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The derivative-dependent functional variable separation for the evolution equations*

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This paper studies variable separation of the evolution equations via the generalized conditional symmetry. To illustrate, we classify the extended nonlinear wave equation $u_{tt} = A(u, u_x)u_{xx} + B(u, u_x, u_t)$ which admits the derivative-dependent functional separable solutions (DDFSSs). We also extend the concept of the DDFSS to cover other variable separation approaches.

Keywords: derivative-dependent functional variable separation, evolution equations, generalized conditional symmetry

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1. Introduction

It is well known that both the Fourier analysis and the variable separation approaches are effective means to deal with linear partial differential equations (PDEs). The inverse scattering transformation method[1] for nonlinear integrable systems is an extension of the Fourier analysis. However, it is quite difficult to extend the variable separation approaches to nonlinear PDEs though certain methods have been developed recently. On the subject of variable separation, Miller applied Lie's classical symmetry[2] to the variable separation for linear PDEs[3,4], Kalnins and Miller introduced the differential Stäckel matrices method[5], Dolye proposed the differential geometry method[6] and Zhdanov *et al* established the ansatz-based method[7] during last decade.

The study on variable separation for the evolution equations made progress recently. Some scholars presented three types of approaches: the first is called the formal variable separation approach or equivalently the nonlinearization of the Lax pairs or symmetry constraints;[8] the second is the multilinear variable separation approach (MLVSA);[9] the third is the functional variable separation approach (FVSA)[10−12] based on the generalized conditional symmetry (GCS). The GCS due to Fokas, Liu and Zhdanov is developed to obtain symmetry reductions and exact solutions to nonlinear PDEs.[13,14]

Recently, we defined a new concept of the derivative-dependent functional separable solutions (DDFSSs)

$$f(u, u_x) = a(x) + b(t) \tag{1}$$

of the evolution equations. In terms of the DDFSS, we established the derivative-dependent functional variable separation approach (DDFVSA). In fact, we applied the DDFVSA to the generalized nonlinear diffusion equations, the KdV type equations and the extended nonlinear wave equations.[15] The DDFSS is a natural extension of the functional separable solution (FSS). The FSSs

$$f(u) = a(x) + b(t) \tag{2}$$

of PDEs has been concerned by few authors.[10−12] If $f(u) = \ln u$, it is the product separable solution; If $f(u) = u$, it is the additive separable solution.

In Ref.[12], we discussed the functional variable separation for the extended nonlinear wave equation

$$u_{tt} = A(u, u_x)u_{xx} + B(u, u_x, u_t), \quad A(u, u_x) \neq 0, \tag{3}$$

which admits the FSSs Eq.(2). In this paper, we intend to extend the concept of FSS to that of DDFSS Eq.(1) for the evolution equations and study

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the derivative-dependent functional variable separation for Eq.(3). The DDFSS Eq.(1) degenerates into the FSS Eq.(2) if $f_{u_x}(u,u_x)=0$, but they are definitely different kinds of constraints because the DDFSS Eq.(1) is a differential constraint while the DDFSS Eq.(2) is a functional one. Since we clarified the case $f_{u_x}(u,u_x)=0$ for Eq.(3) in Ref.[12], for completeness, it is necessary for us to deal with the case $f_{u_x}(u,u_x)\neq 0$ in this paper.

Consider the general mth order nonlinear evolution equation

$$\begin{aligned} E &\equiv E(t,x,u,u_1,u_2,\ldots,u_m,u_t,u_{tt}) \\ &= 0, \end{aligned} \tag{4}$$

where

$$u_k=\partial^k u/\partial x^k, 1\leq k\leq m,$$

E is a smooth function of the indicated variables.

Let

$$V=\eta\frac{\partial}{\partial u}=\eta(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{xxx},u_{xxt},\ldots,)\frac{\partial}{\partial u} \tag{5}$$

be an evolutionary vector field and η its characteristic.

Definition 1 A solution $u=u(x,t)$ is said to be a DDFSS of Eq.(4) if it satisfies Eq.(4) and the ansatz

$$f(u,u_x)=a(x)+b(t), \tag{6}$$

where $f(u,u_x)$ is a smooth function of u and u_x, and $a(x)$, $b(t)$ are some functions of x and t. In particular, when f does not depend on u_x, the solution is said to be a FSS.

Definition 2 The evolutionary vector field Eq.(5) is said to be a GCS of Eq.(4) if and only if

$$V^{(m)}(E)|_{L\bigcap W}=0, \tag{7}$$

where $V^{(m)}$ is the m-th prolongation of Eq.(5), L is the solution manifold of Eq.(4) and W is the set of all the total derivatives of $\eta=0$ with respect to x, namely, $D_x^i\eta=0,\ i=0,1,2,\ldots$, which are the invariant surface condition and its partial derivatives with respect to x by appending to Eq.(4).

It can be derived from Eq.(7) that Eq.(4) admits the GCS Eq.(5) if and only if

$$(E_{u_t}D_t\eta+E_{u_{tt}}D_t^2\eta)|_{L\bigcap W}=0, \tag{8}$$

where D_t and D_t^2 denote the first and second order total derivatives with respect to t respectively.

To sum up the results in Ref.[15], we have the following theorem.

Theorem 1 Equation (4) possesses the DDFSS of Eq.(6) if and only if it admits the GCS

$$\begin{aligned} V= \ \eta\frac{\partial}{\partial u} &= [(f_{uu}u_t+f_{uu_x}u_{xt})u_x+(f_{uu_x}u_t+f_{u_xu_x}u_{xt})u_{xx} \\ &\quad +f_{u_x}u_{xxt}+f_uu_{xt}]\frac{\partial}{\partial u}. \end{aligned} \tag{9}$$

The remaining part of this paper is as follows. In Sec.2, we classify the extended nonlinear wave equation Eq.(3) which admits the DDFSSs. The concept of the DDFSSs is further extended in Sec.3. Sec.4 is summary and discussion.

