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The Jacobi elliptic function-like exact solutions to two kinds of KdV equations with variable coefficients and KdV equation with forcible term*

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By use of an auxiliary equation and through a function transformation, the Jacobi elliptic function wave-like solutions, the degenerated soliton-like solutions and the triangle function wave solutions to two kinds of Korteweg–de Vries (KdV) equations with variable coefficients and a KdV equation with a forcible term are constructed with the help of symbolic computation system Mathematica, where the new solutions are also constructed.

Keywords: auxiliary equation, KdV equation with variable coefficients, KdV equation with a forcible term, Jacobi elliptic function-like exact solutions

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1. Introduction

For the nonlinear evolution equations with ordinary coefficients, many methods[1−13] of constructing the exact solitary wave solutions and periodic wave solutions have been proposed. But the nonlinear evolution equations with ordinary coefficients only approximately reflects the substance movement transformation system. To describe material property one has studied the nonlinear evolution equations with variable coefficients and obtained many research achievements.[14−25] Based on Refs.[1-25], a method of an auxiliary equation with a function transformation is proposed and the new Jacobi elliptic function-like solutions, degenerate soliton-like solutions and triangle function wave solutions to two kinds of KdV equations with variable coefficients and a KdV equation with a forcible term are constructed with the help of the symbolic computation system Mathematica.

2. Method and application procedure

Assume a given nonlinear evolution equation with variable coefficients as follows:

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0. \tag{1}$$

Make transformation $u(x,t) = u(\xi)$, and $\xi = p(t)x + q(t)$, where $p(t)$ and $q(t)$ are functions of t and to be determined, and assume the solutions of Eq.(1) to be in the following form:

$$u(x,t) = g_0(t) + g_1(t)z(\xi), \tag{2}$$

where $g_0(t)$ and $g_1(t)$ are functions of t and to be determined, and $z(\xi)$ is determined by the following auxiliary equation:

$$\left(\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right)^2 = az(\xi) + bz^2(\xi) + cz^3(\xi), \tag{3}$$

and we will obtain the solutions of Eq.(3) as follows: when $a = 4, b = -4(1+k^2)$, and $c = 4k^2$, then

$$z(\xi) = \mathrm{sn}^2(\xi, k); \tag{4}$$

when $a = -4(-1+k^2), b = 4(-1+2k^2)$, and $c = -4k^2$, then

$$z(\xi) = \mathrm{cn}^2(\xi, k); \tag{5}$$

when $a = 4(-1+k^2), b = -4(-2+k^2)$, and $c = -4$, then

$$z(\xi) = \mathrm{dn}^2(\xi, k); \tag{6}$$

when $a = 4k^2, b = -4(1+k^2)$, and $c = 4$, then

$$z(\xi) = \mathrm{ns}^2(\xi, k); \tag{7}$$

when $a = -4k^2, b = 4(-1+2k^2)$, and $c = -4(-1+k^2)$, then

$$z(\xi) = \mathrm{nc}^2(\xi, k); \tag{8}$$

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when $a=-4, b=-4(-2+k^2)$, and $c=4(-1+k^2)$, then

