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Hybrid TS fuzzy modelling and simulation for chaotic Lorenz system*

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The projection of the chaotic attractor observed from the Lorenz system in the X–Z plane is like a butterfly, hence the classical Lorenz system is widely known as the butterfly attractor, and has served as a prototype model for studying chaotic behaviour since it was coined. In this work we take one step further to investigate some fundamental dynamic behaviours of a novel hybrid Takagi–Sugeno (TS) fuzzy Lorenz-type system, which is essentially derived from the delta-operator-based TS fuzzy modelling for complex nonlinear systems, and contains the original Lorenz system of continuous-time TS fuzzy form as a special case. By simply and appropriately tuning the additional parametric perturbations in the two-rule hybrid TS fuzzy Lorenz-type system, complex (two-wing) butterfly attractors observed from this system in the three dimensional (3D) X–Y–Z space are created, which have not yet been reported in the literature, and the forming mechanism of the compound structures have been numerically investigated.

Keywords: Takagi–Sugeno (TS) fuzzy model, chaos, Lorenz system, butterfly attractor

PACC: 0545

1. Introduction

In 1963, Lorenz found the first chaotic attractor when he studied the atmospheric convection,[1] which is a third-order autonomous system with only two multiplication terms but displays very complex dynamical behaviours. The projection of the chaotic attractor observed from the Lorenz system in the $X-Z$ plane is widely known as the butterfly attractor and has served as a paradigm for chaotic research. The dynamics of the Lorenz attractor have been studied in detail by several researchers (see for example Refs.[2,3]) and have been recently revisited in Refs.[4] and [5], where new sets of equations (Chen system and Lü system), not topologically equivalent to the original system, have been proposed, respectively. Very recently, the aforementioned three systems were unified into one chaotic system and can realize the continued transition from one to another.[6] In particular, it was found that the above three attractors have a compound structure by merging together two simple attractors after performing a mirror operation.[6,7] It is notable that the approach to investigate the forming mechanism of the three attractors is essentially derived from the novel idea, first introduced by Elwakil, Kennedy *et al*[8,9] and Ozoguz *et al*.[10] In their works,[8−10] in order to utilize an electronic circuit to realize the well-known butterfly chaotic attractor, a modified Lorenz system is produced, in which one multiplier xy is replaced by an absolute-valued function $f(x) = |x|$ while the other multiplier xz is replaced with a bipolar switching constant. The resulting system is also not topologically equivalent to the Lorenz system, but of similar qualitative dynamics. In their design, an additional control parameter is intentionally used to verify the compound nature of the resulting butterfly-shaped attractor. That is, with tuning the control gain, it is possible to confine the chaotic dynamics to one or another of the butterfly wings of the attractor, forming two simple attractors, which, when merged together, form the whole butterfly. Then the rather detailed analysis of this modified Lorenz system was given in Ref.[11].

Very recently, for the same aim to explore the forming mechanism of its compound structure, we proposed[12] a different and novel hybrid TS fuzzy model for the newly coined chaotic Lü system, which essentially derived from the delta-operator (δ)-based TS fuzzy modelling for complex nonlinear systems.[13,14] The resulting system is topologically equivalent to the Lü system and contains the origi-

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nal Lü system of continuous-time TS fuzzy form as a special case. Different from the investigations mainly carried out for the differential-equation-formulated chaotic systems, the two-rule-based continuous-time TS fuzzy model for the chaotic Lü system is exactly derived, then the exact continuous-time TS fuzzy model is intentionally redescribed in the hybrid or the continuous-time TS-fuzzy-model-based form with uncertainties, and the uncertainty in each local linear state-space model is actually obtained by truncating the third and higher order terms for the δ-operator-based[15] representation of each linear state-space matrix. If we consider the two-uncertainty term in each local linear state-space model as two additional control parameters like in Refs.[5,10,11,16,17], then we get one more additional control parameter. By simply and appropriately tuning the two additional control parameters in the hybrid TS fuzzy chaotic Lü system, not only the compound topological structure obtained by the mirror-operation forming mechanism can be exactly observed, the findings of other abundant and complex dynamical behaviours of Lü attractor, obtained by a new forming mechanism, named the compression and pull forming mechanism, are also reported in Ref.[12].