2. Classification of Eq.(3) which admits the DDFSSs

By Eq.(8) and Theorem 1, we know that classification of Eq.(3) which admits the DDFSSs is equivalent to seeking the GCS of Eq.(9) satisfying

$$(E_{u_t}D_t\eta+E_{u_{tt}}D_t^2\eta)|_{L\bigcap W}=0, \tag{10}$$

while $E\equiv u_{tt}-A(u,u_x)u_{xx}-B(u,u_x,u_t)$ and

$$\begin{aligned} \eta= \ & (f_{uu}u_t+f_{uu_x}u_{xt})u_x+(f_{uu_x}u_t \\ & +f_{u_xu_x}u_{xt})u_{xx}+f_{u_x}u_{xxt}+f_uu_{xt}. \end{aligned} \tag{11}$$

Now we calculate Eq.(10).

Taking into account expressions $D_x^i\eta=0,\ i=0,1,2,\ldots$, and Eq.(3), and excluding the higher order derivatives of u in Eq.(10), we transform the left-handed side of Eq.(10) into an expression of independent derivatives of u, which is

$$
\begin{aligned}
& (E_{u_t} D_t\eta + E_{u_{tt}} D_t^2\eta)|_{L\bigcap W} \\
= \; & (D_t^2\eta - B_{u_t} D_t\eta)|_{L\bigcap W} \\
= \; & [(h_1 u_{xt} + h_{18})u_{xx} + h_2 u_{xt} + h_3]u_{xxx} \\
& + h_4 u_{xx}^4 + (h_6 u_{xt} + h_7)u_{xx}^3 + h_5 u_{xt}^3 + h_8 u_{xt} + h_9 u_{xt}^2 + h_{17} \\
& + (h_{10} u_{xt}^3 + h_{11} u_{xt}^2 + h_{12} u_{xt} + h_{13})u_{xx} + (h_{14} u_{xt} + h_{15} u_{xt}^2 + h_{16})u_{xx}^2, \qquad (12)
\end{aligned}
$$

where $h_i = h_i(u, u_x, u_t)$ are complicated functions containing $f(u, u_x)$, $A(u, u_x)$, $B(u, u_x, u_t)$ and their derivatives with respect to u, u_x and u_t, and $i = 1, 2, \ldots, 18$.

The vanishing of Eq.(12) leads to $f(u, u_x)$, $A(u, u_x)$ and $B(u, u_x, u_t)$ satisfying the following system of PDEs:

$$h_i = h_i(u, u_x, u_t) = 0, \; i = 1, 2, \ldots, 18. \qquad (13)$$

Since the expressions for h_i are quite long, we put them in the Appendix.

Solving Eq.(13) for $f(u, u_x)$, $A(u, u_x, u_t)$ and $B(u, u_x, u_t)$, we obtain the following classification theorem:

Theorem 2 Assume that $f_{u_x}(u, u_x) \neq 0$ in Eq.(6), the equation

$$u_{tt} = A(u, u_x)u_{xx} + B(u, u_x, u_t), \; A(u, u_x) \neq 0$$

admits DDFSS of the Eq.(6) if and only if it is equivalent to one of the following equations, up to translation and dilatation for u:

1)

$$u_{tt} = u_{xx}, \qquad (14)$$

$$f(u_x) = \operatorname{arcsinh}(\tan(u_x)); \qquad (15)$$

2)

$$u_{tt} = \mathrm{e}^{u_x + c_1 u} u_{xx} + (c_1 u_x + b_1)\mathrm{e}^{u_x + c_1 u} + b_2 u_t + b_3, \qquad (16)$$

$$f(u, u_x) = u_x + c_1 u; \qquad (17)$$

3)

$$u_{tt} = \frac{u_{xx}}{c_3 \mathrm{e}^{c_1 u_x} + c_2}, \qquad (18)$$

$$f(u_x) = \ln(c_3 \mathrm{e}^{c_1 u_x} + c_2); \qquad (19)$$

4)

$$u_{tt} = \frac{u_{xx}}{u_x} + b_1 \ln(u_x) + b_2 u_t + b_3 u + b_4, \qquad (20)$$

$$f(u_x) = \ln(u_x); \qquad (21)$$

5)

$$u_{tt} = \frac{u_{xx}}{u_x + 1} + b_1 \ln(u_x + 1) + b_2 u_t + b_3 u + b_4, \qquad (22)$$

$$f(u_x) = \ln(u_x + 1); \qquad (23)$$

6)

$$u_{tt} = \frac{\mathrm{e}^{c_1 u} u_{xx}}{u_x + 1} - (c_1 u_x - b_1)\mathrm{e}^{c_1 u} + c_1 {u_t}^2 + b_2 u_t + b_3, \qquad (24)$$

$$f(u,u_x)=\ln(u_x+1)-c_1u; \tag{25}$$

7)

$$u_{tt}=\frac{(a_1+a_2u)u_{xx}}{{u_x}^2}-2a_2\ln(u_x)+b_1u_t+b_2u+b_3, \tag{26}$$

$$f(u_x)=\ln(u_x) \tag{27}$$

8) $\alpha\neq-1,0$.

