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Stability of dark soliton solutions of the quintic complex Ginzburg–Landau equation in the case of normal dispersion*

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Dark soliton solutions of the one-dimensional complex Ginzburg–Landau equation (CGLE) are analysed for the case of normal group-velocity dispersion. The CGLE can be transformed to the nonlinear Schrödinger equation (NLSE) with perturbation terms under some practical conditions. The main properties of dark solitons are analysed by applying the direct perturbation theory of the NLSE. The results obtained may be helpful for the research on the optical soliton transmission system.

Keywords: refractive index, spectral filtering
PACC: 4265S, 4281D, 0290

1. Introduction

Many nonequilibrium phenomena, such as the generation of spatiotemporal dissipative structures in lasers,[1−3] binary fluid convection,[4,5] and phase transitions may be described by the complex Ginzburg–Landau equation (CGLE). This equation also applies when describing soliton propagation in the optical fibre systems with linear and nonlinear gains and spectral filtering.[6,7] The quintic CGLE can be written in the form[8,9]

$$\mathrm{i}\psi_t + \frac{D}{2}\psi_{xx} + |\psi|^2\psi = \mathrm{i}\delta\psi + \mathrm{i}\alpha|\psi|^2\psi + \mathrm{i}\beta\psi_{xx} + \mathrm{i}\varpi|\psi|^4\psi - \vartheta|\psi|^4\psi, \quad (1)$$

where t is the propagation distance, x is the retarded time, ψ is the normalized envelope of the electric field, $D = \pm 1$ is the dispersion coefficient. α and δ are respectively the linear gain and loss coefficients, β stands for spectral filtering ($\beta > 0$), ϖ represents a higher-order correction to the nonlinear amplification or absorption and ϑ is a possible higher-order correction term to the intensity-dependent refractive index. As is well known, if the right-hand side (RHS) terms of Eq.(1) are set to be zero, which can be regarded as the nonlinear Schrödinger equation (NLSE), then in the anomalous dispersion regime ($D = +1$), the bright soliton solutions of the NLSE exist. In this case, for relatively small values of the RHS of Eq.(1), we can treat it as perturbation. Based on the direct approach of perturbation theory for NLSE,[10,11] in many practical cases, such as the optical fibre and Bose–Einstein condensates,[12,13] the effects of perturbation on the bright soliton have been studied in detail. The stability of solitary wave solution has been analysed in previous papers.[14−18]

In this paper, the CGLE with normal dispersion ($D = -1$) and the coefficients $|\delta|$, β, $|\alpha|$, $|\varpi|$ and $|\vartheta|$ being much less than 1 has been studied. It is found that the dark soliton may exist. As a consequence, it is possible to study the fundamental properties of soliton solutions of Eq.(1) by applying the perturbation method[19] to the dark soliton developed recently. In this paper, we will concentrate on the issue of perturbation and analyse the influences on the dark soliton rooting in the action of linear or nonlinear terms on the RHS of Eq.(1). The evolvements of the characteristic parameters including the soliton height, width, phase and velocity versus the propagation distance and the retarded time have been carried out first when the higher-order correction (ϖ) is left out of account. The numerical results of the soliton height and phase are consistent with our analysis, and the effects on the

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characteristic parameters of the dark soliton under all perturbation interactions have been obtained also by numerical simulation, showing the complicated characteristics of the stability of the dark soliton. The loss of the optical pulse makes the soliton height attenuated, its width widened and velocity declined, however the gain gives an opposite effect. The balance between the loss and the gain keeps the soliton stable. Finally, we give the full soliton solution including the first-order correction.

2. Calculations and analytical procedure

For simplicity, we rewrite Eq.(1) in the following form in the normal dispersion ($D=-1$):

$$\begin{aligned} &\mathrm{i}2\psi_t - \psi_{xx} + 2|\psi|^2\psi = \mathrm{i}\epsilon(d_1\psi_{xx} + d_2\psi \\ &\quad + d_3|\psi|^2\psi + d_4|\psi|^4\psi) - \epsilon d_5|\psi|^4\psi, \end{aligned} \tag{2}$$