In this work, the hybrid TS fuzzy modelling approach is utilized to the classical Lorenz attractor, furthermore, the rich dynamics of the chaotic Lorenz system beyond Refs.[10, 11] is investigated based on this hybrid TS fuzzy model, especially some complex (two-wing) butterfly attractors observed from this system in the three dimensional (3D) X-Y-Z space are created, which can realize the continued transition to the original Lorenz attractor and have not yet been reported before. We validate our proposals via numerical simulations and the forming procedure of these new butterfly attractors are explored in some detail.

2. Hybrid TS fuzzy modelling for Lorenz attractor

The classical Lorenz system of differential equations is described by[1]

$$\begin{aligned} \dot{x}(t) &= a(y(t) - x(t)), \\ \dot{y}(t) &= bx(t) - x(t)z(t) - y(t), \\ \dot{z}(t) &= x(t)y(t) - cz(t), \end{aligned} \tag{1}$$

where $a = 10$, $b = 28$ and $c = 8/3$, and the two multiplier-type nonlinearities (xy and xz) responsible for the generation of chaos.

Interactions between fuzzy logic and chaos theory have been explored since the 1990s. The representation of chaotic systems using TS fuzzy models has a unified approach.[18] Due to the fact that the trajectories either evolve in a strange attractor or on a periodic orbit/equilibrium point, the universal discourse of fuzzy membership functions are in apparent and bounded regions. This leads the TS fuzzy models to exact representation for chaotic systems. By using the fuzzy modelling method suggested in Ref.[19], we can construct an exact TS fuzzy model of the chaotic Lorenz system as follows:

$$\begin{aligned} &\text{Rule } 1: \text{IF } x(t) \text{ is about } F_{\min} \text{ THEN } \dot{X}(t) = A_1 X(t), \\ &\text{Rule } 2: \text{IF } x(t) \text{ is about } F_{\max} \text{ THEN } \dot{X}(t) = A_2 X(t), \end{aligned} \tag{2}$$

where

$$A_1 = \begin{bmatrix} -a & a & 0 \\ b & -1 & -M_1 \\ 0 & M_1 & -c \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & a & 0 \\ b & -1 & -M_2 \\ 0 & M_2 & -c \end{bmatrix},$$

and $X(t) = (x(t), y(t), z(t))^{\mathrm{T}}$ with membership functions $F_{\min}$ and $F_{\max}$ given as:

$$F_{\min} = \frac{-x(t) + M_2}{M_2 - M_1}, \quad F_{\max} = \frac{x(t) - M_1}{M_2 - M_1},$$

where $[M_1, M_2]$ is chosen as [–20,30], because $x(t)$ is seemingly bounded within [–20,30] as shown in Fig.1(a).

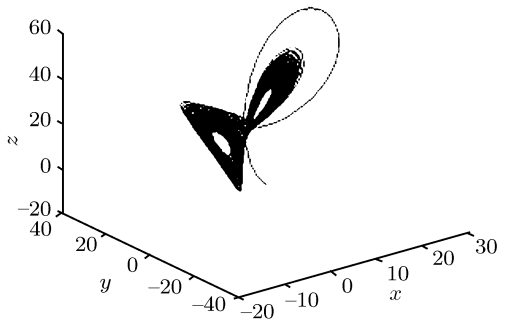


Fig.1. (a) $X-Y-Z$ 3D view of the TS fuzzy Lorenz attractor.

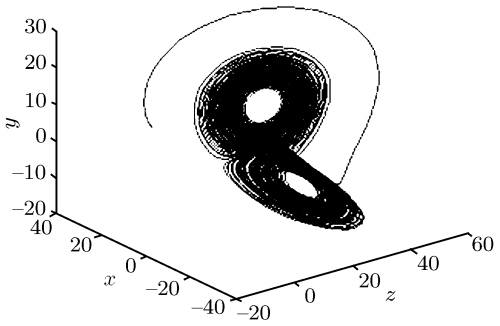


Fig.1. (b) $Z-X-Y$ 3D view of the TS fuzzy Lorenz attractor.

