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# A new method of new exact solutions and solitary wave-like solutions for the generalized variable coefficients Kadomtsev–Petviashvili equation

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Using the solution of general Korteweg–de Vries (KdV) equation, the solutions of the generalized variable coefficient Kadomtsev–Petviashvili (KP) equation are constructed, and then its new solitary wave-like solution and Jacobi elliptic function solution are obtained.

**Keywords:** KdV equation, generalized variable coefficients KP equation, solitary wave-like solution, exact solution

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## 1. Introduction

A generalized variable coefficient Kadomtsev–Petviashvili (KP) equation[1] reads

$$u_{xt} + 6f_1(u_x^2 + uu_{xx}) + f_2 u_{xxxx} + 6fu_x + g^2 u_{yy} + f_3 = 0, \tag{1}$$

where the coefficients $f$, $f_1$, $f_2$, $f_3$ and $g$ are the functions of time. This equation has an extensive physical background.[1] It appears extensively in many nonlinear wave equations. It has important applications in fluid mechanics, plasma physics, gas dynamics and other fields, and it has aroused the interest of many physicists and mathematicians in recent years. There have been proposed quite a number of methods of solving the nonlinear differential equation, such as the Backlund transformation method,[1−6] the truncation expansion method,[7] the Jacobi elliptic function expansion method,[8,9] the mapping approach,[10] the similarity reduction method,[11] the extended hyperbolic-function method,[12−15] the bilinear and variable separation approach,[16−18] the expanded symmetric group approach,[19] and the sine-Gordon travelling wave reduction equation method.[20−25] Recently, the solitary wave-like solutions,[26] which are obtained by using the sine-Gordon travelling wave reduction equation method, are discussed when the external force term $f_3 = 0$. In this paper, using the reduction method of solution for the general Korteweg–de Vries (KdV) equation, we construct solutions for Eq.(1) and obtain new solitary wave-like solutions and other exact solutions.

## 2. Construction of solutions

The nonlinear partial differential equation with variable coefficients and the external force term can be considered to be in the form

$$G(u, u_t, u_{xt}, u_x, u_{xx}, u_y, \cdots, F) = 0, \tag{2}$$

where $F$ is a $t$-dependent external force term. With travelling wave transformation, Eq.(2) becomes an ordinary differential equation. Assume that

$$u = u(\xi), \quad \xi = ax + by + d, \tag{3}$$

where $a$, $b$ and $d$ are the undetermined functions of $t$. And supposing that $\varphi$ is the solution of KdV equation, we want to seek the following solution:

$$u(\xi) = P \cdot x + \sum_{i=1}^{n} Q_i \varphi^i(\xi), \tag{4}$$

where $P$ and $Q_i (i = 1, 2, \cdots, n)$ are the undetermined functions of $t$, and $n$ is determined from the highest derivative term and nonlinear term by balancing Eq.(2). The substitution of the expression (4) into Eq.(2) yields the equation about $x^i \dfrac{\mathrm{d}^j \varphi}{\mathrm{d}\xi^j}$. And assuming their coefficients to be zero, we can determine $P$, $Q_i$, $a$, $b$ and $d$. We apply this method to solving the generalized variable coefficients KP equation (1).

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## 3. Solutions to generalized variable coefficients KP equation

For the general KdV equation

$$\varphi_t + \varphi\varphi_x + \gamma\varphi_{xxx} = 0, \tag{5}$$

its Jacobi elliptic cosine (sine) function solution[9] is

$$\varphi = c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2 \mathrm{cn}^2\xi, \tag{6}$$

where $c$, $\gamma$, $k$ are constants and $m$ is modulus of Jacobi elliptic function. By deriving $\xi$ components of $\varphi$ from the expression (6), we have

$$\begin{aligned}
\varphi_\xi =& \ -24m^2\gamma k^2 \mathrm{dn}\xi\mathrm{sn}\xi\mathrm{cn}\xi, &(7)\\
\varphi_{\xi\xi} =& \ 48m^2\gamma k^2(1+m^2)\mathrm{sn}^2\xi - 72m^4\gamma k^2\mathrm{sn}^4\xi - 24m^2\gamma k^2, &(8)\\
\varphi_{\xi\xi\xi} =& \ 96m^2\gamma k^2(1+m^2-3m^2\mathrm{sn}^2\xi)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi, &(9)\\
\varphi_{\xi\xi\xi\xi} =& \ 96m^2\gamma k^2(1+m^2) + 1440m^4\gamma k^2(m^2 - m^2\mathrm{sn}^2\xi + 1)\mathrm{sn}^4\xi\\
& -192m^2\gamma k^2(1+m^4)\mathrm{sn}^2\xi - 1248m^4\gamma k^2\mathrm{sn}^2\xi. &(10)
\end{aligned}$$