where $\epsilon d_1 = 2\beta$, $\epsilon d_2 = 2\delta$, $\epsilon d_3 = 2\alpha$, $\epsilon d_4 = 2\varpi$, $\epsilon d_5 = 2\vartheta$ and ϵ is small positive constant ($0 < \epsilon << 1$). We introduce a dimensionless variable $t \to \frac{1}{2}t$ and reorganize the field $\psi = \phi \mathrm{e}^{2\mathrm{i}\rho^2 t}$, so that the background wave may keep a stable amplitude in the absence of perturbation. It is very easy to verify that the function $\phi(x,t)$ satisfies the following equation:

$$\begin{aligned} &\mathrm{i}\phi_t - \phi_{xx} + 2(|\phi|^2 - \rho^2)\phi = \mathrm{i}\epsilon(d_1\phi_{xx} + d_2\phi \\ &\quad + d_3|\phi|^2\phi + d_4|\phi|^4\phi) - \epsilon d_5|\phi|^4\phi. \end{aligned} \tag{3}$$

Introducing a slow space variable $\tau = \epsilon t$, we replace the derivative $\partial t \to \partial t + \epsilon\partial\tau$. At the same time, we rewrite $\phi(x,t) = \varphi(x,t) + \epsilon\omega(x,t)$. Introducing a coordinate system moving with the soliton, we have

$$\begin{aligned} &t, \quad x \longrightarrow t, \qquad z = x - x_0 - 2\xi t; \\ &\partial t, \qquad \partial_x \longrightarrow \partial t - 2\xi\partial z, \quad \partial_z. \end{aligned} \tag{4}$$

When the RHS of Eq.(3) equals zero, the following single soliton solution is evident:

$$\begin{aligned} &\varphi = \mathrm{e}^{-\mathrm{i}\theta}(\xi + \mathrm{i}\eta\tanh z)\,, \qquad z = \eta(x - \varsigma)\,, \\ &\varsigma_t = 2\eta, \qquad \rho\mathrm{e}^{\mathrm{i}\theta} = \xi + \mathrm{i}\eta, \end{aligned} \tag{5}$$

where ξ, η, θ and ς are four real parameters which determine the dark soliton height (as well as width), propagation velocity, initial phase of the soliton and initial position of the soliton centre, respectively. Due to the perturbation, the soliton parameters ξ, η, θ and ς are supposed to be functions of the slow space variable τ. From Eqs.(91)–(93) in Ref.[20], we can obtain the soliton parameter equations:

$$\begin{aligned} &\xi_\tau = d_2\xi + d_3\xi^3 + d_4\xi^5 + \frac{1}{3}d_3\xi\eta^2 \\ &\quad + \frac{2}{3}d_4\xi^3\eta^2 + \frac{1}{5}d_4\xi\eta^4, \\ &\xi\theta_\tau + 2\eta^2\varsigma_\tau = 2d_5(\xi^3\eta^2 + \xi\eta^4), \\ &\eta_\tau = \frac{5}{3}d_3\xi^2\eta + \frac{14}{5}d_4\xi^2\eta^3 + \frac{7}{3}d_4\xi^4\eta \\ &\quad + d_2\eta + d_3\eta^3 + d_4\eta^5, \\ &H(t) = (\xi^2 + \eta^2)^{1/2} - \xi, \qquad W(t) = \frac{1}{\eta}, \\ &\theta = \arctan\sigma, \qquad \sigma = \frac{\eta}{\xi}. \end{aligned} \tag{6}$$

The soliton height, width and phase are reflected by $H(t)$, $W(t)$, $\theta(t)$, respectively. From Eq.(6), it is found that the terms including the coefficients d_2, d_3, d_4 can have effects on the soliton height and phase, but including the d_5 term is a means of modulating the effect on the soliton velocity. The terms of linear (d_2) and nonlinear gain or loss (d_3) and higher-order correction to the intensity-dependent refractive index (d_5) may have effects on the soliton stability. In order to analyse clearly how the perturbation terms work on the soliton, we will discuss Eq.(6) in the following two different cases. The effects of two different conditions, excluding the fourth term and including the one of RHS of Eq.(3) are analysed by calculation and numerical simulation.

