Vol 15 No 12, December 2006
1009-1963/2006/15(12)/2796-08

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

A set of Boussinesq-type equations for interfacial internal waves in two-layer stratified fluid *

Song Jin-Bao(宋金宝)†

Institute of Oceanology, Chinese Academy of Sciences, Qingdao 266071, *China*

(Received 29 March 2006; revised manuscript received 20 July 2006)

Many new forms of Boussinesq-type equations have been developed to extend the range of applicability of the classical Boussinesq equations to deeper water in the study of the surface waves. One approach was used by Nwogu (1993. J. Wtrw. Port Coastal and Oc. Eng. 119, 618–638) to improve the linear dispersion characteristics of the classical Boussinesq equations by using the velocity at an arbitrary level as the velocity variable in derived equations and obtain a new form of Boussinesq-type equations, in which the dispersion property can be optimized by choosing the velocity variable at an adequate level. In this paper, a set of Boussinesq-type equations describing the motions of the interfacial waves propagating alone the interface between two homogeneous incompressible and inviscid fluids of different densities with a free surface and a variable water depth were derived using a method similar to that used by Nwogu (1993. J. Wtrw. Port Coastal and Oc. Eng. 119, 618–638) for surface waves. The equations were expressed in terms of the displacements of free surface and density-interface, and the velocity vectors at arbitrary vertical locations in the upper layer and the lower layer (or depth-averaged velocity vector across each layer) of a two-layer fluid. As expected, the equations derived in the present work include as special cases those obtained by Nwogu (1993, J. Wtrw. Port Coastal and Oc. Eng. 119, 618-638) and Peregrine (1967, J. Fluid Mech. 27, 815-827) for surface waves when the density of the upper fluid is taken as zero.

Keywords: two-layer fluid, interfacial internal waves, Boussinesq-type equations

PACC: 0340G, 4735, 5235M, 9210F

1. Introduction

Interfacial internal waves in a density-stratified fluid have received a great deal of attention due to their practical importance. For example, Stokes[1] established the theory of the interfacial wave for two-layer fluid. Lamb[2] described the nature of small-amplitude interfacial waves. Keulegan[3] experimentally observed internal waves at the interface of a two-layer fluid and theoretically derived the first-order solution of the wave speed and profile using Boussinesq's approximation. Long[4] extended Keulegan's theory to the second order, and discussed the limit of the wave amplitude. Thorpe[5] used an expansion technique, analogous to that used by Stokes to study surface waves, to investigate the effects of finite amplitude on internal waves. Tsuji and Nagata[6] applied Stokes' expansion technique to internal waves at the interface between two fluid layers of infinite thickness, and obtained the fifth-order solution. To describe interfacial waves for both the flat top-layer surface and the free-surface boundary conditions, integral equation techniques have been utilized by Grimshaw and Pullin[7] for large-amplitude periodic waves in extreme form, Turner and Vanden-Broeck[8] for solitary waves by the use of the complex velocity potential approach, and Evans and Ford[9] for solitary waves by the use of Green's theorem. Umeyama[10,11] derived the second-order and third-order Stokes solutions for internal waves at the interface between two fluids of slightly different densities, and the theoretical solutions were also tested by Umeyama[10,11] in a 6 m-long laboratory wave tank with a horizontal flap-type wavemaker. Recently, the second-order solutions for random interfacial waves at a constant depth, the two-layer fluid with a rigid lid were derived by Song[12] using an expansion technique, analogous to that used by Longuet-Higgins[13] for random surface waves, and the influences of the uniform background currents and the free-surface boundary condition on the solutions were also studied by Song[14] and Song and Sun.[15] The second-order Stokes solutions and the second-order random solutions for multi-layer fluid were presented by Chen *et al.*[16,17] The wave equations describing interfacial internal waves in a two-layer or multi-layer fluid are nonlinear wave equations and there are many

*Project supported by the National Science Fund for Distinguished Young Scholars, China (Grant No 40425015).

†E-mail: songjb@ms.qdio.ac.cn

methods for solving nonlinear wave equations. For example, perturbation method,[10−17] the homogeneous balance principle,[18−20] expansion method about the Jacobi elliptic function,[21] various tanh method,[22] the general projective Riccati method[23] and so on.

Since the weak dispersion is the most critical limitation for many applications of the classical Boussinesq equations (Boussinesq[24]; Peregrine[25]) in the modelling of surface wave evolutions, many new forms of Boussinesq-type equations have been developed by improving their linear dispersion characteristics. Witting[26] presented a new set of equations valid for a single horizontal dimension. Madsen *et al*[27,28] introduced an additional third-order term into the momentum equation to improve the dispersion accuracy of the equation. Nwogu[29] used the velocity at an arbitrary level as the velocity variable in the derived equations and obtained a new form of Boussinesq-type equations, whose dispersion property could be optimized by choosing the velocity variable at an adequate level. Schaffer and Madsen[30] added third-order terms to the momentum equation and also to the continuity equation so that the frequency dispersion of the equations can be further improved. In the present study, a set of Boussinesq-type equations describing the displacements of free surface and density-interface, and the velocity vectors at arbitrary vertical locations in the upper layer and the lower layer (or depth-averaged velocity vector across each layer) of the two-layer fluid were derived using a method similar to that used to study surface waves by Nwogu.[29] The equations derived here have important potential applications in the understanding of the generation and evolution behaviours of interfacial internal waves, the interactions between the surface waves and internal waves, and the interactions between the internal waves and the topography of ocean bottom.