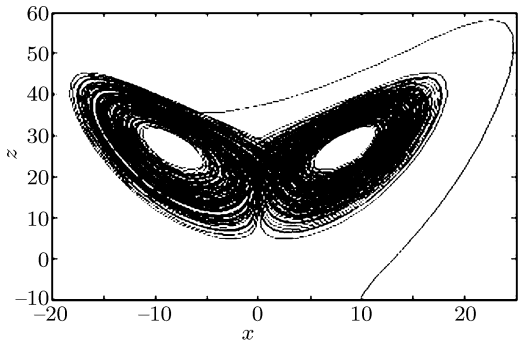


Fig.1. (c) $X-Z$ plane view of the TS fuzzy Lorenz butterfly attractor.

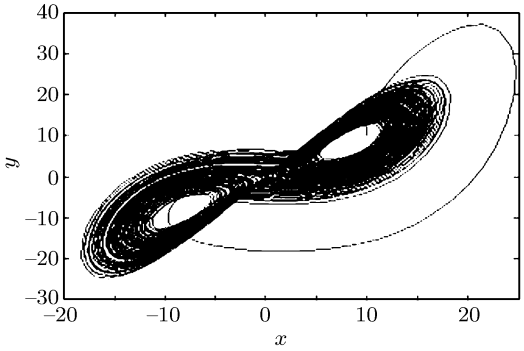


Fig.1. (d) $X-Y$ plane view of the TS fuzzy Lorenz attractor.

In order to study the forming mechanism of the classical chaotic attractor and clarify its topological structure, different from Refs.[10,11] by varying the parameter m in a multiplier-free modify Lorenz system, we consider the following TS fuzzy model with uncertainties like[12]

$$\begin{aligned}&\text{Rule}\ \ 1:\text{IF}\ \ x(t)\ \ \text{is about}\ \ F_{\min}\ \ \text{THEN}\ \ \dot{X}(t)=\left(A_1+\frac{1}{2}T_1A_1^2\right)X(t),\\&\text{Rule}\ \ 2:\text{IF}\ \ x(t)\ \ \text{is about}\ \ F_{\max}\ \ \text{THEN}\ \ \dot{X}(t)=\left(A_2+\frac{1}{2}T_2A_2^2\right)X(t),\end{aligned}\tag{3}$$

where T_1 and T_2 are two constants.

Note the uncertainties $\frac{1}{2}T_iA_i^2,\ i=1,2$ in (3) differ from the time-varying but norm-bounded parametric uncertainties in Ref.[20], where the TS-fuzzy model - based chaotic Lorenz system was discussed, and actually the system parameters are randomly varied within 30% of their nominal values. But in our approach, the sums $\frac{1}{2}TA_i^2,\ i=1,2$ are actually approximate terms by truncating the third and higher order terms for the δ-based representation for the matrices A_i of each local continuous-time linear state space system.[13,14] If we let $T_1=T_2=0$, we get the exact TS fuzzy model of the chaotic Lorenz system as shown by Eq.(2), and T is the sample period for the delta-operator-based TS fuzzy model only when $T_1=T_2>0$.[13,14] So Eq.(3) can be viewed as a hybrid TS fuzzy model, for its "IF-THEN" rules are in the continuous s-domain form and the system matrices of its consequences are in discrete-like δ-domain form. Note the uncertainties $\frac{1}{2}T_iA_i^2, i=1,2$ in Eq.(3) are actually determined by T_1 and T_2, so the two parameters can also be considered as the additional control parameters like Refs.[5,10,11,16,17], and will be intentionally used to investigate the forming mechanism and to explore the abundant and complex dynamics of new butterfly chaotic attractors in the following section.