Remark 1. Suppose d_1, d_2, d_3, $d_5 \neq 0$, $d_4 = 0$, i.e, the effects of the d_4 term of Eq.(3) on a dark soliton are not taken into account. From Eq.(6), we can obtain

$$\xi(t) = \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{c_0 c_1 d_2 \gamma^{1/3} - \gamma}\Big)^{1/2},$$

$$\eta(t) = \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{\gamma} - \frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{c_0 c_1 d_2 \gamma^{1/3} - \gamma}\Big)^{1/2}, \tag{7}$$

$$\rho(t) = \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{\gamma}\Big)^{1/2}, \tag{8}$$

$$H(t) = \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{\gamma}\Big)^{1/2} - \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{c_0 c_1 d_2 \gamma^{1/3} - \gamma}\Big)^{1/2},$$

$$W(t) = \Big[\frac{-\gamma^2 + c_0 c_1 d_2 \gamma^{4/3}}{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}(-2\gamma + c_0 c_1 d_2 \gamma^{\frac{1}{3}})}\Big]^{1/2}, \tag{9}$$

$$\theta(t) = \arctan(c_0 c_1 d_2 \gamma^{-2/3} - 2), \tag{10}$$

$$\varsigma_\tau = v(t) = \Big(\frac{c_0 d_2 \mathrm{e}^{2d_2\epsilon t}}{-\gamma + c_0 c_1 d_2 \gamma^{1/3}}\Big)^{1/2}\Big[\frac{c_0 d_2 d_5 \mathrm{e}^{2d_2\epsilon t}}{\gamma}$$

$$+\frac{d_3}{3}\Big(\frac{1}{-2+c_0c_1d_2\gamma^{-2/3}}\Big)^{1/2}\Big], \tag{11}$$

where $\gamma = 1 - d_3c_0e^{2d_2\epsilon t}$, the constants c_0 and c_1 depend on the initial choices of $\rho(t)$ and $\xi(t)$, respectively. From Eq.(9), (i) if $d_2 < 0$, $d_3 > 0$, $|d_2| < |d_3|$, we can derive $\partial_t H(t) > 0$ and $\partial_t W(t) < 0$; the dark soliton height increases and the width decreases when the nonlinear gain dominates the propagation of the soliton; the opposite state will appear: that is, the soliton height will decrease and the width will increase if the effects of linear or nonlinear loss terms exceeds the gain; the soliton velocity will also degenerate; (ii) if $d_2 < 0$, $d_3 > 0$, $|d_2| \sim |d_3|$, the nonlinear gain offsets the linear loss, and the soliton will keep stable, which has been analysed in Ref.[10]. Figure 1 shows the soliton height (a) and the width (b) versus t, showing the rough coincidence between the numerical results and those by calculation. Figure 1(a) and 1(b) show that the soliton height increases and the width decreases when the nonlinear gain exceeds the linear loss.

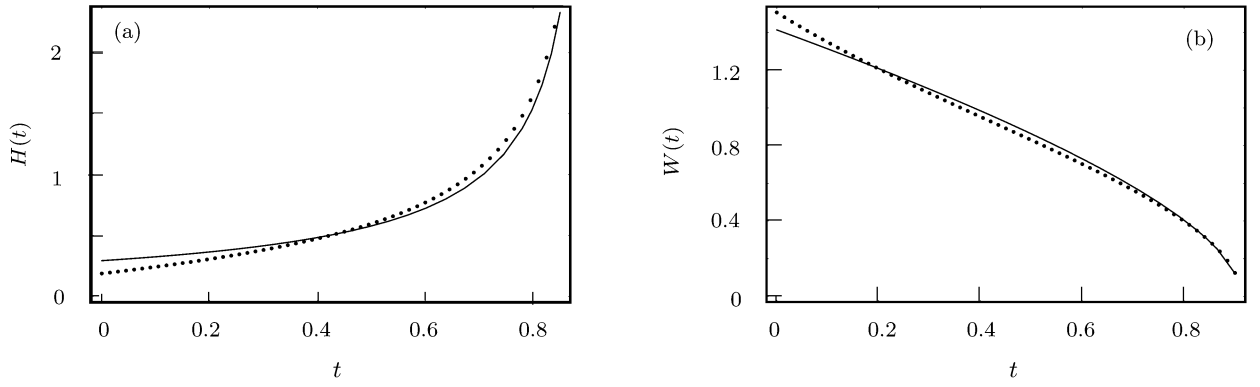


Fig.1. Numerical and analytical results given by Eqs.(6) (solid lines) and (8) (dotted lines) for the soliton height and width versus space variable t. The parameters are selected to be $\xi(0) = \eta(0) = 0.707$, $d_1 = 0.1$, $d_2 = -0.1$, $d_3 = 0.6$, $d_5 = 0.1$.