$$z(\xi)=\mathrm{nd}^2(\xi,k); \tag{9}$$

when $a=4, b=-4(-2+k^2)$, and $c=-4(-1+k^2)$, then

$$z(\xi)=\mathrm{sc}^2(\xi,k); \tag{10}$$

when $a=4, b=4(-1+2k^2)$, and $c=4k^2(-1+k^2)$, then

$$z(\xi)=\mathrm{sd}^2(\xi,k); \tag{11}$$

when $a=-4(-1+k^2), b=-4(-2+k^2)$, and $c=4$, then

$$z(\xi)=\mathrm{cs}^2(\xi,k); \tag{12}$$

when $a=4, b=-4(-1+k^2)$, and $c=4k^2$, then

$$z(\xi)=\mathrm{cd}^2(\xi,k); \tag{13}$$

when $a=4k^2(-1+k^2)^2, b=4(-1+k^2)$, and $c=4$, then

$$z(\xi)=\mathrm{ds}^2(\xi,k); \tag{14}$$

when $a=4k^2, b=-4(1+k^2)$, and $c=4$, then

$$z(\xi)=\mathrm{dc}^2(\xi,k); \tag{15}$$

when $a=-(1-k^2)^2, b=2(1+k^2)$, and $c=-1$, then

$$z(\xi)=\big(k\mathrm{cn}(\xi,k)\pm\mathrm{dn}(\xi,k)\big)^2; \tag{16}$$

when $a=1, b=-2(-1+2k^2)$, and $c=1$, then

$$z(\xi)=\big(\mathrm{ns}(\xi,k)\pm\mathrm{cs}(\xi,k)\big)^2; \tag{17}$$

when $a=1-k^2, b=2(1+k^2)$, and $c=1-k^2$, then

$$z(\xi)=\big(\mathrm{nc}(\xi,k)\pm\mathrm{sc}(\xi,k)\big)^2; \tag{18}$$

when $a=k^4, b=2(-2+k^2)$, and $c=1$, then

$$z(\xi)=\big(\mathrm{ns}(\xi,k)\pm\mathrm{ds}(\xi,k)\big)^2; \tag{19}$$

when $a=k^2, b=2(-2+k^2)$, and $c=k^2$, then

$$z(\xi)=\big(\mathrm{sn}(\xi,k)\pm\mathrm{icn}(\xi,k)\big)^2,$$

and

$$z(\xi)=\frac{\mathrm{dn}^2(\xi,k)}{\Big(\sqrt{1-k^2}\mathrm{sn}(\xi,k)\pm\mathrm{cn}(\xi,k)\Big)^2}; \tag{20}$$

when $a=1, b=-2(-1+2k^2)$, and $c=1$, then

$$z(\xi)=\big(k\mathrm{sn}(\xi,k)\pm\mathrm{idn}(\xi,k)\big)^2; \tag{21}$$

when $a=1-k^2, b=2(1+k^2)$, and $c=1-k^2$, then

$$z(\xi)=\frac{\mathrm{cn}^2(\xi,k)}{\Big(1\pm\mathrm{sn}(\xi,k)\Big)^2}; \tag{22}$$

when $a=-1+k^2, b=2(1+k^2)$, and $c=-1+k^2$, then

$$z(\xi)=\frac{\mathrm{dn}^2(\xi,k)}{\Big(1\pm k\mathrm{sn}(\xi,k)\Big)^2}; \tag{23}$$

when $a=k^2, b=2(-2+k^2)$, and $c=k^2$, then

$$z(\xi)=\frac{k^2\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2}; \tag{24}$$

when $a=1, b=-2(-1+2k^2)$, and $c=1$, then

$$z(\xi)=\frac{\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{cn}(\xi,k)\Big)^2}; \tag{25}$$

when $a=1, b=2(-2+k^2)$, and $c=k^4$, then

$$z(\xi)=\frac{\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2}. \tag{26}$$

Substituting the expression (2) and Eq.(3) into Eq.(1), and setting the coefficients of $z^j(\xi)(j=0,1,2,\ldots)$ and $x^s z^i(\xi)\sqrt{az(\xi)+bz^2(\xi)+cz^3(\xi)}$ $(s=0,1; i=0,1,\ldots)$ to zero yield an over-determined partial differential equation set about $a, b, g_0(t), g_1(t), p(t), q(t)$. We obtain the solutions of the equation set with the help of Mathematica. And then substituting each of solutions of the equation set, i.e. each of the expressions (4)–(26) into the expression (2) respectively, we obtain the Jacobi elliptic function-like solutions, the soliton-like solutions and the triangle function wave solutions to the nonlinear evolution equation (1).

3. The application of the method

3.1. The first kind of KdV equation with the variable coefficients[22−25]

$$u_t+\alpha(t)uu_x+\beta(t)u_{xxx}=0. \tag{27}$$

In Eq.(27), assume that

$$u(x,t)=g_0(t)+g_1(t)z\big(p(t)x+q(t)\big). \tag{28}$$

Substituting Eq.(3) and the expression (28) into Eq.(27), and setting the coefficients of $z^j(\xi)(j=0,1)$, and $x^s z^i(\xi)\sqrt{az(\xi)+bz^2(\xi)+cz^3(\xi)}(s=0,1; i=0,1)$ to zero yields the following over-determined partial differential equation set:

$$\begin{aligned}
g_0'(t)&=0,\\
g_1'(t)&=0,\\
g_1(t)p'(t)&=0,\\
g_1^2(t)p(t)\alpha(t)+3cg_1(t)p^3(t)\beta(t)&=0,\\
g_1(t)g_0(t)p(t)\alpha(t)+bg_1(t)p^3(t)\beta(t)+g_1(t)q'(t)&=0.
\end{aligned}$$

With the aid of Mathematica, we have the following solutions of the equation set:

$$g_0(t)=g_0,\ \ g_1(t)=g_1,\ \ p(t)=p,\ \ \alpha(t)=\frac{-3cp^2\beta(t)}{g_1},\ \ q(t)=-\int\Big(g_0p\alpha(t)+bp^3\beta(t)\Big)dt, \tag{29}$$

where g_0, g_1, p, b and c are constants, $g_0\neq 0$, and $g_1\neq 0$.