3. Complex butterfly attractors evolved from Lorenz system

In order to investigate the compound structure of Lorenz attractor, and clarify its topological structure, the dynamical behaviours of the controlled system (1) were studied in detail by varying the additional control parameter m for $m \geq 0$ and $m < 0$ in Refs.[10,11]. Note there are two additional control parameters T_1 and T_2 in system (3), which gives us more freedom to design with compared to just tuning the one additional control parameter only, and there are six basic combinations of T_1 and T_2 at least, that is ① $T_1 \geq 0$ with $T_2 = 0$; ② $T_1 \leq 0$ with $T_2 = 0$; ③ $T_2 \geq 0$ with $T_1 = 0$; ④ $T_2 \leq 0$ with $T_1 = 0$; ⑤ $T_1 = T_2 \geq 0$; ⑥ $T_1 = T_2 \leq 0$. If we vary the two additional control parameters T_1 and T_2 of the six aforementioned combinations appropriately, the abundant and complex dynamics of chaotic Lorenz attractor beyond those of Refs.[10,11,21] can also be observed. The detailed analysis is omitted here, we only investigate how to arrive at the complex butterfly chaotic attractors from the novel Lorenz-type system (3).

In Eq.(3), if let $T_1 = T_2 = 0$ and we'll obtain the exact TS fuzzy model of the chaotic Lorenz system (2), that the hybrid TS fuzzy model (3) contains the original chaotic Lorenz system as special case. Now, we only discuss the case when $T_1 = T_2 \leq 0$ and vary them, then we can clearly observe the forming procedure of new two-wing butterfly attractors observed from the system (3) in the 3D X-Y-Z space. That is, if we let $T_1 = T_2 \leq 0$ and vary them, it can be seen from Table 1, that (i) when $T_1 = T_2$ is large enough, e.g., $-0.0055 \leq T_1 = T_2 \leq 0$, the system has a complete attractor; (ii) when $T_1 = T_2$ and decreases gradually, the attractor becomes distorted, then, there are period-doubling bifurcations; (iii) when $T_1 = T_2$ is relatively small, the system has a limit cycle and further converges to one point.

Table 1. A summary of the parameter ranges when $T_1 = T_2 \leq 0$ for behaviours of system (3).

For $-0.0055 \leq T_1 = T_2 \leq 0$, the system has a complete attractor.
For $-0.0078 \leq T_1 = T_2 \leq -0.0056$, there are distorted attractors. (See Fig.2(a)).
For $-0.00878 \leq T_1 = T_2 \leq -0.0079$, there are period-doubling bifurcations (See Fig.2(b)).
For $-0.00889 \leq T_1 = T_2 \leq -0.0879$, the system has a limit cycle (See Fig.2(c)).
For $-0.00913 \leq T_1 = T_2 \leq -0.0089$, the system converges to one point.
For $-0.00921 \leq T_1 = T_2 \leq -0.00914$, the system behaviours like fishing nets. (See Fig.2(d)).
For $-0.0099 \leq T_1 = T_2 \leq -0.00922$, the system has transitional attractors. (See Fig.2(e)).
For $-0.0145 \leq T_1 = T_2 \leq -0.01$, the system has limit cycles (See Fig.2(f)).
For $-0.0195 \leq T_1 = T_2 \leq -0.0146$, the system has complex butterfly attractors (See Fig.3(a)–3(f))
For $-0.03 \leq T_1 = T_2 \leq -0.0196$, there are different period-doubling bifurcations
For $-0.07 \leq T_1 = T_2 \leq -0.04$, there are different limit cycles
For $T_1 = T_2 \leq -0.08$, there are periodic windows.

