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Noether–Lie symmetry and conserved quantities of mechanical system in phase space

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In this paper, a new kind of symmetry and its conserved quantities of a mechanical system in phase space are studied. The definition of this new symmetry, i.e. a Noether–Lie symmetry, is presented, and the criterion of this symmetry is also given. The Noether conserved quantity and the generalized Hojman conserved quantity of the Noether–Lie symmetry of the system are obtained. The Noether–Lie symmetry contains the Noether symmetry and the Lie symmetry, and has more generalized significance.

Keywords: Noether–Lie symmetry, mechanical system, conserved quantity, phase space

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1. Introduction

The research on symmetry and conserved quantities of a mechanical system possesses important theoretical and practical significance. The modern methods to find conserved quantities of a mechanical system are mainly the Noether symmetry,[1−5] the Lie symmetry[4−10] and the Mei form invariance (or the Mei symmetry).[11−15] And about conserved quantities there are mainly three kinds:[16] the Noether conserved quantity, the Hojman conserved quantity and a new type of conserved quantity (or the Mei conserved quantity[17]). The previous studies on symmetries and conserved quantities were all carried out from a kind of symmetry to seek a type of conserved quantity.

Recently, the unified symmetry of the Lagrange system was studied in Ref.[18] and the unified symmetry of the holonomic mechanical system was studied in Ref.[19]. The unified symmetry could lead to the three kinds of conserved quantities mentioned above. The Lie-form invariance of the Lagrange system was studied in Ref.[20]. In this paper, we study the Noether–Lie symmetry of mechanical system in phase space and its conserved quantities. The definition and the criterion of the Noether–Lie symmetry of the system are presented. The Noether conserved quantity and the general Hojman conserved quantity deduced from the Noether–Lie symmetry of the system are obtained. The Noether–Lie symmetry contains the Noether symmetry and the Lie symmetry, and has more generalized significance.

2. Definition and criterion of Noether–Lie symmetry

The differential equations of motion for mechanical system of phase space can be written in the form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s, \quad (s = 1, \cdots, s), \tag{1}$$

where $H = H(t, \boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian of the system. Equations (1) can be expressed as follows

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}). \tag{2}$$

Introducing the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{3}$$

where ε is the infinitesimal parameter, and ξ_0, ξ_s, η_s are the generators of infinitesimal transformations.

Definition If a symmetry of mechanical system in phase space is both a Noether symmetry and a Lie symmetry, the symmetry is called Noether–Lie symmetry of the system.

For a mechanical system in phase space Eq.(1), under the infinitesimal transformations Eq.(3), the Noether identity can be written in the form

$$H\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\frac{\partial H}{\partial t}\xi_0+\frac{\partial H}{\partial q_s}\xi_s-p_s\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-Q_s(\xi_s-\dot{q}_s\xi_0)-\frac{\bar{\mathrm{d}}G_N}{\mathrm{d}t}=0, \tag{4}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+g_s\frac{\partial}{\partial q_s}+h_s\frac{\partial}{\partial p_s}, \tag{5}$$

and $G_N=G_N(t,\boldsymbol{q},\boldsymbol{p})$ is a gauge function. The determining equations of the Lie symmetry of the system have the form

$$X^{(1)}\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)=0,\quad X^{(1)}\left(\dot{p}_s+\frac{\partial H}{\partial q_s}-Q_s\right)=0, \tag{6}$$

where

$$X^{(1)}=X^{(0)}+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_s}+\left(\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t}-\dot{p}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{p}_s}, \tag{7}$$

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+\eta_s\frac{\partial}{\partial p_s}. \tag{8}$$

Equations (6) can be also written in the form

$$\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-g_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}=X^{(0)}(g_s),\quad \frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t}-h_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}=X^{(0)}(h_s). \tag{9}$$

According to the above definition, using the relations (4) and (6), we have the following criterion.