Substituting the expressions (29) together with the expressions $(4)-(26)$ into the expression (28) respectively, we obtain Jacobi elliptic function-like exact solutions of Eq (27) as follows:

$$u_{1,1}(x,t)=g_0+g_1\mathrm{sn}^2(\xi,k); u_{2,1}(x,t)=g_0+g_1\mathrm{cn}^2(\xi,k); u_{3,1}(x,t)=g_0+g_1\mathrm{dn}^2(\xi,k);$$

$$u_{4,1}(x,t)=g_0+g_1\mathrm{ns}^2(\xi,k); u_{5,1}(x,t)=g_0+g_1\mathrm{nc}^2(\xi,k); u_{6,1}(x,t)=g_0+g_1\mathrm{nd}^2(\xi,k);$$

$$u_{7,1}(x,t)=g_0+g_1\mathrm{sc}^2(\xi,k); u_{8,1}(x,t)=g_0+g_1\mathrm{sd}^2(\xi,k); u_{9,1}(x,t)=g_0+g_1\mathrm{cs}^2(\xi,k);$$

$$u_{10,1}(x,t)=g_0+g_1\mathrm{cd}^2(\xi,k); u_{11,1}(x,t)=g_0+g_1\mathrm{ds}^2(\xi,k); u_{12,1}(x,t)=g_0+g_1\mathrm{dc}^2(\xi,k);$$

$$u_{13,1}(x,t)=g_0+g_1\Big(k\mathrm{cn}(\xi,k)\pm\mathrm{dn}(\xi,k)\Big)^2; u_{14,1}(x,t)=g_0+g_1\Big(\mathrm{ns}(\xi,k)\pm\mathrm{cs}(\xi,k)\Big)^2;$$

$$u_{15,1}(x,t)=g_0+g_1\Big(\mathrm{nc}(\xi,k)\pm\mathrm{sc}(\xi,k)\Big)^2; u_{16,1}(x,t)=g_0+g_1\Big(\mathrm{ns}(\xi,k)\pm\mathrm{ds}(\xi,k)\Big)^2;$$

$$u_{17,1}(x,t)=g_0+g_1\Big(\mathrm{sn}(\xi,k)\pm\mathrm{icn}(\xi,k)\Big)^2; u_{18,1}(x,t)=g_0+\frac{g_1\mathrm{dn}^2(\xi,k)}{\Big(\sqrt{1-k^2}\mathrm{sn}(\xi,k)\pm\mathrm{cn}(\xi,k)\Big)^2};$$

$$u_{19,1}(x,t)=g_0+g_1\Big(k\mathrm{sn}(\xi,k)\pm\mathrm{idn}(\xi,k)\Big)^2; u_{20,1}(x,t)=g_0+\frac{g_1\mathrm{cn}^2(\xi,k)}{\Big(1\pm\mathrm{sn}(\xi,k)\Big)^2};$$

$$u_{21,1}(x,t)=g_0+\frac{g_1\mathrm{dn}^2(\xi,k)}{\Big(1\pm k\mathrm{sn}(\xi,k)\Big)^2}; u_{22,1}(x,t)=g_0+\frac{g_1k^2\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2};$$

$$u_{23,1}(x,t)=g_0+\frac{g_1\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{cn}(\xi,k)\Big)^2}; u_{24,1}(x,t)=g_0+\frac{g_1\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2},$$

where $\xi=px-\int\Big(g_0p\alpha(t)+bp^3\beta(t)\Big)\mathrm{d}t$, p,b,c,g_1 are constants, $g_0\neq 0$, and $g_1\neq 0$.

The Jacobi elliptic function-like exact solutions above degenerate into the following soliton-like solutions where $k=1$:

$$u_{1,2}(x,t)=g_0+g_1\tanh^2\Big(px-\int\big(g_0p\alpha(t)-8p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{2,2}(x,t)=g_0+g_1\mathrm{sech}^2\Big(px-\int\big(g_0p\alpha(t)+4p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{12p^2\beta(t)}{g_1}\Big);$$

$$u_{3,2}(x,t)=g_0+g_1\coth^2\Big(px-\int\big(g_0p\alpha(t)-8p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{4,2}(x,t)=g_0+\frac{g_1\mathrm{sech}^2\Big(px-\int\big(g_0p\alpha(t)+4p^3\beta(t)\big)\mathrm{d}t\Big)}{\tanh^2\Big(px-\int\big(g_0p\alpha(t)+4p^3\beta(t)\big)\mathrm{d}t\Big)},\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{5,2}(x,t)=g_0+4g_1\mathrm{sech}^2\Big(px-\int\big(g_0p\alpha(t)+4p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{3p^2\beta(t)}{g_1}\Big);$$