This process, in some sense, demonstrates that this is another forming mechanism of chaotic system different from the so-called mirror-operation forming mechanism observed in Refs.[5,10,11,16,17] for Lorenz system, Chen system and Lü system, respectively. Furthermore, when $T_1 = T_2$ decreases gradually, the original Lorenz system seems to gradually evolve into another chaotic attractor; (iv) when $-0.00921 \leq T_1 = T_2 \leq -0.00914$, the system behaves like a fishing net; (v) when $T_1 = T_2$ and decreases gradually, there seems to be transitional attractors, then the transitional attractors become different period-doubling bifurcations. The process (iv) and (v) behaves like a transitional process and when $T_1 = T_2$ continues to decrease, the system evolves into the two-wing complex butterfly chaotic attractors observed in the X-Y-Z 3D space; (vi) when $-0.0195 \leq T_1 = T_2 \leq -0.0146$, the system has complex butterfly attractors and there are abundant and complex dynamical behaviours never observed from the original Lorenz system; and (vii) when $-0.03 \leq T_1 = T_2 \leq -0.0196$, the system behaves in different period-doubling bifurcations, then these different period-doubling bifurcations converge to different limit cycles; when $T_1 = T_2$ is small enough, there are different periodic windows.

According to Figs.2 and 3, just by varying $T_1 = T_2 \leq 0$, one can see that chaotic Lorenz attractor has a compound topological structure not observed by only

varying the parameter m for $m \geq 0$ and $m < 0$ in Refs.[10, 11]. We must note that, in the works of Elwakil, Kennedy[8,9] and Ozoguz *et al*,[10] Lü *et al*,[11] the so-called novel Lorenz-type system or modified Lorenz system, does not contain the original Lorenz system; while if we let $T_1 = T_2 = 0$ in Eq.(3), we'll get the exact TS fuzzy model of the chaotic Lorenz system (2); hence the hybrid TS-fuzzy-model-based Lorenz-type system (3) contains the original Lorenz system as a special case. According to Figs.2 and 3, we can clearly see the original chaotic Lorenz attractor and the complex butterfly attractors (see Fig.3) realize the continuous transition from one to another; hence, can be also realize the continuous transition to the two newly coined Chen attractor and Lü attractor, since the chaotic Lorenz attractor, Chen attractor and Lü attractor can continuously transit from one to another.[6,7,21]

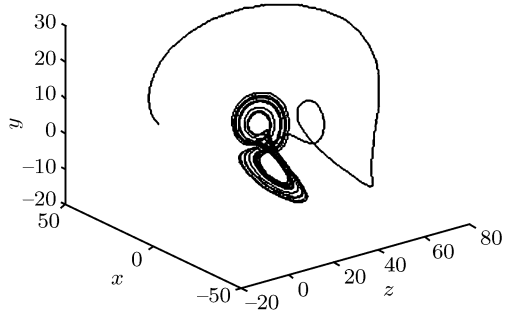


Fig.2. (a) 3D view of the attractor generated at $T_1 = T_2 = -0.0065$.

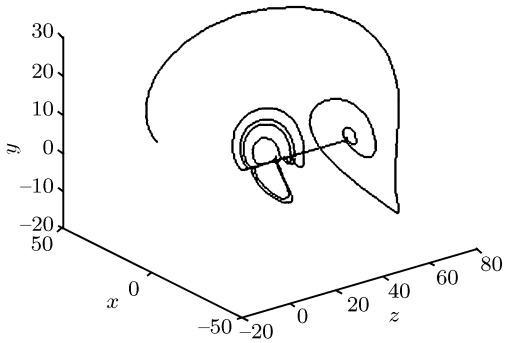


Fig.2. (b) 3D view of the period-doubling bifurcations when $T_1 = T_2 = -0.0086$.

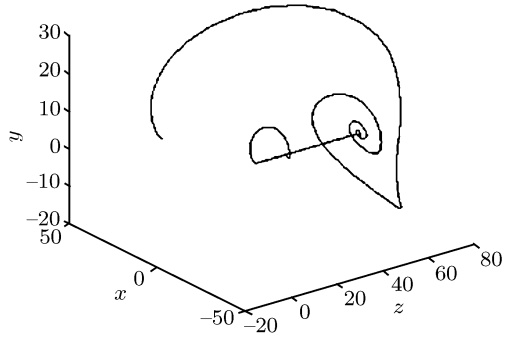


Fig.2. (c) 3D view of the limit cycle when $T_1 = T_2 = -0.00886$.