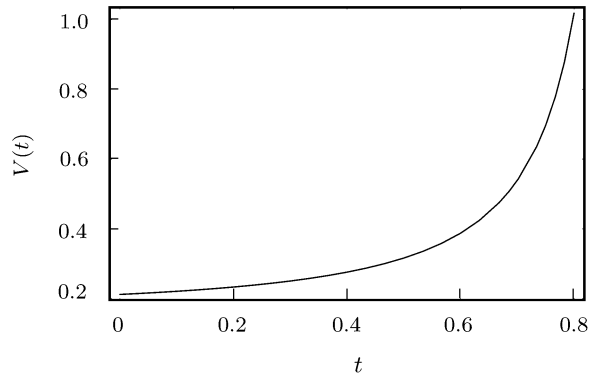


Fig.2. Variation of soliton velocity $V(t)$ with space variable t. The parameters is the same as those in Fig.1.

Figure 2 shows that the soliton velocity is rising with t under the same conditions as in Fig.1, which is in agreement with our previous analysis. The coexistence of the terms d_2, d_3 and d_5 of the RHS of Eq.(3) cause the complication of the stability of solitons.

Remark 2. For $d_1, d_2, d_3, d_4, d_5 \neq 0$, we consider the effects of all the perturbation terms of RHS in Eq.(3) on the dark soliton. In an optical dark soliton transmission system, the gain is used to compensate the loss that the soliton suffers during propagation. However, excess gain will lead to instability of the background. So it is reasonable to insert the d_4 term when its period is smaller than the dispersion distance. In this case, we give different numerical results with three different d_3 and compare them with previous results in Ref.[21].

Employing the last four equations in Eq.(6), we simulate the t-dependence of the soliton height, width and phase in Figs.3(a), 3(b) and 3(c), respectively. From Figs.3(a), 3(b), and 3(c) we can see that the soliton height and phase increase first and then decrease, while the soliton width decreases first and then increases. With the increase of d_3, the curves of the soliton height and phase become more aculeated and their peak values increase, while the curve of the soliton width becomes shorter. In practical cases, the nonlinear gain (loss) is a correction to the linear gain (loss) and its effects on the dark soliton are small.

So the short-haul increase of the soliton height and phase and the decrease of the soliton width are replaced by the long-distance decline and increase, respectively. Our numerical simulation on the soliton height is in qualitative agreement with the result in Ref.[21]. Figure 4 shows the evolution of t-dependence of the soliton velocity $V(t)$, which increases first and decreases finally.