$$u_{6,2}(x,t)=g_0+\frac{g_1\Big(1\pm\mathrm{sech}\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)\Big)^2}{\tanh^2\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)},\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big);$$

$$u_{7,2}(x,t)=g_0+g_1\Bigg(\tanh\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)$$

$$\pm\mathrm{isech}\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2,\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big);$$

$$u_{8,2}(x,t)=g_0+\frac{g_1\tanh^2\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)}{\Bigg(1\pm\mathrm{sech}\Big(px-\int\big(g_0p\alpha(t)-2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2},\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big).$$

The Jacobi elliptic function-like exact solutions above degenerate into the following triangle function solutions where $k=0$:

$$u_{1,3}(x,t)=g_0+g_1\csc^2\Big(px-\int\big(g_0p\alpha(t)-4p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{2,3}(x,t)=g_0+g_1\sec^2\Big(px-\int\big(g_0p\alpha(t)-4p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{3,3}(x,t)=g_0+g_1\tan^2\Big(px-\int\big(g_0p\alpha(t)+8p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{4,3}(x,t)=g_0+g_1\cot^2\Big(px-\int\big(g_0p\alpha(t)+8p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-12p^2\beta(t)}{g_1}\Big);$$

$$u_{5,3}(x,t)=g_0+g_1\Bigg(\csc\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\pm\cot\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2,$$

$$(\alpha(t)=\frac{-3p^2\beta(t)}{g_1});$$

$$u_{6,3}(x,t)=g_0+g_1\Bigg(\sec\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\pm\tan\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2,$$

$$\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big);$$

$$u_{7,3}(x,t)=g_0+4g_1\csc^2\Big(px-\int\big(g_0p\alpha(t)-4p^3\beta(t)\big)\mathrm{d}t\Big),\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big);$$

$$u_{8,3}(x,t)=g_0+\frac{g_1\cos^2\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)}{\Bigg(1\pm\sin\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2},\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big);$$

$$u_{9,3}(x,t)=g_0+\frac{g_1\sin^2\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)}{\Bigg(1\pm\cos\Big(px-\int\big(g_0p\alpha(t)+2p^3\beta(t)\big)\mathrm{d}t\Big)\Bigg)^2},\Big(\alpha(t)=\frac{-3p^2\beta(t)}{g_1}\Big).$$

3.2. The second kind of KdV equation with the variable coefficients[24,25]

$$u_t+\Big(\sigma(t)+\mu(t)x\Big)u_x+\alpha(t)uu_x+\beta(t)u_{xxx}+\nu(t)u=0. \tag{30}$$

In Eq.(30), assume

$$u(x,t)=g_0(t)+g_1(t)z\big(p(t)x+q(t)\big). \tag{31}$$

Substituting Eq.(3) and the expression (31) into Eq.(30), and setting the coefficients of $z^j(\xi)(j=0,1)$, and $x^sz^i(\xi)\sqrt{az(\xi)+bz^2(\xi)+cz^3(\xi)}(s=0,1;i=0,1)$ to zero yield the following over-determined partial differential equation set:

$$g_0(t)\nu(t)+g_0'(t)=0,$$
$$g_1(t)\nu(t)+g_1'(t)=0,$$
$$g_1(t)p'(t)+g_1(t)p(t)\mu(t)=0,$$
$$g_1^2(t)p(t)\alpha(t)+3cg_1(t)p^3(t)\beta(t)=0,$$
$$g_1(t)g_0(t)p(t)\alpha(t)+bg_1(t)p^3(t)\beta(t)+g_1(t)p(t)\sigma(t)+g_1(t)q'(t)=0.$$

With the aid of Mathematica, we obtain the following solutions of the equation set:

$$g_0(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t},\ g_1(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t},\ \frac{g_1(t)\alpha(t)}{\beta(t)p^2(t)}=-3c,$$

$$p(t)=\mathrm{e}^{-\int\mu(t)\mathrm{d}t},\ q(t)=-\int\Big(p(t)g_0(t)\alpha(t)+bp^3(t)\beta(t)+p(t)\sigma(t)\Big)\mathrm{d}t. \tag{32}$$

where b and c constants.