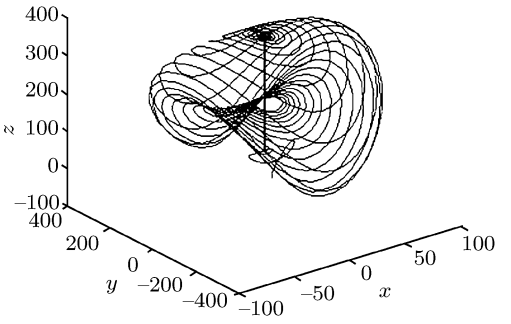


Fig.2. (d) 3D view of the attractor generated at $T_1 = T_2 = -0.00915$.

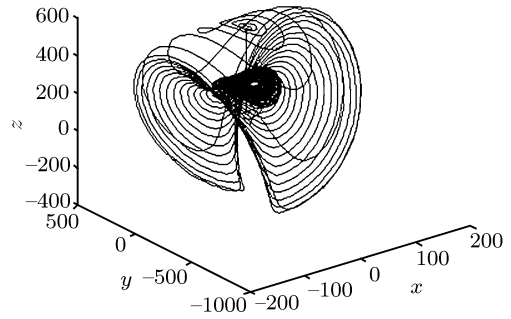


Fig.2. (e) 3D view of the attractor generated at $T_1 = T_2 = -0.00965$

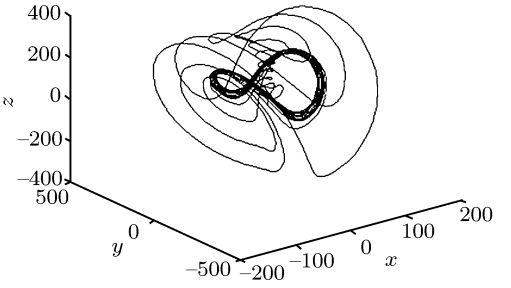


Fig.2. (f) 3D view of the limit cycle when $T_1 = T_2 = -0.014$

From Figs.3(a)–3(f), we can clearly see that the complex butterfly attractors that are somewhat symmetrical with respect to $X = 0$. Different from the projection of the chaotic Lorenz attractor observed in the $X - Z$ plane widely known as the butterfly attractor with its back turned to the observer (See Fig.1(c)), the projection of the complex butterfly attractors observed in the $X - Z$ plane have different shapes (see Fig.3(c)), also, the asymmetric complex butterfly attractors observed in the 3D $X - Y - Z$ space are facing to the observer (see Fig.3(b), 3(e), 3(f) and comparing them with Fig.1(a)) and have more sophisticated topological structure than the classical Lorenz attractor.

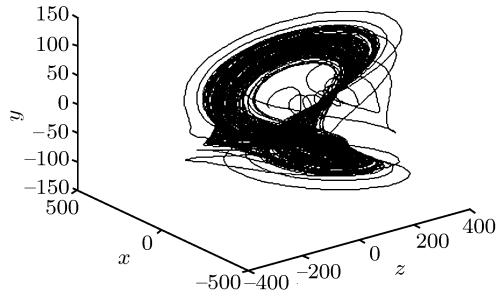


Fig.3. (a) 3D view of the attractor generated at $T_1 = T_2 = -0.015$.

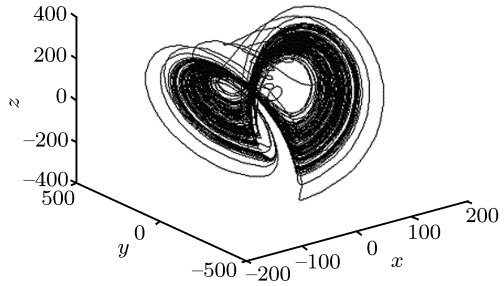


Fig.3. (b) 3D view of the attractor generated at $T_1 = T_2 = -0.015$.