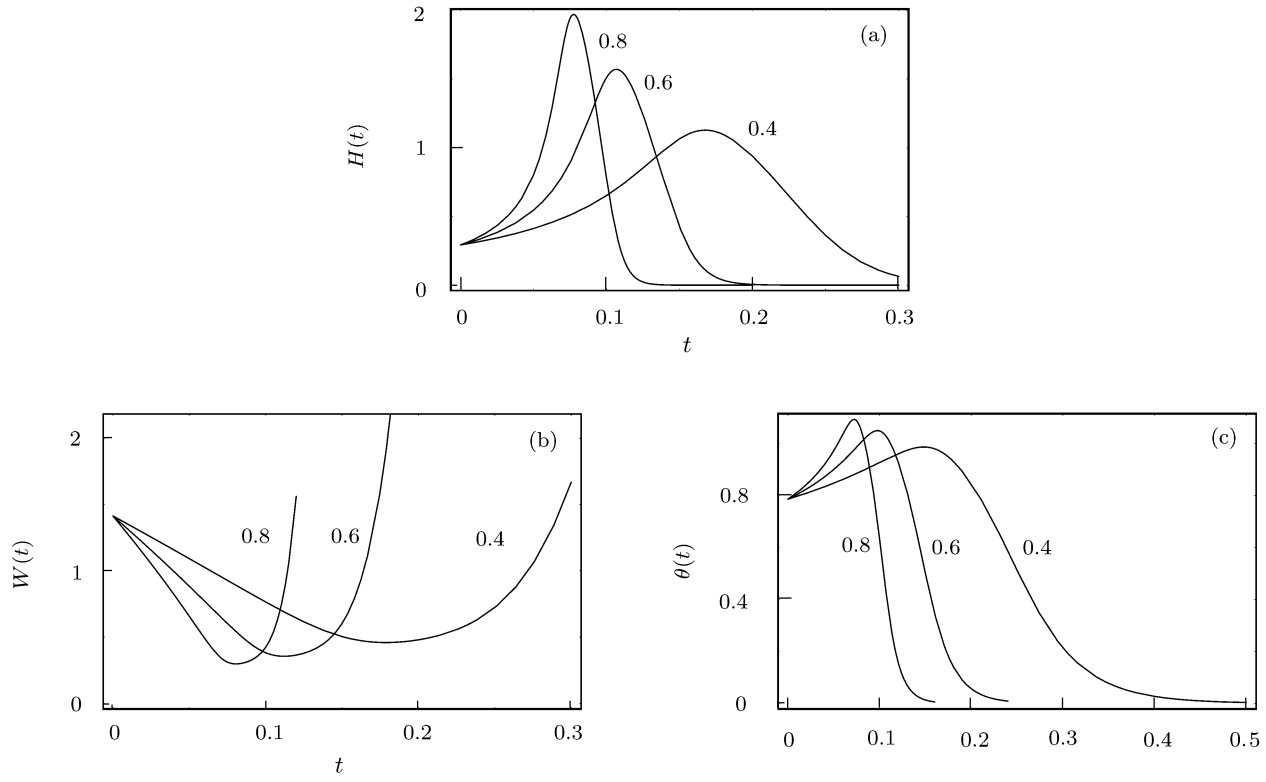


Fig.3. The t-dependence of dark soliton parameters $H(t)$, $W(t)$ and $\theta(t)$ with $\xi(0) = \eta(0) = 0.707$, $d_2 = -0.1$, $d_4 = -0.5$, $d_5 = 0.1$ and three different $d_3 = 0.4$, 0.6 and 0.8.

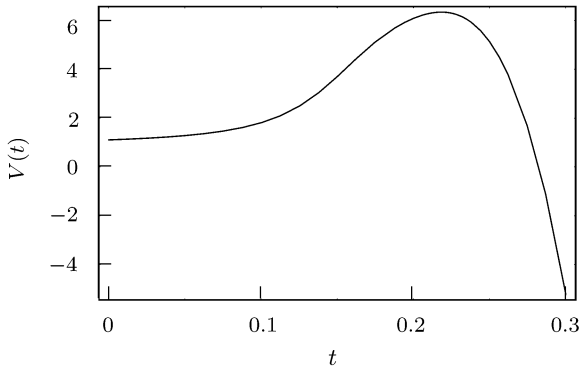


Fig.4. The variation of soliton velocity $V(t)$ with t, $d_3 = 0.4$ and the other parameters are the same as those in Fig.3.

On the basis of Ref.[20], we can obtain the first-order correction for $\phi(x,t)$ of Eq.(3),

$$\omega(z,t) = \frac{\mathrm{i}}{\eta^2} P \int_{-\infty}^{\infty} \mathrm{d}k \Big[\varrho^{(+)}(k) \frac{1}{\lambda^{(+)}} (\mathrm{e}^{-\mathrm{i}\eta^2\lambda^{(+)}t} - 1)\chi^{(+)}(z,k) + \varrho^{(-)}(k) \frac{1}{\lambda^{(-)}} (\mathrm{e}^{-\mathrm{i}\eta^2\lambda^{(-)}t} - 1)\chi^{(-)}(z,k) \Big], \quad (12)$$

where $\chi^{(\pm)}(z,k)$, $\mu^{(\pm)}$ and $\varrho^{(\pm)}(k)$ are defined as follows:

$$\chi^{(\pm)}(z,k)=\frac{\mathrm{e}^{\mathrm{i}kz}}{\sqrt{2\pi k\nu}\mu^{(\pm)}}\begin{pmatrix}-\Big[\dfrac{\mathrm{i}}{2}(\mu^{(\pm)}-k)+\tanh z\Big]^2\mathrm{e}^{-\mathrm{i}\theta}\\ \Big[\dfrac{\mathrm{i}}{2}(\mu^{(\pm)}+k)-\tanh z\Big]^2\mathrm{e}^{\mathrm{i}\theta}\end{pmatrix}, \tag{13}$$

$$\mu^{(\pm)}=-2\sigma+\nu,\qquad \nu=\sqrt{k^2+4}\qquad \lambda^{(\pm)}=k\mu^{(\pm)}, \tag{14}$$