Substituting the expression (32) together with the expressions (4)–(26) into the expression (31) respectively, we obtain the Jacobi elliptic function-like exact solutions of Eq.(30) as follows:

$$u_{1,4}(x,t)=g_0(t)+g_1(t)\mathrm{sn}^2(\xi,k);u_{2,4}(x,t)=g_0(t)+g_1(t)\mathrm{cn}^2(\xi,k);u_{3,4}(x,t)=g_0(t)+g_1(t)\mathrm{dn}^2(\xi,k);$$

$$u_{4,4}(x,t)=g_0(t)+g_1(t)\mathrm{ns}^2(\xi,k);u_{5,4}(x,t)=g_0(t)+g_1(t)\mathrm{nc}^2(\xi,k);u_{6,4}(x,t)=g_0(t)+g_1(t)\mathrm{nd}^2(\xi,k);$$

$$u_{7,4}(x,t)=g_0(t)+g_1(t)\mathrm{sc}^2(\xi,k);u_{8,4}(x,t)=g_0(t)+g_1(t)\mathrm{sd}^2(\xi,k);u_{9,4}(x,t)=g_0(t)+g_1(t)\mathrm{cs}^2(\xi,k);$$

$$u_{10,4}(x,t)=g_0(t)+g_1(t)\mathrm{cd}^2(\xi,k);u_{11,4}(x,t)=g_0(t)+g_1(t)\mathrm{ds}^2(\xi,k);u_{12,4}(x,t)=g_0(t)+g_1(t)\mathrm{dc}^2(\xi,k);$$

$$u_{13,4}(x,t)=g_0(t)+g_1(t)\big(k\mathrm{cn}(\xi,k)\pm\mathrm{dn}(\xi,k)\big)^2;u_{14,4}(x,t)=g_0(t)+g_1(t)\big(\mathrm{ns}(\xi,k)\pm\mathrm{cs}(\xi,k)\big)^2;$$

$$u_{15,4}(x,t)=g_0(t)+g_1(t)\big(\mathrm{nc}(\xi,k)\pm\mathrm{sc}(\xi,k)\big)^2;u_{16,4}(x,t)=g_0(t)+g_1(t)\big(\mathrm{ns}(\xi,k)\pm\mathrm{ds}(\xi,k)\big)^2;$$

$$u_{17,4}(x,t)=g_0(t)+g_1(t)\big(\mathrm{sn}(\xi,k)\pm\mathrm{icn}(\xi,k)\big)^2;u_{18,4}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{dn}^2(\xi,k)}{\Big(\sqrt{1-k^2}\mathrm{sn}(\xi,k)\pm\mathrm{cn}(\xi,k)\Big)^2};$$

$$u_{19,4}(x,t)=g_0(t)+g_1(t)\big(k\mathrm{sn}(\xi,k)\pm\mathrm{idn}(\xi,k)\big)^2;u_{20,4}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{cn}^2(\xi,k)}{\Big(1\pm\mathrm{sn}(\xi,k)\Big)^2};$$

$$u_{21,4}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{dn}^2(\xi,k)}{\Big(1\pm k\mathrm{sn}(\xi,k)\Big)^2};u_{22,4}(x,t)=g_0(t)+\frac{k^2g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2};$$

$$u_{23,4}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{cn}(\xi,k)\Big)^2};u_{24,4}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2},$$

where

$g_0(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t},g_1(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t},\frac{g_1(t)\alpha(t)}{\beta(t)p^2(t)}=-3c,\ p(t)=\mathrm{e}^{-\int\mu(t)\mathrm{d}t},\xi=p(t)x-\int\Big(p(t)g_0(t)\alpha(t)+bp^3(t)\beta(t)+p(t)\sigma(t)\Big)\mathrm{d}t,$ b and c are constants.

The Jacobi elliptic function-like exact solutions above degenerate into the following soliton-like solutions where $k=1$:

$$u_{1,5}(x,t)=g_0(t)+g_1(t)\tanh^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-8p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{2,5}(x,t)=g_0(t)+g_1(t)\mathrm{sech}^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),g_1(t)\alpha(t)=12p^2(t)\beta(t);$$

$$u_{3,5}(x,t)=g_0(t)+g_1(t)\coth^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-8p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{4,5}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sech}^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}{\tanh^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)},g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{5,5}(x,t)=g_0(t)+4g_1(t)\mathrm{sech}^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),g_1(t)\alpha(t)=3p^2(t)\beta(t);$$

$$u_{6,5}(x,t)=g_0(t)+g_1(t)\left(\frac{1\pm\mathrm{sech}\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}{\tanh\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}\right)^2,$$