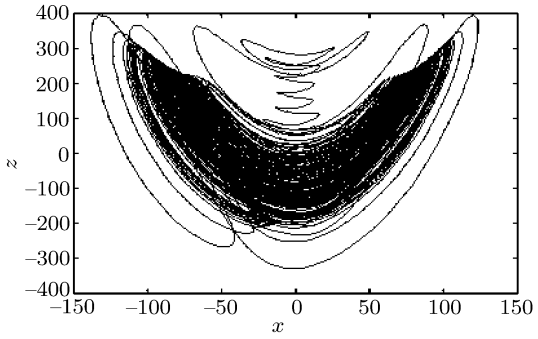


Fig.3. (c) $X-Z$ plane view of the attractor generated at $T_1 = T_2 = -0.015$.

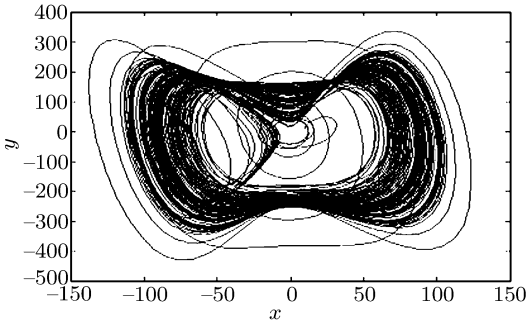


Fig.3. (d) $X-Y$ plane view of the attractor generated at $T_1 = T_2 = -0.015$.

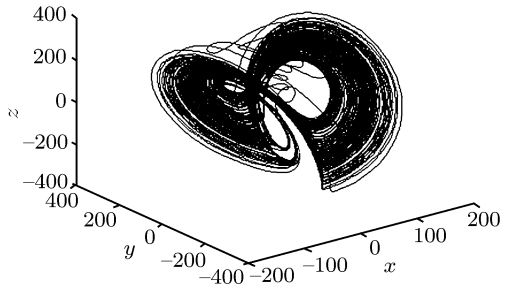


Fig.3. (e) 3D view of the attractor generated at $T_1 = T_2 = -0.0155$.

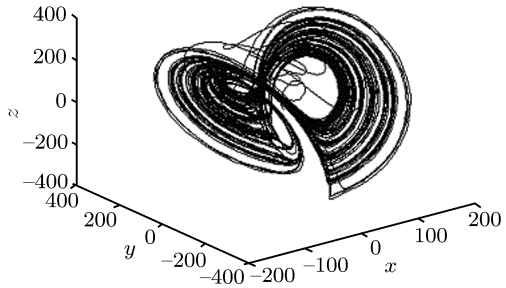


Fig.3. (f) 3D view of the attractor generated at $T_1 = T_2 = -0.01765$.

In addition, if we firstly let the parameters a, b, c vary appropriately, then by virtue of the approach of this paper, very compound and abundant dynamics can also be observed. e.g., if we let $a = 13$, $b = 49$, $c = 2$ with $T_1 = T_2 = -0.0086$ and $T_1 = T_2 = -0.0113$, $a = 6$, $b = 28$, $c = 8/3$, $T_1 = T_2 = -0.012$ in system (3) respectively, we can observe different complex attractors showed in Fig.4(a), 4(b), 4(c), 4(d). Also if we vary T_1 and T_2 differently, there are abundant and complex dynamical behaviours and some typical interesting dynamics can been observed in Figs.5(a) and 5(b). Hence, we believe that the classical chaotic Lorenz system still has many abundant and interesting dynamics to be further discovered and studied by virtue of this novel approach. Nevertheless, more detailed analysis will be needed to further understand the underlying mechanism of this new and effective hybrid modelling method for complex chaotic systems.

It must be noted that this paper actually provides a novel approach to chaotification or anticontrol of chaos from the continuous-time TS fuzzy model via parametric perturbations, which also can be utilized to suppress and generate chaos and distinctively differs from the method used in Refs.[22–24], where different feedback control approaches were used to generate chaos from the continuous-time TS fuzzy model.

