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Integrating factors and conservation theorems of constrained Birkhoffian systems*

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In this paper the conservation theorems of the constrained Birkhoffian systems are studied by using the method of integrating factors. The differential equations of motion of the system are written. The definition of integrating factors is given for the system. The necessary conditions for the existence of the conserved quantity for the system are studied. The conservation theorem and its inverse for the system are established. Finally, an example is given to illustrate the application of the results.

Keywords: integration factor, conservation theorem, constrained Birkhoffian system

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1. Introduction

In 1927, the American mathematician Birkhoff gave a new form of equations of motion and a new integral principle in his work 'Dynamical Systems'.[1] The equations are called Birkhoff's equations on Santilli's suggestion in 1978.[2] Mei and co-workers generalized the study to nonholonomic dynamical systems, and constructed the frame of Birkhoffian dynamics.[3,4] Afterwards, the research of Birkhoffian dynamics was expanded from the classical case to the relativity and rotating relativity domains.[5−7] In recent years, some results for the study of Birkhoffian dynamics were obtained.[8−10] Now, the study of Birkhoffian dynamics which have become an important developed direction in modern mathematical, mechanical and physical domains.

In 1984, the method of integrating factors for finding the conserved quantity of non-conservative mechanical system was derived by Djukic'.[11] This method has the benefits of fewer conditions, easy calculation and wide practical application. Also, it will have a great future in the integration theory of analytical mechanics, and some results have been obtained.[12−14]

In this paper, the theory of integrating factors for differential equations of motion is used to seek the conserved quantities of constrained Birkhoffian systems.

2. Equations of motion of constrained Birkhoffian systems

Suppose that the Birkhoffian systems is subjected to g constraints in the form

$$f_\beta(t, \boldsymbol{a}) = 0, \quad (\beta = 1, 2, \ldots, g). \tag{1}$$

The conditions added to virtual displacements δa^μ by the constraints are

$$\frac{\partial f_\beta}{\partial a^u}\delta a^\mu = 0, \quad (\beta = 1, 2, \ldots, g; \mu = 1, 2, \ldots, 2n). \tag{2}$$

Then the equations of motion of the constrained Birkhoffian systems are[11]

$$\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu}, \quad (\beta = 1, 2 \ldots, g; \mu = 1, 2, \ldots, 2n). \tag{3}$$

where the $R_\gamma = R_\gamma(t, \boldsymbol{a})$ are called the Birkhoff functions, and the $B = B(t, \boldsymbol{a})$ is called the Birkhoff function.

Using the constraint Eq.(1) and the equations of motion (3), and expressing λ_β as the function of t and $\boldsymbol{a}$ before integrating the differential equations of motion, we have

$$\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = P_\mu, \tag{4}$$

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where

$$P_\mu = P_\mu(t, \boldsymbol{a}) = \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu}, \quad (\mu = 1, 2, \ldots, 2n), \tag{5}$$

have been expressed as explicit function of t and $\boldsymbol{a}$.

Eq.(4) is called the free Birkhoff equations, corresponding to the constrained Birkhoffian systems. If the initial conditions satisfy the constraint equations (1), the solutions of Eq.(4) give the motion of the constrained Birkhoffian systems.

3. Integration factors and conservation theorems

3.1. Integrating factors

The invariant

$$\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right]\xi_\mu$$

reduces identically to

$$\begin{aligned} &\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right]\xi_\mu \\ \equiv\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(R_\mu\xi_\mu - B\tau + G\right) + K_\mu\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right], \end{aligned} \tag{6}$$

where τ, G and K_μ are functions of t and $\boldsymbol{a}$. In this case, the $\xi_\mu = \xi_\mu(t, \boldsymbol{a})$ is called integrating factor of the equation of motion Eq.(4).

3.2. Conservation theorem

Combining Eqs.(4) and (6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(R_\mu\xi_\mu - B\tau + G) = -K_\mu\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right], \tag{7}$$

then the following theorem can be obtained.

Theorem 1 If the ξ_μ is integrating factor of Eq.(4), there exists a conserved quantity (first integration)

$$I = R_\mu\xi_\mu - B\tau + G = \text{const} \tag{8}$$

for the free Birkhoffian system Eq.(4), corresponding to the constrained Birkhoffian systems Eqs.(1) and (3).