$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$

$$\begin{aligned}u_{7,5}(x,t)=\ & g_0(t)+g_1(t)\Bigg(\tanh\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\\ &\pm\mathrm{isech}\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Bigg)^2,g_1(t)\alpha(t)=-3p^2(t)\beta(t);\end{aligned}$$

$$u_{8,5}(x,t)=g_0(t)+\frac{g_1(t)\tanh^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}{\Big(1\pm\mathrm{sech}\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Big)^2},$$

$$g_1(t)\alpha(t)=-3p^2(t)\beta(t).$$

The Jacobi elliptic function-like exact solutions above degenerate into the following triangle function solutions where $k=0$:

$$u_{1,6}(x,t)=g_0(t)+g_1(t)\csc^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),$$

$$g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{2,6}(x,t)=g_0(t)+g_1(t)\sec^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),$$

$$g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{3,6}(x,t)=g_0(t)+g_1(t)\tan^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+8p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),$$

$$g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$u_{4,6}(x,t)=g_0(t)+g_1(t)\cot^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+8p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),$$

$$g_1(t)\alpha(t)=-12p^2(t)\beta(t);$$

$$\begin{aligned}u_{5,6}(x,t)=g_0(t)+g_1(t)\Bigg(&\csc\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\\ &\pm\cot\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Bigg)^2,\end{aligned}$$

$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$

$$\begin{aligned}u_{6,6}(x,t)=g_0(t)+g_1(t)\Bigg(&\sec\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\\ &\pm\tan\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Bigg)^2,\end{aligned}$$

$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$

$$u_{7,6}(x,t)=g_0(t)+4g_1(t)\csc^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)-4p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big),$$
$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$
$$u_{8,6}(x,t)=g_0(t)+\frac{g_1(t)\cos^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}{\Big(1\pm\sin\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Big)^2},$$
$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$
$$u_{9,6}(x,t)=g_0(t)+\frac{g_1(t)\sin^2\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)}{\Big(1\pm\cos\Big(p(t)x-\int\big(p(t)g_0(t)\alpha(t)+2p^3(t)\beta(t)+p(t)\sigma(t)\big)\mathrm{d}t\Big)\Big)^2},$$
$$g_1(t)\alpha(t)=-3p^2(t)\beta(t);$$

where $g_0(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t}$, $g_1(t)=\mathrm{e}^{-\int\nu(t)\mathrm{d}t}$, and $p(t)=\mathrm{e}^{-\int\mu(t)\mathrm{d}t}$.

3.3. The KdV equation with a forcible term[22]

$$u_t+\alpha uu_x+\beta u_{xxx}=F(t), \tag{33}$$

where $F(t)$ is an outer-forcible function of t; α and β are constants.

In Eq.(33), assume

$$u(x,t)=g_0(t)+g_1(t)z\big(p(t)x+q(t)\big). \tag{34}$$

Substituting Eq.(3) and the expression (34) into Eq.(33), and setting the coefficients of $z^j(\xi)(j=0,1)$, and $x^sz^i(\xi)\sqrt{az(\xi)+bz^2(\xi)+cz^3(\xi)}$ $(s=0,1;i=0,1)$ to zero yield the following over-determined partial differential equation set:

$$\begin{aligned}-F(t)+g_0'(t)&=0,\\ g_1'(t)&=0,\\ \alpha g_1^2(t)p(t)+3c\beta g_1(t)p^3(t)&=0,\\ g_1(t)p'(t)&=0,\\ \alpha g_1(t)g_0(t)p(t)+b\beta g_1(t)p^3(t)+g_1(t)q'(t)&=0.\end{aligned}$$

With the aid of Mathematica, we have the following solutions of the equation set:

$$g_0(t)=\int F(t)\mathrm{d}t,\ \ g_1(t)=-\frac{3cp^2\beta}{\alpha},\ \ p(t)=p,\ \ q(t)=-\int\Big(p\alpha g_0(t)+b\beta p^3\Big)\mathrm{d}t. \tag{35}$$

where b,c,p,α and β are constants.