2. Basic equations and boundary conditions

Consider the interfacial waves propagating along the interface between two homogeneous incompressible and inviscid fluids of different densities as shown in Fig.1. A Cartesian coordinate system is employed such that the horizontal coordinate axes x and y are fixed along the undisturbed upper boundary, and the vertical axis z is positive upward, ρ_1 and ρ_2 are the densities of the upper and the lower layers, respectively; $\zeta_1(x,y,t)$ is the vertical displacement of free surface, t is the time, $\zeta_2(x,y,t)$ is the distance of the interface measured from the undisturbed free surface, H_1 is the distance between the undisturbed free surface and the undisturbed interface, and $D(x,y)$ denotes the mean water depth.

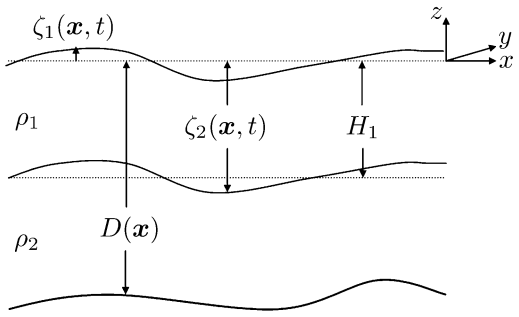


Fig.1. Schematic plot of the two-layer system.

Assuming that the fluid in each layer is irrotational, the governing equations for the fluid motion are the continuity equations, Euler's equations and the irrotationality equations as follows:

$$\nabla\cdot\boldsymbol{u}_i+\frac{\partial w_i}{\partial z}=0,\quad i=1,2, \tag{1}$$

$$\frac{\partial\boldsymbol{u}_i}{\partial t}+(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{u}_i+w_i\frac{\partial\boldsymbol{u}_i}{\partial z}=-\frac{1}{\rho_i}\nabla p_i,\quad i=1,2, \tag{2}$$

$$\frac{\partial w_i}{\partial t}+\boldsymbol{u}_i\cdot\nabla w_i+w_i\frac{\partial w_i}{\partial z}=-\frac{1}{\rho_i}\frac{\partial p_i}{\partial z}-g,\quad i=1,2, \tag{3}$$

$$\frac{\partial\boldsymbol{u}_i}{\partial z}-\nabla w_i=0,\quad \frac{\partial u_i}{\partial y}-\frac{\partial v_i}{\partial x}=0,\quad i=1,2, \tag{4}$$

where g is the gravitational acceleration, $\boldsymbol{u}_i = (u_i, v_i)(i = 1, 2)$ are the horizontal velocity vectors in the upper and lower layers, w_i is the vertical velocity, p_i is the pressure, the subscript 1 will be used for the upper layer and 2 will be used for the lower layer, and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$.

The boundary conditions at the free surface, the density interface and the bottom are

$$w_1 = \frac{\partial \zeta_1}{\partial t} + \boldsymbol{u}_1 \cdot \nabla \zeta_1, \quad z = \zeta_1, \tag{5}$$

$$w_i = -\frac{\partial \zeta_2}{\partial t} - \boldsymbol{u}_i \cdot \nabla \zeta_2, \quad (i = 1, 2), \quad z = \zeta_2, \tag{6}$$

$$p_1 = 0, \quad z = \zeta_1, \tag{7}$$

$$p_1 = p_2, \quad z = -\zeta_2, \tag{8}$$

$$w_2 = -\boldsymbol{u}_2 \cdot \nabla D, \quad z = -D. \tag{9}$$

Introducing nondimensional variables as follows:

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{h_0}, \quad t' = \frac{\sqrt{gh_0}}{L}t, \quad p_i' = \frac{p_i}{\rho_0 g a_0}, \quad \boldsymbol{u}_i' = \frac{h_0}{a_0\sqrt{gh_0}}\boldsymbol{u}_i,$$
$$w_i' = \frac{h_0^2}{a_0 L\sqrt{gh_0}}w_i, \quad \zeta_1' = \frac{\zeta_1}{a_0}, \quad \hat{\zeta}_2' = \frac{\hat{\zeta}_2}{a_0}, \quad H_1' = \frac{H_1}{h_0}, \quad D' = \frac{D}{h_0}, \tag{10}$$

where L is the characteristic wavelength for the horizontal direction, h_0 is the characteristic water depth for the vertical direction, a_0 is the characteristic wave amplitude, ρ_0 is the characteristic density, $\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}\right) = L\nabla$ and $-\zeta_2 = -H_1 + \hat{\zeta}_2$.