Proof: Differentiating Eq.(8) with respect to time t and considering Eqs.(4) and (7), one has

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{d}{\mathrm{d}t}(R_\mu\xi_\mu - B\tau + G) \\ &= -K_\mu\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right] \\ &= 0. \end{aligned} \tag{9}$$

For a given dynamical system, if the ξ_μ is integrating factor of Eq.(4), the necessary condition Eq.(7) must be satisfied for every set of functions ξ_μ, τ, G and K_μ. Now, multiplying two sides of Eq.(4) by $(\xi_\mu - \dot{a}^\mu\tau)$ and making summation for μ, then adding the result to condition Eq.(7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(R_\mu\xi_\mu - B\tau + G) + \left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a^\gamma} - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right](\xi_\mu - \dot{a}^\mu\tau) + \Phi = 0, \tag{10}$$

where

$$\Phi = K_\mu\left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right]. \tag{11}$$

Expanding condition Eq.(10), obviously, the term which contains $\dot{a}^\mu\dot{a}^\gamma$ is zero. Then condition Eq.(10) becomes

$$\left(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{\partial B}{\partial t}\right)\tau + \left(\frac{\partial R_\gamma}{\partial a^\mu}\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu - B\dot{\tau} + R_\mu\dot{\xi}_\mu - P_\mu(\xi_\mu - \dot{a}^\mu\tau) + \dot{G} + \Phi = 0. \tag{12}$$

If the set of functions ξ_μ, τ, G and K_μ satisfies the necessary condition Eq.(12) and the right-hand side of Eq.(8) is reduced to a constant, then we have following theorem.

Theorem 2 For every nonsingular set of functions ξ_μ, τ, G and K_μ, which satisfies the necessary condition Eq.(12), there exists a conserved quantity (first integration) Eq.(8) of the free Birkhoffian system Eq.(4), corresponding to the constrained Birkhoffian systems Eqs.(1) and (3).

4. Inverse theorem

For the corresponding free Birkhoffian system Eq.(4), if the first integral of system has known, then from the inverse theorem of the integrating factors the functions τ, G and the integrating factor ξ_μ can be found.

Suppose that the corresponding free Birkhoffian system (4) has a first integral

$$I = I(t, \boldsymbol{a}) = \text{const}, \tag{13}$$

then, there is

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^\mu}\dot{a}^\mu = 0. \tag{14}$$

Multiplying two sides of Eq.(4) by $(\xi_\mu - \dot{a}^\mu\tau)$ and making summation for μ, then adding the result to Eq.(14), we obtain

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^\mu}\dot{a}^\mu + \left[\left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\dot{a}^\gamma - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu\right](\xi_\mu - \dot{a}^\mu\tau) = 0. \tag{15}$$

Obviously, in Eq.(15), the term which contains $\dot{a}^\mu\dot{a}^\gamma$ is zero. According to the fact that the coefficients of $\dot{a}^\gamma$ in Eq.(15) is zero, we have

$$\frac{\partial I}{\partial a^\gamma} + \left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)\xi_\mu + \left(\frac{\partial B}{\partial a^\gamma} + \frac{\partial R_\gamma}{\partial t} + P_\gamma\right)\tau = 0. \tag{16}$$

Suppose that the system Eq.(4) is non-singular, i.e.

$$\det(\Omega_{\mu\gamma}) \neq 0. \tag{17}$$

Where $(\Omega_{\mu\gamma}) = \left(\frac{\partial R_\gamma}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\gamma}\right)$ is called Birkhoffian tensor. Then the Eq.(16) can also be expressed as

$$\xi_\mu = \Omega^{\mu\gamma}\left[\frac{\partial I}{\partial a^\gamma} + \left(\frac{\partial B}{\partial a^\gamma} + \frac{\partial R_\gamma}{\partial t} + P_\gamma\right)\tau\right] \quad (\mu = 1, 2, \ldots, 2n), \tag{18}$$

where

$$\Omega^{\mu\gamma}\Omega_{\gamma\rho} = \delta_{\mu\rho}(\mu, \gamma, \rho = 1, 2, \ldots, 2n). \tag{19}$$

Finally, let the first integral Eq.(13) equal to the conserved quantity Eq.(8), then we obtain

$$I = R_\mu\xi_\mu - B\tau + G. \tag{20}$$

Theorem 3 If the corresponding free Birkhoffian system Eq.(4) has a first integral Eq.(13), then the integrating factor ξ_μ and the functions τ, G corresponding to the integral Eq.(13) are determined by Eqs.(18) and (20).