Substituting the expression (35) together with the expressions $(4)-(26)$ into the expression (34) respectively, we have the Jacobi elliptic function-like exact solutions of Eq.(33) as follows:

$$u_{1,7}(x,t)=g_0(t)+g_1(t)\mathrm{sn}^2(\xi,k);u_{2,7}(x,t)=g_0(t)+g_1(t)\mathrm{cn}^2(\xi,k);u_{3,7}(x,t)=g_0(t)+g_1(t)\mathrm{dn}^2(\xi,k);$$
$$u_{4,7}(x,t)=g_0(t)+g_1(t)\mathrm{ns}^2(\xi,k);u_{5,7}(x,t)=g_0(t)+g_1(t)\mathrm{nc}^2(\xi,k);u_{6,7}(x,t)=g_0(t)+g_1(t)\mathrm{nd}^2(\xi,k);$$
$$u_{7,7}(x,t)=g_0(t)+g_1\mathrm{sc}^2(\xi,k);u_{8,7}(x,t)=g_0(t)+g_1\mathrm{sd}^2(\xi,k);u_{9,7}(x,t)=g_0(t)+g_1\mathrm{cs}^2(\xi,k);$$
$$u_{10,7}(x,t)=g_0(t)+g_1(t)\mathrm{cd}^2(\xi,k);u_{11,7}(x,t)=g_0(t)+g_1(t)\mathrm{ds}^2(\xi,k);u_{12,7}(x,t)=g_0(t)+g_1(t)\mathrm{dc}^2(\xi,k);$$

$$u_{13,7}(x,t)=g_0(t)+g_1(t)\big(k\mathrm{cn}(\xi,k)\pm\mathrm{dn}(\xi,k)\big)^2;u_{14,7}(x,t)=g_0(t)+g_1(t)\big(\mathrm{ns}(\xi,k)\pm\mathrm{cs}(\xi,k)\big)^2;$$

$$u_{15,7}(x,t)=g_0(t)+g_1(t)\big(\mathrm{nc}(\xi,k)\pm\mathrm{sc}(\xi,k)\big)^2;u_{16,7}(x,t)=g_0(t)+g_1(t)\big(\mathrm{ns}(\xi,k)\pm\mathrm{ds}(\xi,k)\big)^2;$$

$$u_{17,7}(x,t)=g_0(t)+g_1(t)\big(\mathrm{sn}(\xi,k)\pm\mathrm{icn}(\xi,k)\big)^2;u_{18,7}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{dn}^2(\xi,k)}{\Big(\sqrt{1-k^2}\mathrm{sn}(\xi,k)\pm\mathrm{cn}(\xi,k)\Big)^2};$$

$$u_{19,7}(x,t)=g_0(t)+g_1(t)\big(k\mathrm{sn}(\xi,k)\pm\mathrm{idn}(\xi,k)\big)^2;u_{20,7}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{cn}^2(\xi,k)}{\Big(1\pm\mathrm{sn}(\xi,k)\Big)^2};$$

$$u_{21,7}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{dn}^2(\xi,k)}{\Big(1\pm k\mathrm{sn}(\xi,k)\Big)^2};u_{22,7}(x,t)=g_0(t)+\frac{k^2g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2};$$

$$u_{23,7}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{cn}(\xi,k)\Big)^2};u_{24,7}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sn}^2(\xi,k)}{\Big(1\pm\mathrm{dn}(\xi,k)\Big)^2},$$

where $g_0(t)=\int F(t)\mathrm{d}t, g_1(t)=\frac{-3cp^2\beta}{\alpha}$, $p(t)=p,\xi=px-\int\Big(p\alpha g_0(t)+b\beta p^3\Big)\mathrm{d}t; b,c$ and p are constants.

The Jacobi elliptic function-like exact solutions above degenerate into the following soliton-like solutions where $k=1$:

$$u_{1,8}(x,t)=g_0(t)+g_1(t)\tanh^2\Big(px-\int\big(p\alpha g_0(t)-8\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{2,8}(x,t)=g_0(t)+g_1(t)\mathrm{sech}^2\Big(px-\int\big(p\alpha g_0(t)+4\beta p^3\big)\mathrm{d}t\Big),g_1(t)=\frac{12p^2\beta}{\alpha};$$

$$u_{3,8}(x,t)=g_0(t)+g_1(t)\coth^2\Big(px-\int\big(p\alpha g_0(t)-8\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{4,8}(x,t)=g_0(t)+\frac{g_1(t)\mathrm{sech}^2\Big(px-\int\big(p\alpha g_0(t)+4\beta p^3\big)\mathrm{d}t\Big)}{\tanh^2\Big(px-\int\big(p\alpha g_0(t)+4\beta p^3\big)\mathrm{d}t\Big)},g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{5,8}(x,t)=g_0(t)+4g_1(t)\mathrm{sech}^2\Big(px-\int\big(p\alpha g_0(t)+4\beta p^3\big)\mathrm{d}t\Big),g_1(t)=\frac{3p^2\beta}{\alpha};$$

$$u_{6,8}(x,t)=g_0(t)+g_1(t)\left(\frac{1\pm sech\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)}{\tanh\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)}\right)^2,g_1(t)=-\frac{3p^2\beta}{\alpha};$$