By using Eqs.(10), the governing equations and the boundary conditions (1)–(9) can then be expressed in nondimensional form as

$$\mu^2 \nabla \cdot \boldsymbol{u}_i + \frac{\partial w_i}{\partial z} = 0, \quad i = 1, 2, \tag{11}$$

$$\mu^2 \frac{\partial \boldsymbol{u}_i}{\partial t} + \varepsilon\mu^2(\boldsymbol{u}_i \cdot \nabla)\boldsymbol{u}_i + \varepsilon w_i \frac{\partial \boldsymbol{u}_i}{\partial z} = -\frac{1}{\gamma_i}\mu^2 \nabla p_i, \quad i = 1, 2, \tag{12}$$

$$\varepsilon\frac{\partial w_i}{\partial t} + \varepsilon^2 \boldsymbol{u}_i \cdot \nabla w_i + \frac{\varepsilon^2}{\mu^2} w_i \frac{\partial w_i}{\partial z} = -\frac{\varepsilon}{\gamma_i}\frac{\partial p_i}{\partial z} - 1, \quad i = 1, 2, \tag{13}$$

$$\frac{\partial \boldsymbol{u}_i}{\partial z} - \nabla w_i = 0, \quad \frac{\partial u_i}{\partial y} - \frac{\partial v_i}{\partial x} = 0, \quad i = 1, 2, \tag{14}$$

$$w_1 = \mu^2 \frac{\partial \zeta_1}{\partial t} + \varepsilon\mu^2 \boldsymbol{u}_1 \cdot \nabla \zeta_1, \quad z = \varepsilon\zeta_1, \tag{15}$$

$$w_i = \mu^2 \frac{\partial \hat{\zeta}_2}{\partial t} + \varepsilon\mu^2 \boldsymbol{u}_i \cdot \nabla \hat{\zeta}_2 \quad (i = 1, 2), \quad z = -H_1 + \varepsilon\hat{\zeta}_2, \tag{16}$$

$$p_1 = 0, \quad z = \varepsilon\zeta_1, \tag{17}$$

$$p_1 = p_2, \quad z = -H_1 + \varepsilon\hat{\zeta}_2, \tag{18}$$

$$w_2 = -\mu^2 \boldsymbol{u}_2 \cdot \nabla D, \quad z = -D, \tag{19}$$

where $\varepsilon = \frac{a_0}{h_0}$, $\mu = \frac{h_0}{L}$, $\gamma_i = \frac{\rho_i}{\rho_0}$ and we have dropped the primes for dimensionless variables.

Integrating the continuity equation (11) for $i = 1$ across the upper-fluid layer (from the density interface to the free surface) and for $i = 2$ across the lower-fluid layer (from the seabed to the density interface), sep-

arately, and applying the boundary conditions (15), (16) and (19), we have

$$\frac{\partial(\zeta_1-\hat{\zeta}_2)}{\partial t}+\nabla\cdot\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}\boldsymbol{u}_1\mathrm{d}z=0, \tag{20}$$

$$\frac{\partial\hat{\zeta}_2}{\partial t}+\nabla\cdot\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}\boldsymbol{u}_2\mathrm{d}z=0. \tag{21}$$

Similarly, the horizontal momentum equations (12) and (13) can be integrated over the depth to give

$$\begin{aligned}&\mu^2\frac{\partial}{\partial t}\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}\boldsymbol{u}_1\mathrm{d}z+\varepsilon\mu^2\frac{\partial}{\partial x}\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}u_1\boldsymbol{u}_1\mathrm{d}z+\varepsilon\mu^2\frac{\partial}{\partial y}\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}v_1\boldsymbol{u}_1\mathrm{d}z\\&+\frac{\mu^2}{\gamma_1}\nabla\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}p_1\mathrm{d}z+\frac{\varepsilon\mu^2}{\gamma_1}\nabla\hat{\zeta}_2p_1|_{z=-H_1+\varepsilon\hat{\zeta}_2}=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&\mu^2\frac{\partial}{\partial t}\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}\boldsymbol{u}_2\mathrm{d}z+\varepsilon\mu^2\frac{\partial}{\partial x}\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}u_2\boldsymbol{u}_2\mathrm{d}z+\varepsilon\mu^2\frac{\partial}{\partial y}\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}v_2\boldsymbol{u}_2\mathrm{d}z\\&+\frac{\mu^2}{\gamma_2}\nabla\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}p_2\mathrm{d}z-\frac{\varepsilon\mu^2}{\gamma_2}\nabla\hat{\zeta}_2p_2|_{z=-H_1+\varepsilon\hat{\zeta}_2}-\frac{\mu^2}{\gamma_2}\nabla Dp_2|_{z=-D}=0,\end{aligned} \tag{23}$$

where Eqs.(15)–(19) have been used.