Criterion Under the infinitesimal transformations (3), if there exists such a gauge function $G_N(t,\boldsymbol{q},\boldsymbol{p})$ that the infinitesimal generators ξ_0, ξ_s and η_s satisfy the condition

$$\begin{aligned}&\left\{X^{(1)}\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\right\}^2+\left\{X^{(1)}\left(\dot{p}_s+\frac{\partial H}{\partial q_s}-Q_s\right)\right\}^2\\&+\left\{H\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\frac{\partial H}{\partial t}\xi_0+\frac{\partial H}{\partial q_s}\xi_s-p_s\frac{\bar{d}\xi_s}{\mathrm{d}t}-Q_s(\xi_s-\dot{q}_s\xi_0)-\frac{\bar{\mathrm{d}}G_N}{\mathrm{d}t}\right\}^2=0,\end{aligned} \tag{10}$$

then the symmetry is a Noether–Lie symmetry of mechanical system in phase space.

3. Conserved quantities deduced from Noether–Lie symmetry

The Noether–Lie symmetry of mechanical system in phase space can lead to the Noether conserved quantity and the general Hojman conserved quantity.

Proposition 1 For mechanical system in phase space Eq.(1), the Noether–Lie symmetry will lead to the Noether conserved quantity

$$I_N=p_s\xi_s-H\xi_0+G_N=\text{const}. \tag{11}$$

Proof Because the Noether–Lie symmetry is certainly a Noether symmetry from the Definition, there exists such a gauge function $G_N(t,\boldsymbol{q},\boldsymbol{p})$ that the Noether identity Eq.(4) holds. Substituting Eq.(4) into the derivative of Eq.(11) with respect to time t, we can obtain Proposition 1.

Proposition 2 For the mechanical system in phase space Eq.(1), under the infinitesimal transformations Eq.(3), if there exists such a function $u=u(t,\boldsymbol{q},\boldsymbol{p})$ that

$$\frac{\partial g_s}{\partial q_s}+\frac{\partial h_s}{\partial p_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln u=0, \tag{12}$$

then the Noether–Lie symmetry will lead to conserved quantity

$$\begin{aligned}I_H=&\ \frac{1}{u}\frac{\partial(u\xi_0)}{\partial t}+\frac{1}{u}\frac{\partial(u\xi_s)}{\partial q_s}+\frac{1}{u}\frac{\partial(u\eta_s)}{\partial p_s}-\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\=&\ \text{const}.\end{aligned} \tag{13}$$

Proof We have

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_H}{\mathrm{d}t} = & \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial t}\xi_0\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial t}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial q_s}\xi_s\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s}\\ & +\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial q_s}\eta_s\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \eta_s}{\partial q_s}-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right), \qquad (14)\end{aligned}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial t} = \frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}-\frac{\partial \xi_0}{\partial q_k}\frac{\partial g_k}{\partial t}-\frac{\partial \xi_0}{\partial p_k}\frac{\partial h_k}{\partial t}, \qquad (15)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\frac{\partial \xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s}-\frac{\partial \xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s}, \qquad (16)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \eta_s}{\partial p_s} = \frac{\partial}{\partial p_s}\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t}-\frac{\partial \eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s}-\frac{\partial \eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}. \qquad (17)$$

Because the Noether–Lie symmetry is certainly a Lie symmetry from the Definition, so we have the determining equation of the Lie symmetry Eq.(6) or Eq.(9). Substituting Eqs.(15)–(17) into Eq.(14) and using Eq.(9), we can obtain