$$u_{tt}={u_x}^{\alpha}u_{xx}+b_1{u_x}^{\alpha+1}+b_2u_t+b_3u+b_4, \tag{28}$$

$$f(u_x)=\ln(u_x); \tag{29}$$

9)

$$u_{tt}=u{u_x}^{\alpha}u_{xx}-\frac{2{u_x}^{\alpha+2}}{\alpha+2}+b_1u_t+b_2u+b_3, \tag{30}$$

$$f(u_x)=\ln(u_x); \tag{31}$$

10)

$$\begin{aligned}u_{tt}=&\frac{(u_x+1)^{\alpha}u_{xx}}{\mathrm{e}^{\alpha c_1u}}+\frac{[-c_1(\alpha+1){u_x}^2+(b_1-c_1\alpha)u_x+b_1+c_1](u_x+1)^{\alpha}}{(\alpha+1)\mathrm{e}^{\alpha c_1u}}\\&+c_1{u_t}^2+b_2u_t+b_3,\end{aligned} \tag{32}$$

$$f(u,u_x)=\ln(u_x+1)-c_1u; \tag{33}$$

11)

$$u_{tt}=\frac{(u_x+1)^{\alpha}}{\mathrm{e}^{a_1u}}\left[u_{xx}-\frac{a_1(\alpha{u_x}^2+(\alpha-1)u_x-1)}{(\alpha+1)(\alpha+2)}\right]+\frac{a_1}{\alpha+1}{u_t}^2+b_1u_t+b_2, \tag{34}$$

$$f(u,u_x)=\ln(u_x+1)-\frac{a_1u}{\alpha+1}; \tag{35}$$

12)

$$\begin{aligned}u_{tt}=&\frac{(a_1u+a_2)u_{xx}}{u_x+a_3(a_1u+a_2)}+b_1u_t\\&-\frac{2a_1u_x^2-(a_1u+a_2)[(b_2-a_1a_3)u_x+b_2a_3(a_1u+a_2)]}{u_x+a_3(a_1u+a_2)},\end{aligned} \tag{36}$$

$$f(u,u_x)=-\ln[u_x+a_3(a_1u+a_2)]; \tag{37}$$

13) $-b_2c_2+b_4c_1=0$,

$$\begin{aligned}u_{tt}=&c_3(u_x+c_1u+c_2)^{\alpha}u_{xx}+b_3u_t\\&-\frac{c_3[(c_1(\alpha+1)-b_1)(c_1u+c_2)-b_1u_x](u_x+c_1u+c_2)^{\alpha}}{\alpha+1}+b_2u+b_4,\end{aligned} \tag{38}$$

$$f(u,u_x)=\ln(u_x+c_1u+c_2); \tag{39}$$

14)

$$u_{tt}=(a_1u+a_2)u_xu_{xx}-\frac{2}{3}a_1{u_x}^3+\frac{1}{3}a_1b_1u^3+a_2b_1u^2+b_2u_t+b_3u+b_4, \tag{40}$$

$$f(u,u_x)=\ln(u_x); \tag{41}$$

15)

$$u_{tt} = a_1 u_x u_{xx} - a_1 \frac{u u_x^3}{1+u^2} + b_1 {u_x}^2 + b_2 u^2 + b_3 u_t + b_4 u + b_5, \tag{42}$$

$$f(u, u_x) = \ln(u_x); \tag{43}$$

16)

$$u_{tt} = (u+a_1)(u_x + c_1(u+a_1))u_{xx} + b_2 u_t - \frac{2}{3}u_x^3 - \frac{1}{3}c_1(u+a_1)[2u_x^2 + c_1(u+a_1)(u_x + c_1(u+a_1)) + b_1], \tag{44}$$

$$f(u, u_x) = \ln(u_x + c_1(u+a_1)); \tag{45}$$

17) $|c_1|^2 + |c_2|^2 \neq 0$,

$$u_{tt} = (u_x + c_1 u + c_2)^2 u_{xx} + b_2 u_t + b_1(c_1 u + c_2) + \frac{1}{12}c_1[7u_x - 5(c_1 u + c_2)](u_x + c_1 u + c_2)^2, \tag{46}$$

$$f(u, u_x) = \ln(u_x + c_1 u + c_2); \tag{47}$$

18) $c_1 \neq 0$,

$$u_{tt} = (u+a_1)(u_x + c_1(u+a_1))^2 u_{xx} + b_2 u_t + b_1 c_1 (u+a_1) - \frac{1}{4}[2u_x^2 - c_1(u+a_1)u_x + c_1^2(u+a_1)^2](u_x + c_1(u+a_1))^2, \tag{48}$$

$$f(u, u_x) = \ln(u_x + c_1(u+a_1)); \tag{49}$$

19) $c_1 \neq 0$,

$$u_{tt} = (u+a_1)(u_x + c_1(u+a_1))^3 u_{xx} + b_2 u_t + b_1 c_1 (u+a_1) - \frac{1}{5}[2u_x^2 - 2c_1(u+a_1)u_x + c_1^2(u+a_1)^2](u_x + c_1(u+a_1))^3, \tag{50}$$

$$f(u, u_x) = \ln(u_x + c_1(u+a_1)); \tag{51}$$

20)

$$u_{tt} = (c_1 + a_1 u)(u_x + c_1 + a_1 u)^\alpha u_{xx} + b_1(c_1 + a_1 u) + \frac{a_1[-2u_x^2 + (c_1 + a_1 u)(\alpha - 1)u_x - (c_1 + a_1 u)^2](u_x + c_1 + a_1 u)^\alpha}{\alpha + 2}, \tag{52}$$

$$f(u, u_x) = \ln(u_x + c_1 + a_1 u); \tag{53}$$

21)

$$u_{tt} = \frac{u_{xx}}{u_x + a_1 u + a_2} - \frac{a_1(a_1 u + a_2)}{u_x + a_1 u + a_2} + b_2 u_t + b_1 u + b_3, \tag{54}$$

$$f(u, u_x) = \ln(u_x + a_1 u + a_2); \tag{55}$$

where α, a_i, b_i, c_i are arbitrary constants in their definition sets, and $i = 1, 2, \ldots$.