Integrating the continuity equation (11) from z to $z=\varepsilon\zeta_1$ in the upper layer and from $z=-D$ to z in the down layer, separately, and applying the boundary conditions (15) and (19), we have

$$w_1=\mu^2\left\{\frac{\partial\zeta_1}{\partial t}+\nabla\cdot\int_z^{\varepsilon\zeta_1}\boldsymbol{u}_1\mathrm{d}z\right\}, \tag{24}$$

$$w_2=-\mu^2\nabla\cdot\int_{-D}^{z}\boldsymbol{u}_2\mathrm{d}z. \tag{25}$$

Similarly, integrating the momentum equation (13) from z to $z=\varepsilon\zeta_1$ in the upper layer and from z to $z=-H_1+\varepsilon\hat{\zeta}_2$ in the down layer, respectively, and applying the boundary condition (17), we have

$$p_1(\boldsymbol{x},z,t)=\gamma_1\zeta_1-\frac{\gamma_1}{\varepsilon}z+\gamma_1\frac{\partial}{\partial t}\int_z^{\varepsilon\zeta_1}w_1\mathrm{d}z+\varepsilon\gamma_1\left[\frac{\partial}{\partial x}\int_z^{\varepsilon\zeta_1}u_1w_1\mathrm{d}z+\frac{\partial}{\partial y}\int_z^{\varepsilon\zeta_1}v_1w_1\mathrm{d}z\right]-\frac{\varepsilon\gamma_1}{\mu^2}w_1^2, \tag{26}$$

$$\begin{aligned}p_2(\boldsymbol{x},z,t)=\;&p_2|_{z=-H_1+\varepsilon\hat{\zeta}_2}+\gamma_2\hat{\zeta}_2-\frac{\gamma_2}{\varepsilon}(z+H_1)+\gamma_2\frac{\partial}{\partial t}\int_z^{-H_1+\varepsilon\hat{\zeta}_2}w_2\mathrm{d}z\\&+\varepsilon\gamma_2\left[\frac{\partial}{\partial x}\int_z^{-H_1+\varepsilon\hat{\zeta}_2}u_2w_2\mathrm{d}z+\frac{\partial}{\partial y}\int_z^{-H_1+\varepsilon\hat{\zeta}_2}v_2w_2\mathrm{d}z\right]-\frac{\varepsilon\gamma_2}{\mu^2}w_2^2.\end{aligned} \tag{27}$$

Substituting Eqs.(24) and (25) into the irrotationality condition (14) yields

$$\frac{\partial\boldsymbol{u}_1}{\partial z}=\mu^2\left\{\nabla\frac{\partial\zeta_1}{\partial t}+\varepsilon\nabla(\nabla\zeta_1\cdot\boldsymbol{u}_1)|_{z=\varepsilon\zeta_1}+\int_z^{\varepsilon\zeta_1}\nabla(\nabla\cdot\boldsymbol{u}_1)\mathrm{d}z+\varepsilon\nabla\zeta_1(\nabla\cdot\boldsymbol{u}_1)|_{z=\varepsilon\zeta_1}\right\}, \tag{28}$$

$$\frac{\partial \boldsymbol{u}_2}{\partial z} = -\mu^2 \left\{ \nabla(\nabla D \cdot \boldsymbol{u}_2)|_{z=-D} + \int_{-D}^{z} \nabla(\nabla \cdot \boldsymbol{u}_2)\mathrm{d}z + \nabla D(\nabla \cdot \boldsymbol{u}_2)|_{z=-D} \right\}. \tag{29}$$

All the foregoing equations are exact, and valid for all orders of ε and μ.

3. A set of Boussinesq-type equations

There are different methods of deriving the Boussinesq or Boussinesq-type equations. One approach was used by Nwogu[29] to improve the linear dispersion characteristics of the classical Boussinesq equations (Boussinesq[24]; Peregrine[25]). In this section, a method similar to that employed by Nwogu[29] is used to derive a set of Boussinesq-type equations for the two-layer fluid system. The horizontal velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are expanded into Taylor series about arbitrary depths $z = z_{\alpha_i}(i = 1, 2)$ of the upper layer and the lower layer as follows:

$$\begin{aligned} \boldsymbol{u}_i(\boldsymbol{x}, z, t) = \ & \boldsymbol{u}_i|_{z=z_{\alpha_i}} + (z - z_{\alpha_i}) \frac{\partial \boldsymbol{u}_i}{\partial z}\Big|_{z=z_{\alpha_i}} \\ & + \frac{1}{2}(z - z_{\alpha_i})^2 \frac{\partial^2 \boldsymbol{u}_i}{\partial z^2}\Big|_{z=z_{\alpha_i}} + \cdots \end{aligned} \tag{30}$$

Substituting Eqs.(28)–(30) into Eqs.(24) and (25) yields

$$w_1 = \mu^2 \left\{ \frac{\partial \zeta_1}{\partial t} + \nabla \cdot [(\varepsilon\zeta_1 - z)\boldsymbol{u}_{\alpha_1}] \right\} + O(\mu^4), \tag{31}$$

$$w_2 = -\mu^2 \nabla \cdot [(z + D)\boldsymbol{u}_{\alpha_2}] + O(\mu^4), \tag{32}$$

where $\boldsymbol{u}_{\alpha_i} = \boldsymbol{u}_i|_{z=z_{\alpha_i}}(i = 1, 2)$.