Here, we have (2n+1) algebraic equations with (2n+2) unknown functions in most case, thus the set of functions ξ_μ, τ and G did not satisfy the uniqueness.

5. Example

The motion of a mechanical system can be described by the fourth order Birkhoff equations as follows.

The Birkhoffian and the Birkhoff functions are respectively

$$B = \frac{1}{2}\{(a^1)^2 + (a^3)^2 + (a^4)^2\},$$
$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = R_4 = 0. \tag{21}$$

The constraints acting on the system are

$$f_1 = a^1a^3 - c_1^2 = 0, \quad f_2 = a^1 + a^4 - c_2 = 0. \tag{22}$$

We try to find the conserved quantities of the system. Firstly, we study the direct problem to obtain the conserved quantities of the system. The first step

is to establish corresponding free Birkhoff equations. Equations (3) gives

$$-\dot{a}^3 - a^1 = \lambda_1 a^3 + \lambda_2, \quad -\dot{a}^4 = 0,$$
$$\dot{a}^1 - a^3 = \lambda_1 a^1, \quad \dot{a}^2 - a^4 = \lambda_2. \tag{23}$$

From Eqs.(22) and (23), we obtain

$$\lambda_1 = -\frac{a^3}{a^1}, \quad \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{24}$$

then

$$P_1 = \lambda_1 a^3 + \lambda_2 = -a^1, \quad P_2 = 0,$$
$$P_3 = -a^3, \quad P_4 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{25}$$

Substituting Eq.(25) into Eq.(23), we have

$$-\dot{a}^3 - a^1 = -a^1, \quad -\dot{a}^4 = 0,$$
$$\dot{a}^1 - a^3 = -a^3, \quad \dot{a}^2 - a^4 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{26}$$

The second step is to seek the conserved quantities of corresponding free Birkhoffian system Eq.(26).

According to the necessary condition Eq.(12) we have

$$(\dot{a}^1 - a^3)\xi_3 + (\dot{a}^2 - a^4)\xi_4 - a^1\xi_1 - B\dot{\tau} + a^3\dot{\xi}_1 + a^4\dot{\xi}_2 + \dot{G} + a^1(\xi_1 - \dot{a}^1\tau) + a^3(\xi_3 - \dot{a}^3\tau)$$
$$+ \left(a^1 - \frac{(a^3)^2}{a^1}\right)(\xi_4 - \dot{a}^4\tau) - \dot{a}^3 K_1 + \dot{a}^1 K_3 + \left(\dot{a}^2 - a^4 + a^1 - \frac{(a^3)^2}{a^1}\right) K_4 = 0. \tag{27}$$

Equation (27) have solutions

$$\tau = 0, \quad \xi_2 = 1, \quad \xi_1 = \xi_3 = \xi_4 = 0, \quad K_1 = K_2 = K_3 = K_4 = 0, \quad G = 0. \tag{28}$$
$$\tau = 0, \quad \xi_1 = 1, \quad \xi_2 = \xi_3 = \xi_4 = 0, \quad K_1 = K_2 = K_3 = K_4 = 0, \quad G = 0. \tag{29}$$
$$\tau = 0, \ \xi_4 = 0, \ \xi_1 = a^3, \ \xi_2 = a^4, \ \xi_3 = -a^1, \ K_1 = K_2 = K_3 = K_4 = 0, \ G = \frac{1}{2}[(a^1)^2 - (a^3)^2 - (a^4)^2]. \tag{30}$$

Then the conserved quantities Eq.(8) of theorem 2 give, respectively

$$I_1 = a^4 = \text{const}, \tag{31}$$

$$I_2 = a^3 = \text{const}, \tag{32}$$

$$I_3 = \frac{1}{2}[(a^1)^2 + (a^3)^2 + (a^4)^2]. \tag{33}$$

Next, we study the inverse problem. From Eq.(21), we obtain

$$(\Omega_{\mu\gamma}) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad (\Omega^{\mu\gamma}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{34}$$