It is worth noting that Eq.(16) with $c_1 = b_i = 0$, $i = 1, 2, 3$, is just the Toda equation, and it is integrable and can be solved by the inverse scattering method.

To construct the DDFSSs of any equation in Theorem 2, the algorithm is as follows: first, we solve the DDFSS ansatz $f(u, u_x) = a(x) + b(t)$ for u with $f(u, u_x)$ replaced by its corresponding expression in Theorem 2; then substitute this expression of u into the corresponding equation in Theorem 2, it leads to

$a(x)$ and $b(t)$ yielding a two-dimensional dynamic system; finally, manage to split it into some system of ordinary differential equations (ODEs) so as to fix functions $a(x)$, $b(t)$ and some integration functions $s(t)$. These equations enjoy abundant exact solutions since $a(x)$, $b(t)$ and $s(t)$ satisfy ODEs with rich exact solutions. To show this, one can refer to examples in Ref.[15].

3. Further remarks on the DDFSSs

Now we outline the procedure of DDFVSA for some evolution equations. Firstly, let the certain combination of different measurements of fields and their partial derivatives possess additive or multiplicative separable solution; secondly, determine its corresponding GCS according to the solution ansatz; thirdly, determine all the possible equations and the possible combinations of functions via the GCS and the theory of group, obtain all the equivalent classes of the original equations; finally, solve out the DDFSSs respectively.

One may pose the following two questions. (1) What is the relationship between any form of solution ansatz and its corresponding symmetry? (2) Can we provide some symmetry interpretation to any form of solution ansatz?

We think that any form of the solution ansatz may correspond to one form of GCS, though finding its corresponding GCS is not always easy. Therefore, it is possible to unify other variable separation approaches in terms of the GCS by extending the DDFSS ansatz. We show this by the following remarks.

Remark 1 A DDFSS Eq.(6) of Eq.(4) corresponds to the GCS of Eq.(9).[15]

Actually, if one differentiates Eq.(6) with respect to x and t, one gets

$$\eta \equiv (f_{uu}u_t + f_{uu_x}u_{xt})u_x + (f_{uu_x}u_t + f_{u_xu_x}u_{xt})u_{xx} + f_{u_x}u_{xxt} + f_u u_{xt} = 0. \tag{56}$$

The corresponding GCS is just the GCS Eq.(9). Specially, if $f_{u_x} \equiv 0$, the DDFSS is just the FSS, its GCS reads

$$V = \eta\frac{\partial}{\partial u} = (f_{uu}u_t u_x + f_u u_{xt})\frac{\partial}{\partial u}. \tag{57}$$

if $f(u, u_x) \equiv u$, the DDFSS is just the additive separable solution, its GCS is

$$V = \eta\frac{\partial}{\partial u} = u_{xt}\frac{\partial}{\partial u}. \tag{58}$$

Remark 2 Galaktionov's nonlinear variable separation method.

Galaktionov[16] proposed a generalization of the usual variable separation approach by assuming a general form for the solution ansatz

$$u = \phi(t)(\psi(t) + \theta(x)). \tag{59}$$

In general, the method is said to be the 'nonlinear separation of variables'. We note that the nonlinear separable solution (NSS) is a special case of the following generalized nonlinear separable solution (GNSS)

$$u = a(x) + b(t) + \phi(t)\theta(x). \tag{60}$$

So Galaktionov's NSS corresponds to $a(x) = 0$ and $b(t) = \phi(t)\psi(t)$ in GNSS. Differentiate Eq.(60) with respect to x and t, then make logarithm transformation for both sides, and differentiate it with respect to x and t again, we obtain

$$\eta \equiv [\ln(u_{xt})]_{xt} = 0. \tag{61}$$

So the GCS relating to the GNSS is

$$V = \eta\frac{\partial}{\partial u} = [\ln(u_{xt})]_{xt}\frac{\partial}{\partial u}. \tag{62}$$

Therefore the GNSS corresponds to the DDFSS of the form

$$f(u_{xt}) = \ln(u_{xt}) = p(x) + q(t). \tag{63}$$

Remark 3 The multi-linear variable separation approach(MLVSA).[9]

The MLVSA has been used successfully to derive interesting exact solutions to several higher-dimensional integrable systems.[9] The variable separation ansatz for the multi-linear form takes

$$u = a_0 + a_1 p(x,t) + a_2 q(y,t) + a_3 p(x,t)q(y,t). \tag{64}$$

In the same way, we obtain its GCS

$$V = \eta\frac{\partial}{\partial u} = [\ln(u_x)]_{xy}\frac{\partial}{\partial u} \tag{65}$$

or

$$V = \eta\frac{\partial}{\partial u} = [\ln(u_y)]_{xy}\frac{\partial}{\partial u}. \tag{66}$$

Therefore the multi-linear variable separation ansatz Eq.(64) corresponds to the DDFSS in the form

$$f(u_x) = \ln(u_x) = \phi(x) + \psi(t) \tag{67}$$

or

$$f(u_y) = \ln(u_y) = \phi(x) + \psi(t). \tag{68}$$

In the end, to cover other variable separation approaches, we need to extend the definition of DDFSS. For some (2+1)-dimensional evolution equations, we have the following definition.