Using Eqs.(20), (21) and (30), we have

$$\frac{\partial \zeta_1}{\partial t} = -H_1 \nabla \cdot \boldsymbol{u}_{\alpha_1} - \nabla \cdot (D\boldsymbol{u}_{\alpha_2}) + H_1 \nabla \cdot \boldsymbol{u}_{\alpha_2} + O(\varepsilon, \mu^2), \tag{33}$$

and Eq.(31) can be rewritten as follows:

$$w_1 = \mu^2 \{-(H_1 + z)\nabla \cdot \boldsymbol{u}_{\alpha_1} - \nabla \cdot (D\boldsymbol{u}_{\alpha_2}) + H_1 \nabla \cdot \boldsymbol{u}_{\alpha_2}\} + O(\varepsilon^2, \varepsilon\mu^2, \mu^4). \tag{34}$$

Using the irrotationality condition (14), (32) and (34), we have

$$\begin{aligned} \boldsymbol{u}_1(\boldsymbol{x}, z, t) = \ & \boldsymbol{u}_{\alpha_1} + \mu^2 (z - z_{\alpha_1}) \{-H_1 \nabla(\nabla \cdot \boldsymbol{u}_{\alpha_1}) - \nabla[\nabla \cdot (D\boldsymbol{u}_{\alpha_2})] + H_1 \nabla(\nabla \cdot \boldsymbol{u}_{\alpha_2})\} \\ & - \frac{1}{2}\mu^2 (z^2 - z_{\alpha_1}^2) \nabla(\nabla \cdot \boldsymbol{u}_{\alpha_1}) + O(\varepsilon^2, \varepsilon\mu^2, \mu^4), \end{aligned} \tag{35}$$

$$\begin{aligned} \boldsymbol{u}_2(\boldsymbol{x}, z, t) = \ & \boldsymbol{u}_{\alpha_2} - \mu^2 (z - z_{\alpha_2}) \nabla[\nabla \cdot (D\boldsymbol{u}_{\alpha_2})] \\ & - \frac{1}{2}\mu^2 (z^2 - z_{\alpha_2}^2) \nabla(\nabla \cdot \boldsymbol{u}_{\alpha_2}) + O(\varepsilon^2, \varepsilon\mu^2, \mu^4). \end{aligned} \tag{36}$$

Substituting Eqs.(32), (34)–(36) into Eqs.(26) and (27), and using Eq.(18) yields

$$\begin{aligned} p_1 = \ & \gamma_1 \zeta_1 - \frac{\gamma_1}{\varepsilon} z + \gamma_1 \mu^2 \left\{ -z \left[-H_1 \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_1}}{\partial t} - \nabla \cdot \frac{\partial (D\boldsymbol{u}_{\alpha_2})}{\partial t} + H_1 \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_2}}{\partial t} \right] \right. \\ & \left. + \frac{1}{2} z^2 \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_1}}{\partial t} \right\} + O(\varepsilon^2, \varepsilon\mu^2, \mu^4), \end{aligned} \tag{37}$$

$$\begin{aligned} p_2 = \ & \gamma_1 \mu^2 H_1 \left[-\frac{1}{2} H_1 \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_1}}{\partial t} - \nabla \cdot \frac{\partial (D\boldsymbol{u}_{\alpha_2})}{\partial t} + H_1 \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_2}}{\partial t} \right] \\ & + \gamma_2 \hat{\zeta}_2 - \frac{\gamma_2}{\varepsilon} z + \gamma_2 \mu^2 \left[(z + H_1) \nabla \cdot \frac{\partial (D\boldsymbol{u}_{\alpha_2})}{\partial t} + \frac{1}{2}(z^2 - H_1^2) \nabla \cdot \frac{\partial \boldsymbol{u}_{\alpha_2}}{\partial t} \right] \\ & + \gamma_1 (\zeta_1 - \hat{\zeta}_2) + \frac{(\gamma_1 - \gamma_2)}{\varepsilon} H_1 + O(\varepsilon^2, \varepsilon\mu^2, \mu^4). \end{aligned} \tag{38}$$

Assuming that $O(\varepsilon)=O(\mu^2)\ll 1$ and $O(\nabla D)=O(1)$, substituting Eqs.(32), (34)–(38) into Eqs.(20)–(23), and retaining terms up to $O(\varepsilon)$ and $O(\mu^2)$, we have

$$\frac{\partial(\zeta_1-\hat{\zeta}_2)}{\partial t}+\nabla\cdot\left\{\left[H_1+\varepsilon(\zeta_1-\hat{\zeta}_2)\right]\boldsymbol{u}_{\alpha_1}\right\}+\mu^2\nabla\cdot\left\{\left(\frac{1}{3}H_1^3+\frac{1}{2}H_1z_{\alpha_1}^2+H_1^2z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_1})\right.$$
$$\left.+\left(\frac{1}{2}H_1^2+H_1z_{\alpha_1}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]-\left(\frac{1}{2}H_1^3+H_1^2z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2})\right\}=0, \tag{39}$$

$$\frac{\partial\hat{\zeta}_2}{\partial t}+\nabla\cdot[(D-H_1+\varepsilon\hat{\zeta}_2)\boldsymbol{u}_{\alpha_2}]+\mu^2\nabla\cdot\left\{(D-H_1)\left(\frac{1}{2}D+\frac{1}{2}H_1+z_{\alpha_2}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]\right.$$
$$\left.+\frac{1}{6}(D-H_1)(3z_{\alpha_2}^2-D^2-H_1^2-DH_1)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2})\right\}=0, \tag{40}$$