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_H}{\mathrm{d}t} = & \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial t}\xi_0\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial q_s}\xi_s\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{u}\frac{\partial u}{\partial p_s}\eta_s\right)+\frac{\partial g_s}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\frac{\partial h_s}{\partial p_s}\frac{\bar{d}\xi_0}{\mathrm{d}t}\\ & +\xi_0\frac{\partial^2 g_s}{\partial q_s\partial t}+\xi_0\frac{\partial^2 h_s}{\partial p_s\partial t}+\frac{\partial^2 g_s}{\partial q_s\partial q_k}\xi_k+\frac{\partial^2 g_s}{\partial q_s\partial p_k}\eta_k+\frac{\partial^2 h_s}{\partial p_s\partial q_k}\xi_k+\frac{\partial^2 h_s}{\partial p_s\partial p_k}\eta_k. \qquad (18)\end{aligned}$$

Finding the partial differential of Eq.(12) with respect to t, q_k and p_k respectively, and substituting the results into Eq.(18), noting Eq.(9), we have

$$\frac{\bar{\mathrm{d}}I_H}{\mathrm{d}t} = 0. \qquad (19)$$

We call the conserved quantity (13) general Hojman conserved quantity of mechanical system in phase space.

If $\xi_0 = 0$, the conserved quantity Eq.(13) becomes

$$I_H = \frac{1}{u}\frac{\partial(u\xi_s)}{\partial q_s}+\frac{1}{u}\frac{\partial(u\eta_s)}{\partial p_s} = \text{const}. \qquad (20)$$

Which is the Hojman conserved quantity of mechanical system in phase space.[21]

For the Hamilton system, the conserved quantities Eqs.(13) and (20) become the results in the Refs.[22] and [16] respectively.

4. Illustrative example

The Hamiltonian of a mechanical system is

$$H = \frac{1}{2}(p_1^2+p_2^2), \qquad (21)$$

the non-potential generalized forces are $Q_1 = 0$, $Q_2 = p_1$. Try to study its Noether–Lie symmetry and conserved quantity.

Equations (1) and (2) give

$$\begin{aligned}g_2 = \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2,\\ h_1 = \dot{p}_1 = -\frac{\partial H}{\partial q_1}+Q_1 = 0,\\ h_2 = \dot{p}_2 = -\frac{\partial H}{\partial q_2}+Q_2 = p_1. \qquad (22)\end{aligned}$$

Choose the infinitesimal generators as

$$\begin{aligned}&\xi_0 = 1, \quad \xi_1 = (p_2-q_1)^2\\ &\xi_2 = 1, \quad \eta_1 = 0, \quad \eta_2 = 0. \qquad (23)\end{aligned}$$

For the infinitesimal generators Eq.(23), when taking $G_N = p_1q_2 - q_1$, the Criterion equation of the Noether–Lie symmetry Eq.(10) is satisfied, so the infinitesimal generators Eq.(23) are Noether–Lie symmetrical. From Eq.(11) we can obtain

$$\begin{aligned}I_N = & \ p_1(p_2-q_1)^2+\frac{1}{2}(p_2^2-p_1^2)+p_1q_2-q_1\\ & = \text{const}. \qquad (24)\end{aligned}$$

Equation (13) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln u = 0, \qquad (25)$$

which has the solution

$$u = \mathrm{e}^{p_1}. \qquad (26)$$

Substituting Eqs.(23) and (26) into Eq.(13), we obtain

$$I_H = 2(p_1-q_1) = \text{const}. \qquad (27)$$

5. Conclusion

A new kind of symmetry and its conserved quantities of a mechanical system in phase space are studied in this paper. The definition of this new symmetry, i.e. a Noether–Lie one is presented, and the criterion of this symmetry is also given. The Noether conserved quantity and the generalized Hojman conserved quantity of the Noether–Lie symmetry of the system are obtained. An example is given to illustrate the application of the result.

The previous studies on symmetries and conserved quantities of mechanical system in phase space were all carried out from a kind of symmetry to seek a type of conserved quantity. The Noether–Lie symmetry shown in the paper contains the Noether symmetry and the Lie symmetry, and has more generalized significance. By the Noether–Lie symmetry, we have obtained two types of conserved quantity, i.e. the Noether conserved quantity and the generalized Hojman conserved quantity of mechanical system in phase space.

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