Balancing the highest order derivative terms with nonlinear terms of Eq.(1) yields $n_1 = 2$. Balancing the highest order derivative terms with nonlinear terms of Eq.(5) leads to $n_2 = 2$. Thus, $n_1 = n_2$. Because $\varphi$ is the solution of Eq.(5), we can determine $n = 1$ in the expression (4). So we can assume that Eq.(1) has the solution as follows:

$$u(\xi) = P\cdot x + Q\cdot\varphi(\xi), \tag{11}$$

where $\xi = ax + by + d$, and $P$, $Q_i$, $a$, $b$, and $d$ are undetermined functions of $t$. Substituting the expression (11) into Eq.(1), replacing $\varphi$ and its derivative terms with the terms on the right-hand side of the expressions (6)–(10), combining the homogeneous term of $x^h y^i \mathrm{sn}^j\xi\mathrm{cn}^k\xi\mathrm{dn}^l\xi$ (where $h = 0, 1$ and $i = 0, 1$; for $l = 0$, then $j = 0, 2, 4, 6$ and $k = 0$; for $l = 1$ then $j =$ 1,3 and $k = 1$), and considering the coefficients of $x^h y^i \mathrm{sn}^j\xi\mathrm{cn}^k\xi\mathrm{dn}^l\xi$ to be zero, we can obtain the equation group as follows (Note that here the same equations and same common factors are not listed. If the expressions (6) and (11) are directly substituted into Eq.(1), the following results may not be obtained, which is because the software Maple may not be simplified as we desired.):

$$\begin{aligned}
&6f_1P^2 + P_t + 6fP - 24Qam^2\gamma k^2 d_t + 96f_2Qa^4m^4\gamma k^2 - 24g^2Qb^2m^2\gamma k^2 + 96f_2Qa^4m^2\gamma k^2\\
&-144f_1Q^2a^2cm^2\gamma k^2 - 576f_1Q^2a^2\gamma^2k^4m^2 - 576f_1Q^2a^2m^4\gamma^2k^4 + f_3 = 0, &(12)\\
&-24Qam^2\gamma k^2 b_t = 0, &(13)\\
&6Pf_1a + a_t = 0, &(14)\\
&-24Qk^2\gamma f_1a^2m^4 - 156Qk^2\gamma f_1a^2m^2 - 24Qk^2\gamma f_1a^2 - 6Qf_1a^2cm^2 - 6Qf_1a^2c\\
&+4f_2a^4m^4 - g^2b^2m^2 - am^2d_t + 26f_2a^4m^2 - ad_t + 4f_2a^4 - b^2g^2 = 0, &(15)\\
&-6f_1\gamma k^2Q + f_2a^2 = 0, &(16)\\
&-120Qk^2\gamma f_1a^2m^2 - 120Qk^2\gamma f_1a^2 - 6Qf_1a^2c + 20f_2a^4m^2 - ad_t + 20f_2a^4 - b^2g^2 = 0, &(17)\\
&Q_ta + a_tQ + 12f_1PQa + 6fQa = 0. &(18)
\end{aligned}$$

Solving Eqs.(12)–(18), we have the solutions as follows:

$$P = -\frac{a_t}{6f_1a}, \quad Q = \frac{f_2a^2}{6f_1\gamma k^2}, \tag{20}$$

$$a = k_1\mathrm{e}^{\int\left(-6f-\frac{f_2'}{f_2}+\frac{f_1'}{f_1}\right)\mathrm{d}t}, \quad b = k_1, \tag{19}$$

$$d = -\int \frac{f_2 a^4 c + g^2 k_1^2 \gamma k^2}{a\gamma k^2} \mathrm{d}t + k_2. \tag{21}$$