$$\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}=-\nabla\zeta_1-\mu^2\left\{\left(\frac{1}{2}z_{\alpha_1}^2+H_1z_{\alpha_1}\right)\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}\right)+z_{\alpha_1}\nabla\left[\nabla\cdot\frac{\partial(D\boldsymbol{u}_{\alpha_2})}{\partial t}\right]\right.$$
$$\left.-H_1z_{\alpha_1}\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}\right)\right\}-\varepsilon(\boldsymbol{u}_{\alpha_1}\cdot\nabla)\boldsymbol{u}_{\alpha_1}, \tag{41}$$

$$\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}=-\nabla\hat{\zeta}_2-\gamma\nabla(\zeta_1-\hat{\zeta}_2)-\varepsilon(\boldsymbol{u}_{\alpha_2}\cdot\nabla)\boldsymbol{u}_{\alpha_2}$$
$$-\mu^2\left\{-\frac{\gamma}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}\right)+[z_{\alpha_2}+(1-\gamma)H_1]\nabla\left[\nabla\cdot\frac{\partial(D\boldsymbol{u}_{\alpha_2})}{\partial t}\right]\right.$$
$$\left.+\left[\frac{1}{2}z_{\alpha_2}^2+\left(\gamma-\frac{1}{2}\right)H_1^2\right]\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}\right)\right\}, \tag{42}$$

where $\gamma=\dfrac{\gamma_1}{\gamma_2}=\dfrac{\rho_1}{\rho_2}$, Eqs.(39) and (40) have been used in the derivations of Eqs.(41) and (42).

If the density of the upper layer is small enough and can be neglected (i.e., $\rho_1=0$), then $\gamma=0$, and its mode may be described as an oscillation of the lower fluid which is almost the same as if its upper surface were free, as stated by Lamb.[4] For this extreme case, moving the coordinate system down to $z=-H_1$ along the vertical axis (i.e., $z\to z+H_1$), and denoting $\eta=\hat{\zeta}_2$, $h=D-H_1$, $\boldsymbol{u}_\alpha=\boldsymbol{u}_{\alpha_2}$ and $z_\alpha=z_{\alpha_2}+H_1$, Eqs.(40) and (42), respectively, reduce to the Boussinesq-type equations (25a) and (25b) presented by Nwogu in Ref.[29] for surface waves.

We denote the depth-averaged velocity vector across each layer as follows:

$$\bar{\boldsymbol{u}}_1=\frac{1}{H_1+\varepsilon(\zeta_1-\hat{\zeta}_2)}\int_{-H_1+\varepsilon\hat{\zeta}_2}^{\varepsilon\zeta_1}\boldsymbol{u}_1\mathrm{d}z,\quad \bar{\boldsymbol{u}}_2=\frac{1}{D-H_1+\varepsilon\hat{\zeta}_2}\int_{-D}^{-H_1+\varepsilon\hat{\zeta}_2}\boldsymbol{u}_2\mathrm{d}z. \tag{43}$$

Thus, substituting Eqs.(35) and (36) into the above equations yields

$$\bar{\boldsymbol{u}}_1=\boldsymbol{u}_{\alpha_1}+\mu^2\left\{\left(\frac{1}{3}H_1^2+\frac{1}{2}z_{\alpha_1}^2+H_1z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_1})\right.$$
$$\left.+\left(\frac{1}{2}H_1+z_{\alpha_1}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]-\left(\frac{1}{2}H_1^2+H_1z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2})\right\}, \tag{44}$$

$$\bar{\boldsymbol{u}}_2=\boldsymbol{u}_{\alpha_2}+\mu^2\left\{\left(\frac{1}{2}D+\frac{1}{2}H_1+z_{\alpha_2}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]\right.$$
$$\left.+\frac{1}{6}\left(3z_{\alpha_2}^2-D^2-H_1^2-DH_1\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2}))\right\}, \tag{45}$$

and the variables ζ_1, $\hat{\zeta}_2$, $\bar{\boldsymbol{u}}_1$ and $\bar{\boldsymbol{u}}_2$ satisfy the following equations:

$$\frac{\partial(\zeta_1-\hat{\zeta}_2)}{\partial t}+\nabla\cdot\{[H_1+\varepsilon(\zeta_1-\hat{\zeta}_2)]\bar{\boldsymbol{u}}_1\}=0, \tag{46}$$

$$\frac{\partial\hat{\zeta}_2}{\partial t}+\nabla\cdot[(D-H_1+\varepsilon\hat{\zeta}_2)\bar{\boldsymbol{u}}_2]=0, \tag{47}$$