$$\begin{aligned}\varrho^{(\pm)}(k)=&\pm\Big(\frac{\pi}{2k\nu}\Big)^{1/2}\Big\{\Big[\frac{1}{2}\mu^{(\pm)}k-\frac{1}{\mu^{(\pm)}}\Big(\frac{k^3}{6}-\frac{2k}{3}\Big)\Big](d_3\eta^2\xi+2d_4\xi^3\eta^2+2d_4\xi\eta^4)\\&+2\mathrm{i}\sigma(d_5\xi^4\eta+2d_5\xi^2\eta^3+d_5\eta^5)+\frac{\mathrm{i}}{4}\mu^{(\pm)}k^2(2d_5\xi^2\eta^3+2d_5\eta^5)+\frac{1}{\mu^{(\pm)}}k(k^2+4)\\&\times\Big(\frac{(\mu^{(\pm)})^2}{12}-\frac{k^2}{60}+\frac{1}{15}\Big)d_4\xi^2\eta^4-\frac{\mathrm{i}}{35}k^2(k^2+4)(\mu^{(\pm)}+\sigma)d_5\eta^5-2\mathrm{i}(d_5\xi^5+2d_5\xi^3\eta^2+d_5\xi\eta^4)\\&-k(d_2\eta+d_3\xi^2\eta+d_4\xi^4\eta+d_3\eta^3+2d_4\xi^2\eta^3+d_4\eta^5)-\frac{1}{6}k(k^2+4)(2d_1\eta^3+d_3\eta^3\\&+2d_4\xi^2\eta^3+2d_4\eta^5)+\frac{\mathrm{i}}{12}k^2(k^2+4)d_5\xi\eta^4+\frac{1}{12}k(k^2+4)(0.3k^2+0.8)d_4\eta^5\Big\}.\end{aligned} \tag{15}$$

Owing to the analytically unsolvable integration implied in Eq.(12), we can give a numerical analysis about the first-order solution $\omega(x,t)$ of Eq.(12). We select the small positive constant $\epsilon=0.01$, $\xi=\eta=0.707$, $d_1=1$, $d_2=-1$, $d_3=1$, $d_4=-1$ and $d_5=1$, which make the coefficients β, δ, α, ϖ, ϑ of the RHS in Eq.(1) satisfied the premise of the direct perturbation approach. Substituting the above numerical values into Eqs.(12)–(15), we can display $|\omega|^2$ and $|\phi|^2$ versus z as shown in Figs.5(a) and 5(b).