And $f_3$ must satisfy

$$\begin{aligned} f_3 = \ & -(-f_1'' f_2 f_1 + 18 f f_1^2 f_2' - 3 f_2' f_1 f_1' \\ & -24 f f_2 f_1 f_1' + 3 f_1'^2 f_2 + 6 f' f_2 f_1^2 \\ & + f_2'' f_1^2 + 72 f^2 f_2 f_1^2) \frac{1}{6 f_1^3 f_2}, \end{aligned} \tag{22}$$

where

$$f_1' = \frac{\mathrm{d}f_1}{\mathrm{d}t}, \ \text{ and } \ f_1'' = \frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2}.$$

From the expressions (19) to (22), the exact solution of Eq.(1) can be determined as follows:

$$u_1 = -\frac{a_t}{6 f_1 a} x + \frac{f_2 a^2}{6 f_1 \gamma k^2} \left[ c + 4(1 - 2m^2)\gamma k^2 + 12 m^2 \gamma k^2 \mathrm{cn}^2 \xi \right], \tag{23}$$

$$\xi = k_1 \mathrm{e}^{\int \left( -6f - \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) \mathrm{d}t} x + k_1 y - \int \frac{f_2 a^4 c + g^2 k_1^2 \gamma k^2}{a \gamma k^2} \mathrm{d}t + d_2, \tag{24}$$

where $k_1$ and $k_2$ are integration constants.

## 4. Discussion on solutions

**Case 1**

When $m \to 1$, $\mathrm{sn}\xi \to \tanh\xi$ or $\mathrm{cn}\xi \to \mathrm{sech}\xi$, and from the expressions (23) and (24), the solitary wave-like solution of Eq.(1) can be obtained as follows:

$$\begin{aligned} u_2 = \ & -\frac{a_t}{6 f_1 a} x + \frac{f_2 a^2}{6 f_1 \gamma k^2} \left( c - 4\gamma k^2 \right. \\ & \left. + 12 \gamma k^2 \mathrm{sech}^2 \xi \right). \end{aligned} \tag{25}$$

**Case 2**

If Eq.(1) is of Painlevé type, then we obtain $f_1 = f_2$. From the expression (22), we have

$$f_3 = \frac{f f_1' - f' f_1 - 12 f^2 f_1}{f_1^2}. \tag{26}$$

The expression (26) is one of the necessary conditions under which Eq.(1) is of Painlevé type. Then Eq.(1) becomes

$$\begin{aligned} & u_{xt} + 6 f_1 (u_x^2 + u u_{xx}) + f_2 u_{xxxx} + 6 f u_x + g^2 u_{yy} \\ & + \frac{f f_1' - f' f_1 - 12 f^2 f_1}{f_1^2} = 0. \end{aligned} \tag{27}$$

From the expressions (19) to (21), we can obtain

$$a = k_1 \mathrm{e}^{-\int 6f \mathrm{d}t}, \quad b = k_1, \tag{28}$$

$$d = \int -\frac{c f_1 k_1^3 \mathrm{e}^{\int -24 f \mathrm{d}t} + g^2 k_1 \gamma k^2}{\mathrm{e}^{\int -6 f \mathrm{d}t} \gamma k^2} \mathrm{d}t + k_2, \tag{29}$$

$$P = \frac{f}{f_1}, \quad Q = \frac{k_1^2 \mathrm{e}^{-\int 12 f \mathrm{d}t}}{6 \gamma k^2}. \tag{30}$$

From the expressions (28) to (30), we can obtain an exact solution of Eq.(27) as follows:

$$\begin{aligned} u_3 = \ & \frac{f x}{f_1} + \frac{k_1^2 \mathrm{e}^{-\int 12 f \mathrm{d}t}}{6 \gamma k^2} [c + 4(1 - 2m^2)\gamma k^2 \\ & + 12 m^2 \gamma k^2 \mathrm{cn}^2 \xi], \end{aligned} \tag{31}$$

where

$$\begin{aligned} \xi = \ & k_1 \mathrm{e}^{-\int 6 f \mathrm{d}t} x + k_1 y \\ & - \int \frac{f_1 k_1^4 c \mathrm{e}^{\int -24 f \mathrm{d}t} + g^2 k_1^2 \gamma k^2}{k_1 \gamma k^2 \mathrm{e}^{-\int -6 f \mathrm{d}t}} \mathrm{d}t + k_2, \end{aligned}$$