$$\begin{aligned}\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}=&-\nabla\zeta_1-\varepsilon(\bar{\boldsymbol{u}}_1\cdot\nabla)\bar{\boldsymbol{u}}_1+\mu^2\left\{\frac{1}{3}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}\right)+\frac{1}{2}H_1\nabla\left[\nabla\cdot\frac{\partial(D\bar{\boldsymbol{u}}_2)}{\partial t}\right]\right.\\&\left.-\frac{1}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_2}{\partial t}\right)\right\},\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial\bar{\boldsymbol{u}}_2}{\partial t}=&-\nabla\hat{\zeta}_2-\gamma\nabla(\zeta_1-\hat{\zeta}_2)-\varepsilon(\bar{\boldsymbol{u}}_2\cdot\nabla)\bar{\boldsymbol{u}}_2\\&+\mu^2\left\{\frac{\gamma}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}\right)+\left[\frac{1}{2}D+\left(\gamma-\frac{1}{2}\right)H_1\right]\nabla\left[\nabla\cdot\frac{\partial(D\bar{\boldsymbol{u}}_2)}{\partial t}\right]\right.\\&\left.+\frac{1}{6}[2(1-3\gamma)H_1^2-D^2-DH_1]\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_2}{\partial t}\right)\right\}.\end{aligned} \tag{49}$$

For the extreme case of $\rho_1=0$, denoting $\eta=\hat{\zeta}_2$, $h=D-H_1$, and $\bar{\boldsymbol{u}}_\alpha=\bar{\boldsymbol{u}}_{\alpha_2}$, Eqs.(47) and (49), respectively, reduce to the classical Boussinesq equations derived by Peregrine[25] for surface waves (see the dimensionless form of Eqs.(14) and (13) presented by Peregrine in Ref.[25] or the equations (26a) and (26b) restated by Nwogu in Ref.[29]). Thus, the equations derived from the present work include as special cases those obtained by Nwogu[29] and Peregrine[25] for surface waves.

The dimensional form of Eqs.(39)–(42) and (46)–(49) are

$$\begin{aligned}&\frac{\partial(\zeta_1-\hat{\zeta}_2)}{\partial t}+\nabla\cdot\left\{[H_1+\zeta_1-\hat{\zeta}_2]\boldsymbol{u}_{\alpha_1}\right\}+\nabla\cdot\left\{\left(\frac{1}{3}H_1^3+\frac{1}{2}H_1z_{\alpha_1}^2+H_1^2z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_1})\right.\\&\left.+\left(\frac{1}{2}H_1^2+H_1z_{\alpha_1}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]-\left(\frac{1}{2}H_1^3+H_1^2z_{\alpha_1}\right)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2})\right\}=0,\end{aligned} \tag{50}$$

$$\begin{aligned}&\frac{\partial\hat{\zeta}_2}{\partial t}+\nabla\cdot\left[(D-H_1+\hat{\zeta}_2)\boldsymbol{u}_{\alpha_2}\right]+\nabla\cdot\left\{(D-H_1)\left(\frac{1}{2}D+\frac{1}{2}H_1+z_{\alpha_2}\right)\nabla[\nabla\cdot(D\boldsymbol{u}_{\alpha_2})]\right.\\&\left.+\frac{1}{6}(D-H_1)(3z_{\alpha_2}^2-D^2-H_1^2-DH_1)\nabla(\nabla\cdot\boldsymbol{u}_{\alpha_2})\right\}=0,\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}=&-g\nabla\zeta_1-\left(\frac{1}{2}z_{\alpha_1}^2+H_1z_{\alpha_!}\right)\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}\right)-z_{\alpha_1}\nabla\left[\nabla\cdot\frac{\partial(D\boldsymbol{u}_{\alpha_2})}{\partial t}\right]\\&+H_1z_{\alpha_1}\nabla\left(\nabla\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}\right)-(\boldsymbol{u}_{\alpha_1}\cdot\nabla)\boldsymbol{u}_{\alpha_1},\end{aligned} \tag{52}$$

$$\begin{aligned}\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}=&-g\nabla\hat{\zeta}_2-\gamma g\nabla(\zeta_1-\hat{\zeta}_2)-(\boldsymbol{u}_{\alpha_2}\cdot\nabla)\boldsymbol{u}_{\alpha_2}+\frac{\gamma}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_1}}{\partial t}\right)\\&-[z_{\alpha_2}+(1-\gamma)H_1]\nabla\left[\nabla\cdot\frac{\partial(D\boldsymbol{u}_{\alpha_2})}{\partial t}\right]-\left[\frac{1}{2}z_{\alpha_2}^2+\left(\gamma-\frac{1}{2}\right)H_1^2\right]\nabla\left(\nabla\cdot\frac{\partial\boldsymbol{u}_{\alpha_2}}{\partial t}\right),\end{aligned} \tag{53}$$

$$\frac{\partial(\zeta_1-\hat{\zeta}_2)}{\partial t}+\nabla\left\{\left[H_1+\zeta_1-\hat{\zeta}_2\right]\bar{\boldsymbol{u}}_1\right\}=0, \tag{54}$$

$$\frac{\partial\hat{\zeta}_2}{\partial t}+\nabla\cdot\left[(D-H_1+\hat{\zeta}_2)\bar{\boldsymbol{u}}_2\right]=0, \tag{55}$$

$$\begin{aligned}\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}=&-g\nabla\zeta_1+\frac{1}{3}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}\right)+\frac{1}{2}H_1\nabla\left[\nabla\frac{\partial(D\bar{\boldsymbol{u}}_2)}{\partial t}\right]\\&-\frac{1}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_2}{\partial t}\right)-(\bar{\boldsymbol{u}}_1\cdot\nabla)\bar{\boldsymbol{u}}_1,\end{aligned} \tag{56}$$