Definition 3 A solution $u = u(x, y, t)$ is said to be a DDFSS of equation

$$E(t, x, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots, u_m, u_t, u_{tt}) = 0, \tag{69}$$

if it satisfies Eq.(69) and the ansatz

$$f(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}, \ldots, u_n) = p(x) + q(y) + r(t), \tag{70}$$

where E is a smooth function of the indicated variables and u_j, $3 \le j \le m$, $n \le m$ denote all the j-th order partial derivatives with respect to x and y.

4. Summary and discussion

In this paper, we have developed the DDFVSA to the evolution equations in terms of the GCS. For instance, we have classified the extended nonlinear wave equation which admits the DDFSSs. Moreover, We have extended the concept of the DDFSS so as to cover other variable separation approaches.

There are still the following three interesting topics to be addressed later.

(1) How to apply the DDFVSA to some types of evolution equations or system of evolution equations? For example,

$$u_{xt} = F(x, t, u, u_x, u_{xx}, u_{xxx}, \ldots).$$

(2) How to unify other variable separation approaches by the DDFVSA? For instance, how to recover the results from the MLVSA by the DDFVSA?

(3) It is interesting to extend the DDFVSA and apply it to some differential-difference equations.

Appendix. Expressions of h_i in Eq.(13)

In the following $f = f(u, u_x)$, $A = A(u, u_x)$ and $B = B(u, u_x, u_t)$, the subscripts $1, 2, 3$ of f, A and B denote whose partial derivatives with respect to u, u_x, and u_t respectively.

$$h_1 = -\frac{3f_2A_2^2}{A} - 3f_{22}A_2 + 3f_2A_{22} = 0, \tag{A1}$$

$$h_2 = -\left[\left(2\frac{A_1f_2}{A} + 3f_{12}\right)A_2 - 2A_{12}f_2\right]u_x - \left(3f_1 + \frac{B_2f_2}{A}\right)A_2$$
$$+2f_{12}A - \frac{2f_{22}f_1A}{f_2} + (2B_{33}A + 2A_1 + B_{22})\,f_2 = 0, \tag{A2}$$

$$h_3 = \left[-3A_2u_xf_{11} - \frac{2f_{12}f_1A}{f_2} + \left(2A_{11}u_x - \frac{2{A_1}^2u_x}{A} - \frac{A_1B_2}{A}\right)f_2\right.$$
$$\left.+f_2B_{12} + 2Af_{11}\right]u_t + 2B_{13}Af_2u_x + f_2BB_{23} = 0, \tag{A3}$$

$$h_4 = Af_2B_{223} = 0, \tag{A4}$$

$$\begin{aligned} h_5 = &\left[\frac{(2f_{22}f_{12} - f_{122}f_2)\,A_2}{f_2A} - \frac{3f_{12}f_{222}}{f_2} + f_{1222} + \frac{2f_{22}\,(2f_{22}f_{12} - f_{122}f_2)}{{f_2}^2}\right] u_x \\ &-\frac{\left({f_2}^2B_{33} - 2f_{22}f_1 + 2f_{12}f_2\right)A_2}{f_2A} + f_2B_{233} - \frac{3f_1f_{222}}{f_2} + f_{22}B_{33} \\ &+\frac{4f_1{f_{22}}^2}{{f_2}^2} - \frac{4f_{22}f_{12}}{f_2} + 3f_{122} = 0, \end{aligned} \tag{A5}$$

$$\begin{aligned} h_6 = &-\frac{{A_2}^2f_{22}}{A} - \left(\frac{A_{22}}{A}f_2 + \frac{{f_{22}}^2}{f_2}\right)A_2 + \frac{5f_{22}Af_{222}}{f_2} \\ &+A_{222}f_2 - f_{22}A_{22} - Af_{2222} - \frac{4{f_{22}}^3A}{{f_2}^2} = 0, \end{aligned} \tag{A6}$$

$$\begin{aligned} h_7 = &\left[\frac{3f_{12}Af_{222}}{f_2} + \left(A_{122} - \frac{A_1A_{22}}{A}\right)f_2 - \frac{f_{22}\,(A_1f_2 + f_{12}A)\,A_2}{f_2A}\right. \\ &\left.-2A_{22}f_{12} - \frac{4{f_{22}}^2Af_{12}}{{f_2}^2} + \left(A_{12} + \frac{2Af_{122}}{f_2}\right)f_{22} - Af_{1222}\right] u_t \\ &+2f_2B_{23}A_2 + B_{223}Af_2 + f_2B_{223}B + f_{22}B_{23}A = 0, \end{aligned} \tag{A7}$$