with $g$ being an arbitrary function of $t$, which is different from that in Ref.[1] where $g$ is given by the following expression:

$$g = \mathrm{e}^{\int \frac{f_1' - 24 f f_1}{2 f_1} \mathrm{d}t}; \tag{32}$$

and a solitary wave-like solution of Eq.(27)

$$\begin{aligned} u_4 = \ & \frac{f x}{f_1} + \frac{k_1^2 \mathrm{e}^{-\int 12 f \mathrm{d}t}}{6 \gamma k^2} (c - 4\gamma k^2 \\ & + 12 \gamma k^2 \mathrm{sech}^2 \xi). \end{aligned} \tag{33}$$

**Case 3**

When $f_1 = f_2 = 1$ in Eq.(27), we can have the variable coefficient generalized Kadomtsev–Petviashvili (VCGKP) equation[11]

$$\begin{aligned} & u_{xt} + 6(u_x^2 + u u_{xx}) + u_{xxxx} + 6 f u_x \\ & + g^2 u_{yy} - (f' + 12 f^2) = 0, \end{aligned} \tag{34}$$

where $g$ is an arbitrary function of $t$, which is different from that in Ref.[1] where $g$ is given as follows:

$$g = \mathrm{e}^{\int -12 f \mathrm{d}t}. \tag{35}$$

From the expressions (28) to (30), we can obtain

$$d = -\int \frac{ck_1^3 \mathrm{e}^{\int -24f\mathrm{d}t} + g^2 k_1 \gamma k^2}{\mathrm{e}^{\int -6f\mathrm{d}t}\gamma k^2}\mathrm{d}t + k_2, \quad (36)$$

$$P = f, \quad Q = \frac{k_1^2 \mathrm{e}^{-\int 12f\mathrm{d}t}}{6\gamma k^2},$$

$$a = k_1 \mathrm{e}^{-\int 6f\mathrm{d}t}, \quad b = k. \quad (37)$$

Thus we can determine a new exact solution of VCGKP equation

$$u_5 = fx + \frac{k_1^2 \mathrm{e}^{-\int 12f\mathrm{d}t}}{6\gamma k^2}\left[c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2 \mathrm{cn}^2\xi\right], \quad (38)$$

where

$$\xi = k_1 \mathrm{e}^{-\int 6f\mathrm{d}t} x + k_1 y - \int \frac{k_1^3 c \mathrm{e}^{\int -24f\mathrm{d}t} + g^2 k_1 \gamma k^2}{\gamma k^2 \mathrm{e}^{-\int 6f\mathrm{d}t}}\mathrm{d}t + k_2;$$

and a solitary wave-like solution

$$u_6 = fx + \frac{k_1^2 \mathrm{e}^{-\int 12f\mathrm{d}t}}{6\gamma k^2}(c - 4\gamma k^2 + 12\gamma k^2 \mathrm{sech}^2\xi). \quad (39)$$

**Case 4**

When $f = 0$ and $g = 1$ in Eq.(34), then it becomes a KP equation

$$u_{xt} + 6(u_x^2 + uu_{xx}) + u_{xxxx} + u_{yy} = 0. \quad (40)$$

From the expressions (28), (38) and (39), we can determine a new exact solution of KP equation

$$u_7 = \frac{k_1^2}{6\gamma k^2}[c + 4(1-2m^2)\gamma k^2 + 12m^2\gamma k^2 \mathrm{cn}^2\xi], \quad (41)$$

where

$$\xi = k_1 x + k_1 y - \frac{k_1^3 c + k_1 \gamma k^2}{\gamma k^2} t + k_2,$$

and a solitary wave solution

$$u_8 = \frac{k_1^2}{6\gamma k^2}(c - 4\gamma k^2 + 12\gamma k^2 \mathrm{sech}^2\xi). \quad (42)$$

## 5. Summary

Using the Jacobi elliptic cosine (sine) function solution of general KdV equation, we construct the solution of Eq.(1). As a result, we successfully obtain new solitary wave-like solutions and exact solutions for a generalized variable coefficients KP equation. Then we discuss its evolutional equation under three particular conditions, and obtain corresponding new solitary wave-like solutions and exact solutions of VCGKP and KP equations that are of Painlevé type.

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