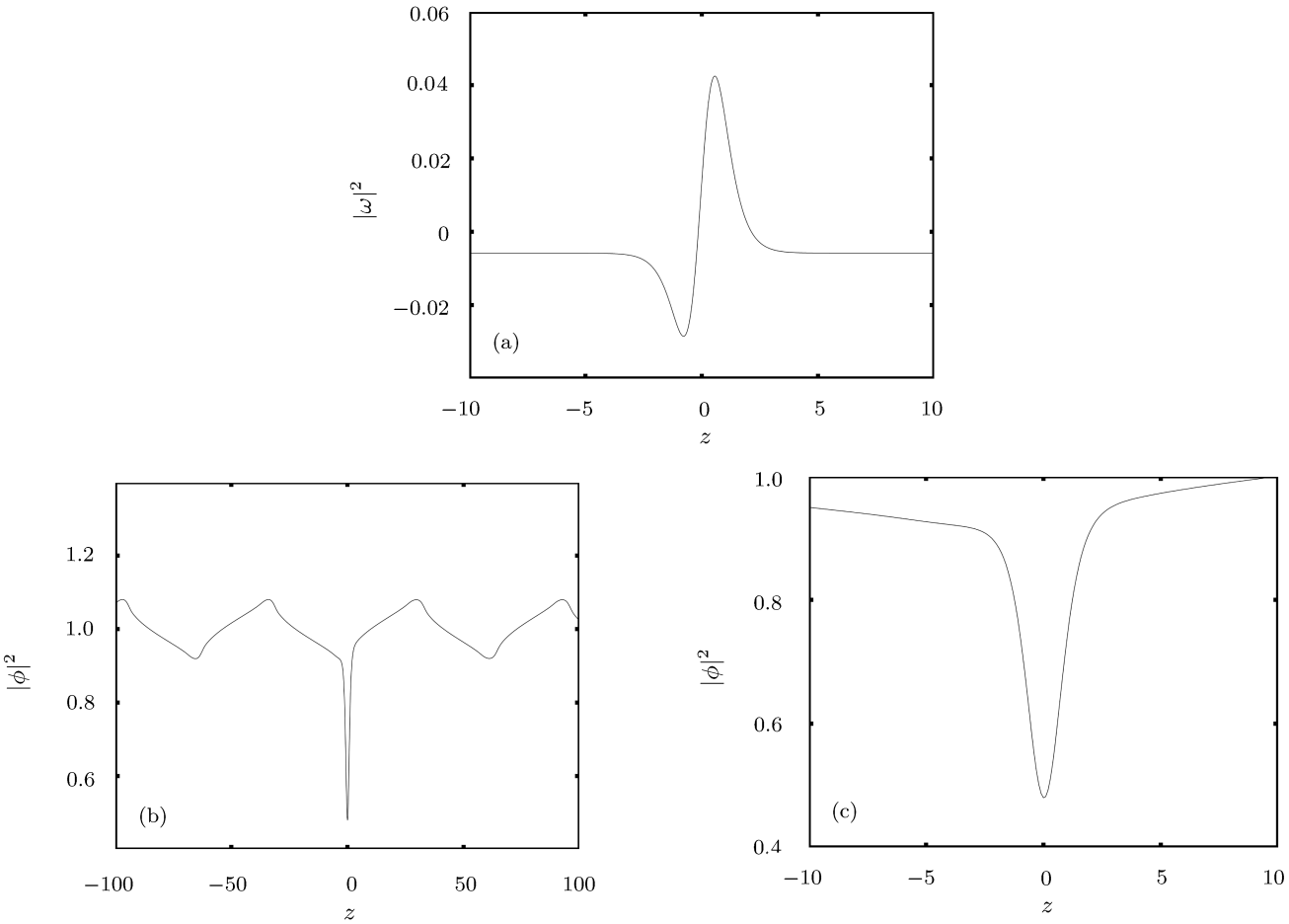


Fig.5. The first-order correction solution $|\omega|^2$ and the full solution $|\phi|^2$ versus z, with $\epsilon=0.01$, $\xi=\eta=0.707$, $d_1=1$, $d_2=-1$, $d_3=1$, $d_4=-1$, $d_5=1$.

From Fig.5(a), we can see clearly the curve of the first-order-correction. This result corresponds to the course of collapse and revival of the envelope of electric field of the optical pulse. Although the first-order-correction is small, it still has an obvious correction to the full wave shape as shown in Fig.5(b). The small vibrations of the full wave shape corresponds to the operations of losses and gains: the gains make the vibration upward and the losses make it downward. The operations of them lead to the small rise or fall of the verge of the dark soliton. Figure 5(c) shows the full wave $|\phi|^2$ versus variable z in the vicinity of the central location. The hollow in the central location corresponds to the centre of dark soliton.

3. Discussion

In the above analysis, the properties of the dark soliton, including the height, the width and the phase have been studied in detail by applying the direct approach. We have regarded the spectral filtering, linear and nonlinear gain (loss) and the higher-order correction as perturbations. The expressions of the soliton parameters have been given and analysed. The gain makes the soliton height $H(t)$ and velocity $v(t)$ increase, the width $W(t)$ decrease. The opposite effects are caused by the loss. The balance between the gain and the loss makes the soliton stable.

The cubic CGLE is a good model for describing optical transmission systems with guiding filters.[8,9] The use of nonlinear gain ($d_3 > 0$) in these systems allows the reduction or suppression of the growth of linear radiation.[21,22] In our theory, through controlling the coefficients of the RHS of Eq.(3), we can obtain the stable soliton, which is perhaps helpful for the experiment in which small perturbation effects exist.

We also have obtained the full solution including the first-order correction of the quintic CGLE in the normal dispersion case. The expressions of the soliton parameters can be written, which include the coefficients of the CGLE. Although the first-order correction is small, it still has an obvious correction to the full wave shape, and causes the process of collapse and revival of the envelope of optical pulse in propagation.

4. Conclusion

In conclusion, there always exist various factors in experiments which affect the soliton of the CGLE in the normal dispersion ($D = -1$). We have learned that the effects of the five kinds of perturbations on the dark soliton are remarkable. The CGLE can be transformed to an ordinary NLS with perturbation terms and the effects of each term on the dark soliton can be given in concrete expressions and numerical analysis. Our result is also helpful for the understanding of the optical soliton of the CGLE.

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