Suppose that Eq.(21) has a first integral Eq.(31), then the Eqs.(18) and (20) give

$$\xi_1 = 0, \quad \xi_2 = 1 + \left[a^4 - a^1 + \frac{(a^3)^2}{a^1}\right]\tau, \quad \xi_3 = 0, \quad \xi_4 = 0, \quad a^4\left[a^4 - a^1 + \frac{(a^3)^2}{a^1}\right]\tau - B\tau + G = 0. \tag{35}$$

Taking $G = 0$, from Eq.(35) we obtain

$$\xi_1 = \xi_3 = \xi_4 = 0, \quad \xi_2 = 1, \quad \tau = 0.$$

Next, suppose that Eq.(21) has a first integral Eq.(32), as calculated above, we have

$$\xi_1 = 1, \quad \xi_2 = \left[a^4 - a^1 + \frac{(a^3)^2}{a^1}\right]\tau, \quad \xi_3 = 0, \quad \xi_4 = 0, \quad a^4\left[a^4 - a^1 + \frac{(a^3)^2}{a^1}\right]\tau - B\tau + G = 0. \tag{36}$$

Taking $G = 0$, from Eq.(36), we obtain

$$\xi_1 = 0, \quad \xi_2 = \xi_3 = \xi_4 = 0, \quad \tau = 0.$$

Finally, suppose that Eq.(21) has still a first integral Eq.(33), as calculated above, we have

$$\xi_1 = a^3, \quad \xi_2 = a^4 + \left[a^4 - a^1 + \frac{(a^3)^2}{a^1}]\tau\right], \quad \xi_3 = -a^1, \quad \xi_4 = 0,$$
$$a^3\xi_1 + a^4\xi_2 - \frac{1}{2}[(a^1)^2 + (a^3)^2 + (a^4)^2]\tau + G = \frac{1}{2}[(a^1)^2 - (a^3)^2 - (a^4)^2]. \tag{37}$$

Taking $G = \frac{1}{2}[(a^1)^2 - (a^3)^2 - (a^4)^2]$, from Eq.(37), we obtain

$$\tau = 0, \quad \xi_1 = a^3, \quad \xi_2 = a^4, \quad \xi_3 = -a^1, \quad \xi^4 = 0.$$

These values are consistent with the result in Ref.[15].

References

[1] Birkhoff G D 1927 *Dynamical Systems* (Providence, RI: AMS)

[2] Santilli R M 1983 *Foundations of Theoretical Mechanics* II (New York: Springer)

[3] Mei F X and Wu H 1993 *The frame of the Noether's theory of Birkhoffian Systems.* In: Proc. of ICN M-II. Beijing, 718–720

[4] Mei F X, Shi R C, Zhang Y F and Wu H B 1996 *Dynamics of Birkhoffian System* (Beijing Institute of Technology Press) (in Chinese)

[5] Fu J L, Chen L Q, Luo S K, Chen X W and Wang X W 2001 *Acta Phys. Sin.* **50** 2289 (in Chinese)

[6] Fu J L 1999 *J. Shangqiu Teachers College* **15** 15 (in Chinese)

[7] Fu J L and Chen L Q 2004 *Phys. Lett.* **A** 324 (2/3) 95–103

[8] Zhang Y and Ge W K 2003 *Acta Phys. Sin.* **52** 2363 (in Chinese)

[9] Chen X W, Zhang R C and Mei F X 2000 *Acta Mech. Sin.* **16** 282 (in Chinese)

[10] Fu J L and Wang X M 2000 *Acta Phys. Sin.* **49** 1023 (in Chinese)

[11] Dj S and Sutela T 1984 *Int. J. Non-Linear Mech.* **19** 331

[12] Li R J, Qiao Y F and Liu Y 2002 *Chin. Phys.* **11** 760

[13] Qiao Y F, Zhang Y L and Han G C 2002 *Chin. Phys.* **11** 988

[14] Mei F X 1996 *J. Beijing Institute of Technology* **16** 245 (in Chinese)

[15] Mei F X 1999 *Applications of Lie Group and Lie Algebras to Constrained Mechanical Systems.* (Beijing: Science Press) 90–475 (in Chinese)