$$\begin{aligned} h_8 = &\left\{\left[3f_{1112} + \frac{(2f_{11}f_{12} - f_{111}f_2)\,A_2}{f_2A} - \frac{2f_{112}A_1}{A} + \frac{4{f_{12}}^3 + 8f_{12}f_{22}f_{11}}{{f_2}^2}\right.\right. \\ &+\left(\frac{2A_1f_{22}f_{11}}{A} - 6f_{122}f_{11} - 7f_{112}f_{12} + \frac{2A_1{f_{12}}^2}{A} - 2f_{111}f_{22}\right){f_2}^{-1}\Big] u_x \\ &-\frac{2f_{11}A_1}{A} - \frac{4f_{11}f_{12}}{f_2} + 3f_{111} - \frac{\left(-2f_2A_1f_{12} + 3f_2Af_{112} - 4{f_{12}}^2A\right)f_1}{{f_2}^2A}\Bigg\} {u_t}^2 \\ &+\left\{\left[-3B_{23}f_{11} + \left(-B_3{f_{12}}^2 - 2B_3f_{11}f_{22}\right){f_2}^{-1} + 2B_{13}f_{12} + f_{112}B_3\right.\right. \\ &\left.+\frac{f_{11}B_3A_2}{A} + \left(-\frac{2A_1B_{13}}{A} + 2B_{113}\right)f_2\right] u_x + f_{11}B_3 - B_3f_{12}f_1{f_2}^{-1}\Bigg\} u_t \\ &+\left[\frac{3A_1{f_{12}}^2 + 5Af_{112}f_{12}}{f_2} - A_{11}f_{12} - Af_{1112} - 2f_{112}A_1 - \frac{4A{f_{12}}^3}{{f_2}^2}\right] {u_x}^3 \\ &+\left[-\frac{(B_1f_{12} + f_2B_{11})\,A_2}{A} + f_{22}B_{11} - 4f_{11}A_1 + B_{112}f_2 - f_{12}B_{12}\right. \\ &+\frac{B_2{f_{12}}^2 - (5f_{22}B_1 - 6Af_{11})\,f_{12}}{f_2} - \left(A_{11} - \frac{4A_1f_{12} + 3Af_{112}}{f_2} + \frac{6{f_{12}}^2A}{{f_2}^2}\right)f_1 \\ &\left.+3f_{122}B_1 - 3Af_{111} - f_{112}B_2\right] {u_x}^2 + \left[-\frac{(f_1B_1 + f_{11}B)\,A_2}{A} + 3Bf_{112}\right. \\ &+\left(2B_{133}B + 2B_{11} + 2B_3B_{13} + 2B_{33}B_1\right)f_2 + \left(\frac{A_1}{f_2} - \frac{2f_{12}A}{{f_2}^2}\right){f_1}^2 \\ &\times\frac{(B_2f_{12} + 2Af_{11} - 5f_{22}B_1)\,f_1}{f_2} - \frac{3B{f_{12}}^2 + 2f_{11}Bf_{22}}{f_2} \\ &\left.-f_{11}B_2 - f_1B_{12} + 6B_1f_{12}\right] u_x + 3f_{11}B - \frac{3Bf_{12}f_1}{f_2} = 0, \end{aligned} \tag{A8}$$

$$\begin{aligned} h_9 = &\left[\left(-\frac{3f_{222}f_{11}}{f_2} - \frac{2\left(f_{112}f_2 - f_{11}f_{22} - {f_{12}}^2\right)A_2}{f_2A} + 3f_{1122} - \frac{A_1f_{122}}{A}\right.\right. \\ &\left.\left.-\frac{2(4f_{12}Af_{122} - f_{12}A_1f_{22} + 2f_{112}f_{22}A)}{f_2A} + \frac{4f_{22}\left(f_{11}f_{22} + 2{f_{12}}^2\right)}{{f_2}^2}\right) u_x\right. \end{aligned}$$

$$+6f_{112}+\left(B_{133}-\frac{A_1B_{33}}{A}\right)f_2+\left(B_{33}-\frac{2A_1}{A}\right)f_{12}+\left(\frac{2f_{12}f_1}{f_2A}-\frac{2f_{11}}{A}\right)A_2$$
$$-\frac{2\left(-f_2A_1f_{22}+3f_2Af_{122}-4f_{22}f_{12}A\right)f_1}{{f_2}^2A}-\frac{4({f_{12}}^2+f_{11}f_{22})}{f_2}\Bigg]u_t$$
$$+\left[-\frac{2f_2B_{13}A_2}{A}+2B_{123}f_2-2f_{12}B_{23}+2B_{13}f_{22}+2B_3f_{122}-\frac{4f_{22}B_3f_{12}}{f_2}\right]u_x$$
$$-2f_1B_{23}+\left(2B_{13}+2B_3B_{33}\right)f_2-\frac{4f_{22}B_3f_1}{f_2}+f_2B_{333}B+4B_3f_{12}=0, \tag{A9}$$

$$h_{10}=\frac{\left(-f_{222}f_2+2{f_{22}}^2\right)A_2}{f_2A}+\frac{4{f_{22}}^3}{{f_2}^2}+f_{2222}-\frac{5f_{222}f_{22}}{f_2}=0, \tag{A10}$$

$$h_{11}=\Bigg[-\frac{2\left(-2f_{22}f_{12}+f_{122}f_2\right)A_2}{f_2A}-\left(\frac{5f_{12}}{f_2}+\frac{A_1}{A}\right)f_{222}+3f_{1222}$$
$$-\frac{2f_{22}\left(5Af_{122}-A_1f_{22}\right)}{f_2A}+\frac{12f_{12}{f_{22}}^2}{{f_2}^2}\Bigg]u_t-\frac{2f_2B_{23}A_2}{A}-\frac{8{f_{22}}^2B_3^2}{f_2}f_{222}$$
$$+f_2B_{333}A+2B_{223}f_2=0, \tag{A11}$$