$$\frac{\partial \bar{\boldsymbol{u}}_2}{\partial t} = -g\nabla\hat{\zeta}_2 - \gamma g\nabla(\zeta_1 - \hat{\zeta}_2) - (\bar{\boldsymbol{u}}_2\cdot\nabla)\bar{\boldsymbol{u}}_2 + \frac{\gamma}{2}H_1^2\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_1}{\partial t}\right) + \left[\frac{1}{2}D + \left(\gamma - \frac{1}{2}\right)H_1\right]\nabla\left[\nabla\cdot\frac{\partial(D\bar{\boldsymbol{u}}_2)}{\partial t}\right] + \frac{1}{6}\left[2(1-3\gamma)H_1^2 - D^2 - DH_1\right]\nabla\left(\nabla\cdot\frac{\partial\bar{\boldsymbol{u}}_2}{\partial t}\right), \quad (57)$$

where the dimensional $\bar{\boldsymbol{u}}_1$ and $\bar{\boldsymbol{u}}_2$ are

$$\bar{\boldsymbol{u}}_1 = \frac{1}{H_1 + \zeta_1 - \hat{\zeta}_2}\int_{-H_1+\hat{\zeta}_2}^{\zeta_1}\boldsymbol{u}_1\mathrm{d}z, \quad \bar{\boldsymbol{u}}_2 = \frac{1}{D - H_1 + \hat{\zeta}_2}\int_{-D}^{-H_1+\hat{\zeta}_2}\boldsymbol{u}_2\mathrm{d}z. \quad (58)$$

4. Conclusions

A set of Boussinesq-type equations describing the displacements of free surface and density-interface, and the velocity vectors at arbitrary vertical locations in the upper layer and the lower layer (or depth-averaged velocity vector across each layer) of a two-layer fluid have been derived using a method similar to that used to study surface waves by Nwogu.[29] From the derivations and the results, it is noted that the higher-order Boussinesq-type equations with higher-order nonlinear effects can be derived using the same method as that presented in this paper. It is also noted that the equations derived from the present work include as special cases those obtained by Nwogu[29] and Peregrine[25] for surface waves when the density of the upper fluid is taken as zero.

References

[1] Stokes G G 1847 *Trans. Camb. Phil. Soc.* **8** 441
[2] Lamb H 1932 *Hydrodynamics* (New York: Cambridge University Press)
[3] Keulegan M J 1953 *J. Res. Natl. Bur. Stand.* **51** 133
[4] Long R R 1956 *Tellus* **8** 460
[5] Thorpe S A 1968 *Phil. Trans. Roy. Soc.* **263**A 563
[6] Tsuji Y and Nagata Y 1973 *J. Oceanogr. Soc. Jpn.* **29** 61
[7] Grimshaw R and Pullin D 1986 *Phys. Fluids* **29** 2802
[8] Turner R and Vanden-Broeck J 1988 *Phys. Fluids* **31** 2486
[9] Evans W A B and Ford M J 1996 *Phys. Fluids* **8** 2032
[10] Umeyama M 1998 Memoirs Tokyo *Met. Univ.* **48** 137
[11] Umeyama M 2000 Memoirs Tokyo *Met. Univ.* **50** 120
[12] Song J B 2004 *Geophysical Research Letters* **31** (15): Art. No. L15302 AUG 6
[13] Longuet-Higgins M S 1963 *J. Fluid Mech.* **17** 459
[14] Song J B 2005 *China Ocean Eng.* **19** 333
[15] Song J B and Sun Q 2006 *Acta Oceanol. Sin.* **25** 15
[16] Chen X G, Song J B and Sun Q 2005 *Acta Phys. Sin.* **54** 5699 (in Chinese)
[17] Chen X G and Song J B 2006 *Chin. Phys.* **15** 756
[18] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)
[19] Fan E G 2000 *Acta Phys. Sin.* **49** 1409 (in Chinese)
[20] Zhang J L, Wang M L and Wang Y M 2003 *Chin. Phys.* **12** 245
[21] Liu S K, Fu Z T, Liu Sh D and Zhao Q 2001 *Acta Phys. Sin.* **50** 2068 (in Chinese)
[22] Chen Y, Yan Z Y, Li B and Zhang H Q 2003 *Chin. Phys.* **12** 1
[23] Chen Y and Li B 2004 *Chin. Phys.* **13** 302
[24] Boussinesq J 1872 *J. Math. Pures Appl.* **17** 55
[25] Peregrine D H 1967 *J. Fluid Mech.* **27** 815
[26] Witting J M 1984 *J. Comput. Phys.* **56** 203
[27] Madsen P A, Murray R and Sorensen O R 1991 *Coast. Eng.* **15** 371
[28] Madsen P A and Sorensen O R 1992 *Coast. Eng.* **18** 183
[29] Nwogu O 1993 *J. Wtrw. Port Coastal and Oc. Eng.* **119** 618
[30] Schaffer H A and Madsen P A 1995 *Coast. Eng.* **26** 1