$$\begin{aligned}u_{7,8}(x,t)=\ & g_0(t)+g_1(t)\Bigg(\tanh\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)\\ & \pm\mathrm{isech}\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)\Bigg)^2,g_1(t)=-\frac{3p^2\beta}{\alpha};\end{aligned}$$

$$u_{8,8}(x,t)=g_0(t)+\frac{g_1(t)\tanh^2\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)}{\Big(1\pm\mathrm{sech}\Big(px-\int\big(p\alpha g_0(t)-2\beta p^3\big)\mathrm{d}t\Big)\Big)^2},g_1(t)=-\frac{3p^2\beta}{\alpha}.$$

The Jacobi elliptic function-like exact solutions above degenerate into the following triangle function solutions where $k=0$:

$$u_{1,9}(x,t)=g_0(t)+g_1(t)\csc^2\Big(px-\int\big(p\alpha g_0(t)-4\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{2,9}(x,t)=g_0(t)+g_1(t)\sec^2\Big(px-\int\big(p\alpha g_0(t)-4\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{3,9}(x,t)=g_0(t)+g_1(t)\tan^2\Big(px-\int\big(p\alpha g_0(t)+8\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$u_{4,9}(x,t)=g_0(t)+g_1(t)\cot^2\Big(px-\int\big(p\alpha g_0(t)+8\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{12p^2\beta}{\alpha};$$

$$\begin{aligned}u_{5,9}(x,t)=\ & g_0(t)+g_1(t)\Bigg(\csc\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\\ & \pm\cot\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\Bigg)^2,g_1(t)=-\frac{3p^2\beta}{\alpha};\end{aligned}$$

$$\begin{aligned}u_{6,9}(x,t)=\ & g_0(t)+g_1(t)\Bigg(\sec\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\\ & \pm\tan\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\Bigg)^2,g_1(t)=-\frac{3p^2\beta}{\alpha};\end{aligned}$$

$$u_{7,9}(x,t)=g_0(t)+4g_1(t)\csc^2\Big(px-\int\big(p\alpha g_0(t)-4\beta p^3\big)\mathrm{d}t\Big),g_1(t)=-\frac{3p^2\beta}{\alpha};$$

$$u_{8,9}(x,t)=g_0(t)+\frac{g_1(t)\cos^2\Big(px-\int\Big(p\alpha g_0(t)+2\beta p^3\Big)\mathrm{d}t\Big)}{\Big(1\pm\sin\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\Big)^2},g_1(t)=-\frac{3p^2\beta}{\alpha};$$

$$u_{9,9}(x,t)=g_0(t)+\frac{g_1(t)\sin^2\Big(px-\int\Big(p\alpha g_0(t)+2\beta p^3\Big)\mathrm{d}t\Big)}{\Big(1\pm\cos\Big(px-\int\big(p\alpha g_0(t)+2\beta p^3\big)\mathrm{d}t\Big)\Big)^2},g_1(t)=-\frac{3p^2\beta}{\alpha}.$$

where $g_0(t)=\int F(t)\mathrm{d}t, p, \alpha$ and β are constants.

4. Discussion and conclusions

In this paper, the method of the auxiliary equation with a function-transformation is proposed, and the new Jacobi elliptic function exact solutions, the degenerated soliton-like solutions and the triangle function wave solutions to two kinds of KdV equation with variable coefficients and a KdV equation a forcible term, are constructed. The Jacobi elliptic function exact solutions, the degenerated soliton-like solutions and the triangle function wave solutions are $u_{1,1}(x,t)\sim u_{2,1}(x,t)$, $u_{1,2}(x,t)\sim u_{2,2}(x,t)$,$u_{3,2}(x,t)$,$u_{5,2}(x,t)$,$u_{2,3}(x,t)$; $u_{1,4}(x,t)\sim u_{2,4}(x,t)$, $u_{1,5}(x,t)\sim u_{2,5}(x,t)$, $u_{3,5}(x,t)$,$u_{5,5}(x,t)$,$u_{2,6}(x,t)$; $u_{1,7}(x,t)\sim u_{2,7}(x,t)$, $u_{1,8}(x,t)\sim u_{2,8}(x,t)$,$u_{3,8}(x,t)$,$u_{5,8}(x,t)$,$u_{2,9}(x,t)$. They were obtained in Refs.[22-25]. Besides the solutions, some new solutions are also obtained in the paper. The new method is of significance to constructing the Jacobi elliptic function exact solutions to the nonlinear evolution equations with variable coefficients.

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