$$h_{12}=\Bigg[3f_{1122}-\frac{2f_{122}A_1}{A}-\frac{10f_{12}Af_{122}-4f_{12}A_1f_{22}+5f_{112}f_{22}A}{f_2A}$$
$$+\left(-\frac{f_{112}}{A}+\frac{2{f_{12}}^2}{f_2A}\right)A_2+\frac{12{f_{12}}^2f_{22}}{{f_2}^2}\Bigg]{u_t}^2+\Bigg\{2B_{123}f_2-\frac{3f_{22}B_3f_{12}}{f_2}$$
$$+\frac{B_3f_{12}A_2}{A}+B_3f_{122}+\left(-\frac{2A_1f_2}{A}-f_{12}\right)B_{23}\Bigg\}u_t$$
$$+\Bigg[\left(-\frac{A_{11}f_2}{A}-\frac{A_1f_{12}}{A}-3f_{112}+\frac{4{f_{12}}^2}{f_2}\right)A_2-\frac{12Af_{22}{f_{12}}^2}{{f_2}^2}+A_{112}f_2$$
$$-3A_{12}f_{12}-3Af_{1122}-f_{122}A_1+\frac{\left(A_1f_{22}+10Af_{122}\right)f_{12}+5f_{112}f_{22}A}{f_2}\Bigg]{u_x}^2$$
$$+\Bigg[\Bigg(\left(-\frac{A_1}{A}+\frac{5f_{12}}{f_2}\right)f_1-\frac{2f_2B_{12}}{A}-\frac{B_2f_{12}}{A}-\frac{f_{22}B_1}{A}-6f_{11}\Bigg)A_2+f_{22}B_{12}$$
$$+3B_1f_{222}-\left(A_1+B_{22}\right)f_{12}+\frac{8{f_{12}}^2A-3f_{22}B_2f_{12}+4Af_{11}f_{22}-5{f_{22}}^2B_1}{f_2}$$
$$+\left(-3A_{12}+\frac{-A_1f_{22}+6Af_{122}}{f_2}-\frac{12f_{22}f_{12}A}{{f_2}^2}\right)f_1$$
$$-6Af_{112}+\left(2B_{122}+2B_{33}A_1+2B_{133}A+2A_{11}\right)f_2+B_2f_{122}\Bigg]u_x$$
$$-\frac{\left(B_1{f_2}^2-{f_1}^2A+B_2f_1f_2+Bf_{12}f_2\right)A_2}{f_2A}-\frac{2{f_1}^2Af_{22}}{{f_2}^2}$$
$$+4B_2f_{12}+3f_2B_{12}+3Bf_{122}+\left(-A_1-B_{22}+\frac{2f_{12}A-4f_{22}B_2}{f_2}\right)f_1$$
$$+\left(2B_{33}B_2+2B_{233}B\right)f_2+2B_3f_2B_{23}+f_{22}B_1-\frac{5f_{22}Bf_{12}}{f_2}=0, \tag{A12}$$

$$h_{13}=\Bigg\{\Bigg[-3Af_{1112}-f_{112}A_1+\left(\frac{4f_{11}f_{12}}{f_2}-3f_{111}\right)A_2+\left(A_{11}-\frac{{A_1}^2}{A}\right)f_{12}$$
$$+\frac{A_1{f_{12}}^2+7Af_{112}f_{12}+2f_{22}Af_{111}+6Af_{11}f_{122}}{f_2}+\left(-\frac{A_{11}A_1}{A}+A_{111}\right)f_2$$
$$-4A_{12}f_{11}-\frac{4f_{12}A\left({f_{12}}^2+2f_{11}f_{22}\right)}{{f_2}^2}\Bigg]u_t+B_{13}Af_{12}+\left[2B_{13}A_1+B_{113}A\right]f_2\Bigg\}{u_x}^2$$

$$
\begin{aligned}
&+\left\{\left[\frac{f_{11}f_1A_2}{f_2}+\left(A_{11}-\frac{{A_1}^2}{A}-\frac{A_1f_{12}}{f_2}+\frac{-2Af_{11}f_{22}-2{f_{12}}^2A}{{f_2}^2}\right)f_1\right.\right.\\
&+\left(-2\frac{A_1f_2}{A}+f_{12}\right)B_{12}+2B_{112}f_2-\frac{A_1B_2f_{12}}{A}-2B_{22}f_{11}+f_{22}B_{11}\\
&+3f_{122}B_1+\left(-\frac{f_{22}B_1}{A}-f_{11}\right)A_1-f_{11}B_{33}A+f_{112}B_2\\
&\left.+\frac{-B_2{f_{12}}^2+\left(-5f_{22}B_1+4Af_{11}\right)f_{12}-2B_2f_{11}f_{22}}{f_2}\right]u_t\\
&\times\left(2B_{123}B+2B_{13}B_2\right)f_2+f_{22}B_{13}B+B_{13}Af_1+\left(f_{12}B+2f_2B_1\right)B_{23}\}\,u_x\\
&+\frac{\left(2{f_{12}}^2f_2A_1-5f_{12}f_2Af_{112}-f_{112}A_1{f_2}^2+4A{f_{12}}^3+f_{1112}A{f_2}^2\right){u_t}^3}{A{f_2}^2}\\
&+\left[-f_{112}B_3+\left(\frac{B_3A_1}{A}-B_{13}\right)f_{12}+B_3{f_{12}}^2{f_2}^{-1}\right]{u_t}^2\\
&+\left[-\frac{5B{f_{12}}^2}{f_2}+3Bf_{112}+\left(B_{11}-\frac{B_1A_1}{A}\right)f_2+\left(-\frac{2B_2f_{12}}{f_2}-\frac{A_1B_2}{A}\right)f_1\right.\\
&\left.+2f_{11}B_2+f_1B_{12}+\left(B_1-B_{33}B-\frac{BA_1}{A}\right)f_{12}\right]u_t+B\left(f_2B_{13}+f_1B_{23}\right)=0, \qquad \text{(A13)}
\end{aligned}
$$

