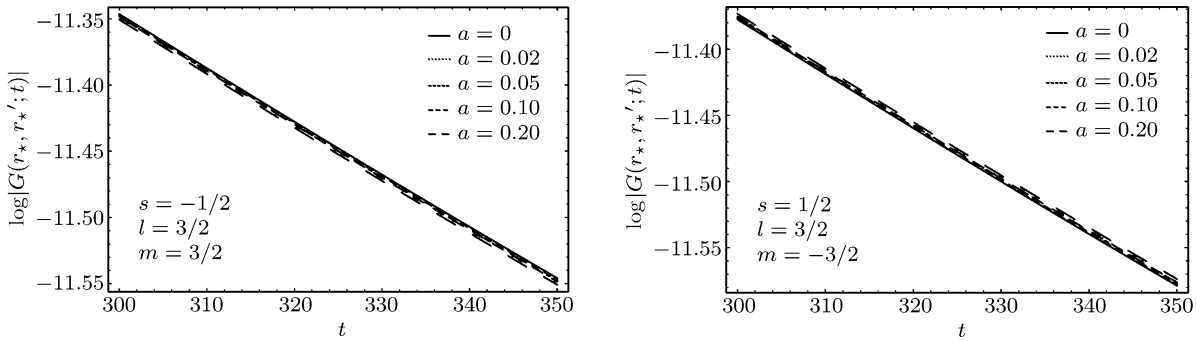


Fig.2. $\ln|G^{\mathrm{C}}(r_\star, r'_\star; t)|$ versus t for different a and m with $\mu = 0.01$ and $\lambda < 0$. When $m > 0$, the rotating parameter a speeds the decay up; and $m < 0$, slows it down.

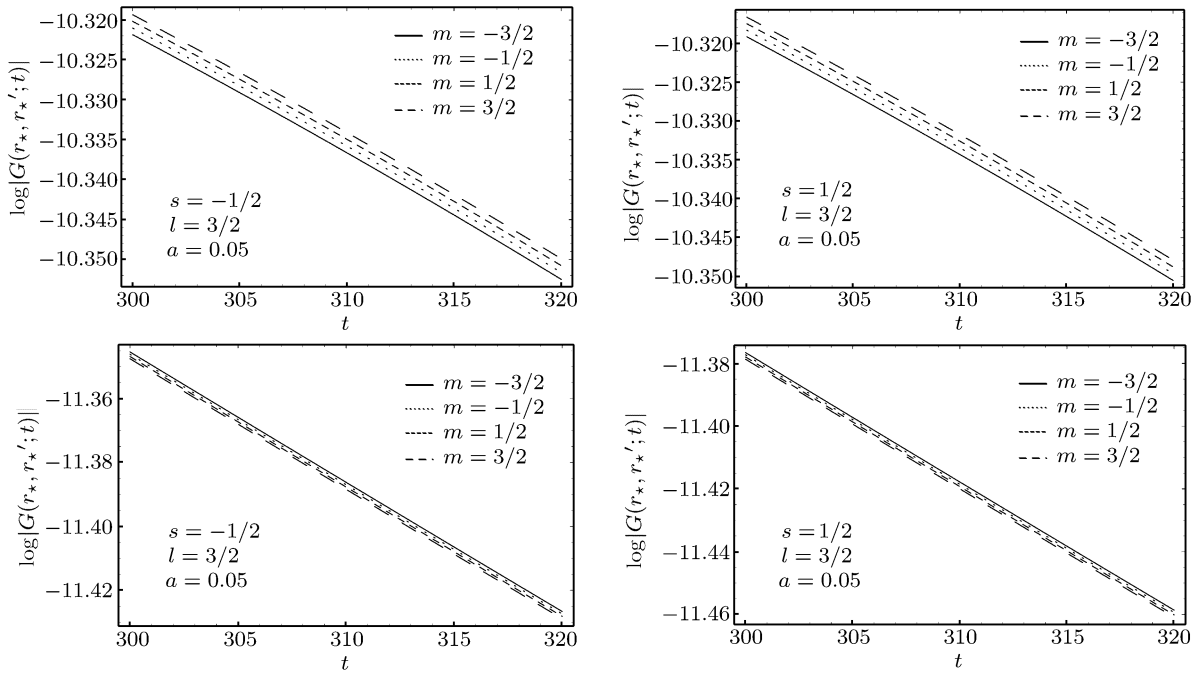


Fig.3. $\ln|G^C(r_\star, r'_\star; t)|$ versus t for different m and λ, with $\mu = 0.01$ and $a = 0.05$. The top two figures are for the $\lambda > 0$ case, which show that the decay rate increases when parameter m ranges from $\frac{3}{2}$ to $-\frac{3}{2}$. The bottom two for $\lambda < 0$, which show that it decreases as m ranges from $\frac{3}{2}$ to $-\frac{3}{2}$.

4. Summary

The intermediate late-time behaviour of massive Dirac fields in the background of a rotating black hole is studied using the black hole Green's function. We use numerical method to study how the decay rate is affected by the rotation of the black hole and the results are presented by graphs, as the integrand in Eq.(26) depends on ω and the decay rate cannot be worked out analytically. From the graphs we find that the asymptotic behaviour of the massive Dirac fields is dominated by a decaying tail without any oscillation, which is different from the oscillatory decaying tails of the massive scalar field. The difference in the late time tail between the massive Dirac and massive scalar fields is caused by the factor $\lambda\omega/\mu$ in the parameter b. We also find that the larger the parameter a is, the more slowly the perturbation decays for $m\lambda > 0$; but the larger the parameter a is, the more quickly it decays for $m\lambda < 0$. In addition, we also note that the decay rate increases as parameter m ranges from $\frac{3}{2}$ to $-\frac{3}{2}$ for the $\lambda > 0$ case, but it decreases as m ranges from $\frac{3}{2}$ to $-\frac{3}{2}$ for $\lambda < 0$.

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