$$
\begin{aligned}
h_{14}=&\left[\left(A_1+\frac{5f_{12}A}{f_2}\right)f_{222}-\frac{12{f_{22}}^2Af_{12}}{{f_2}^2}+\left(-\frac{2A_{12}f_2}{A}+\frac{3f_{22}f_{12}}{f_2}-3f_{122}-\frac{A_1f_{22}}{A}\right)A_2\right.\\
&\left.-\frac{f_{12}{A_2}^2}{A}-2f_{12}A_{22}-3Af_{1222}+\frac{2f_{22}\left(5Af_{122}-A_1f_{22}\right)}{f_2}+2A_{122}f_2-f_{22}A_{12}\right]u_x\\
&+f_2B_{222}-\frac{f_1{A_2}^2}{A}+\left[\left(-\frac{B_{22}}{A}-\frac{A_1}{A}+2B_{33}\right)f_2-3f_{12}-\frac{f_{22}B_2}{A}\right]A_2\\
&+\left(\frac{3f_1A}{f_2}+2B_2\right)f_{222}+\left(-2A_{22}-\frac{6A{f_{22}}^2}{{f_2}^2}\right)f_1+\frac{6f_{22}f_{12}A-4B_2{f_{22}}^2}{f_2}\\
&+\left(2B_{233}A+3A_{12}\right)f_2-3Af_{122}=0, \qquad \text{(A14)}
\end{aligned}
$$

$$
h_{15}=2f_2B_{223}=0, \qquad \text{(A15)}
$$

$$
\begin{aligned}
h_{16}=&\left\{\left[\left(-\frac{A_1f_{12}}{A}-3f_{112}+\frac{f_{11}f_{22}+2{f_{12}}^2}{f_2}\right)A_2-\frac{{A_1}^2f_{22}}{A}-A_{12}f_{12}-3Af_{1122}\right.\right.\\
&+\frac{3Af_{11}f_{222}}{f_2}+f_{122}A_1-3A_{22}f_{11}+\left(2A_{112}-\frac{2A_{12}A_1}{A}\right)f_2+f_{22}A_{11}\\
&\left.+\frac{\left(-2f_{22}A_1+8Af_{122}\right)f_{12}+4Af_{112}f_{22}}{f_2}-\frac{4f_{22}A\left(f_{11}f_{22}+2{f_{12}}^2\right)}{{f_2}^2}\right]u_t\\
&\left.+2f_2B_{13}A_2+2B_{123}Af_2+f_{22}B_{13}A+\left(2A_1f_2+f_{12}A\right)B_{23}\right\}u_x\\
&+\left[\left(A_{11}-\frac{B_{22}A_1}{A}+B_{122}-\frac{{A_1}^2}{A}\right)f_2+\frac{2f_{22}\left(-2f_{12}B_2+Af_{11}\right)}{f_2}\right.\\
&-\left(\frac{A_1}{A}+\frac{f_{12}}{f_2}\right)f_1A_2-\left(A_1+B_{33}A+B_{22}\right)f_{12}\\
&\left.+\frac{f_{22}\left(B_{12}A-B_2A_1\right)}{A}+2B_2f_{122}+\left(A_{12}-\frac{2f_{12}Af_{22}}{{f_2}^2}\right)f_1\right]u_t\\
&+\left(B_{13}A+B_{223}B\right)f_2+\left(f_1A+2B_2f_2+f_{22}B\right)B_{23}=0, \qquad \text{(A16)}
\end{aligned}
$$

$$
h_{17}=\left[-Af_{1111}-A_{11}f_{11}+\frac{\left(3A_1f_{11}+2Af_{111}\right)f_{12}+3Af_{112}f_{11}}{f_2}-2f_{111}A_1-\frac{4Af_{11}{f_{12}}^2}{{f_2}^2}\right]u_t{u_x}^3
$$

$$+\left\{\left[f_2B_{111}+3f_{112}B_1-B_2f_{111}+\left(-\frac{B_1A_1}{A}+2B_{11}\right)f_{12}-2B_{12}f_{11}-\frac{A_1B_{11}f_2}{A}\right.\right.$$
$$\left.+\left(\frac{A_1f_{11}}{f_2}-\frac{2Af_{11}f_{12}}{{f_2}^2}\right)f_1+\frac{-2{f_{12}}^2B_1+B_2f_{11}f_{12}-3f_{22}B_1f_{11}+2{f_{11}}^2A}{f_2}\right]u_t$$
$$\left.-\frac{A_1B_{11}f_2}{A}+B_{13}Bf_{12}+\left(B_{113}B+2B_{13}B_1\right)f_2\right\}{u_x}^2$$
$$+\left\{\left(-\frac{f_{111}A_1}{A}-\frac{-2f_{12}A_1f_{11}+2f_{12}Af_{111}+3Af_{112}f_{11}}{Af_2}+f_{1111}+\frac{4f_{11}{f_{12}}^2}{{f_2}^2}\right){u_t}^3\right.$$
$$+\left[\left(\frac{B_3A_1}{A}+\frac{B_3f_{12}}{f_2}-B_{13}\right)f_{11}-B_3f_{111}\right]{u_t}^2$$
$$+\left[-\frac{f_{11}B\left(A_1f_2+B_{33}Af_2+5f_{12}A\right)}{Af_2}+\left(-\frac{B_1A_1}{A}+B_{11}-\frac{2f_{12}B_1}{f_2}\right)f_1\right.$$
$$\left.\left.+3Bf_{111}+3B_1f_{11}\right]u_t+B_{13}Bf_1\right\}u_x=0, \tag{A17}$$

$$h_{18}=\left[\left(-\frac{3A_1f_2}{A}-3f_{12}\right)A_2+3A_{12}f_2\right]u_t+3f_2B_{23}A=0. \tag{A18}$$

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