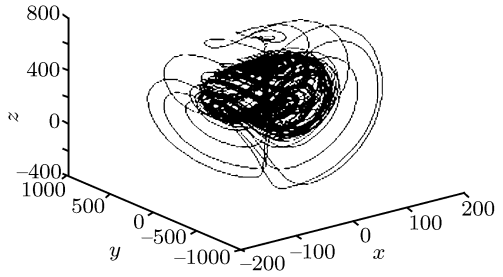


Fig.4. (a) 3D view of the attractor when $a = 13, b = 49, c = 2$, $T_1 = T_2 = -0.0086$.

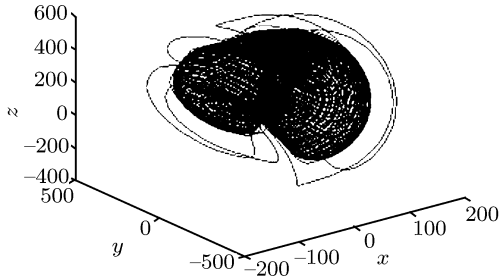


Fig.4. (b) 3D view of the attractor when $T_1 = T_2 = -0.0113$.

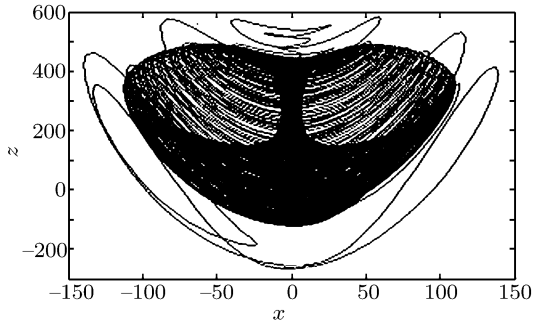


Fig.4. (c) $X - Y$ plane view of the attractor when $T_1 = T_2 = -0.0113$.

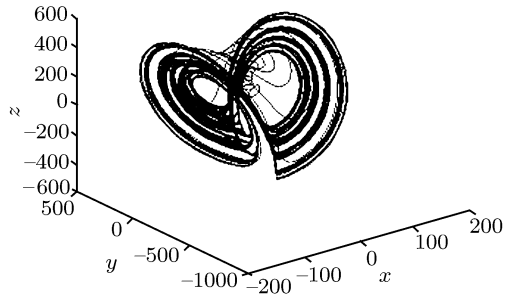


Fig.4. (d) 3D view of the attractor when $a = 6, b = 28, c = 8/3$, $T_1 = T_2 = -0.012$.

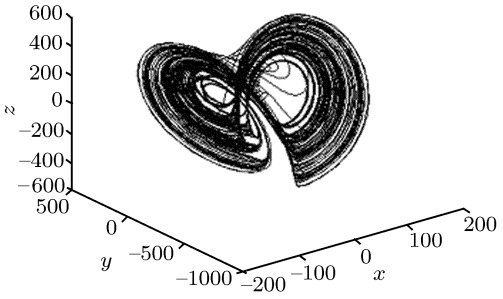


Fig.5. (a) 3D view when $a = 6, b = 20, c = 8/3$, $T_1 = -0.00955, T_2 = -0.0123$.

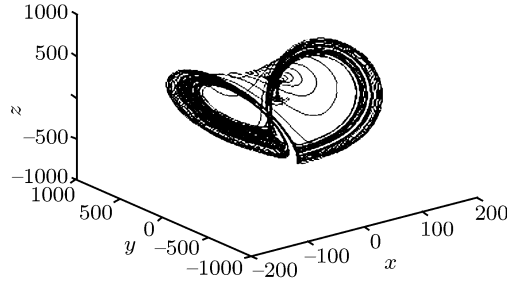


Fig.5. (b) 3D view of the attractor when $T_1 = -0.0123, T_2 = -0.00955$.

4. Conclusion

Despite the chaotic Lorenz system has been intensively studied since it was firstly coined in 1963, it is well known as butterfly chaotic attractor observed from the projection of this system in the $X - Z$ plane. In this article, we have taken one step further to investigate some fundamental dynamic behaviours of a novel hybrid TS fuzzy Lorenz-type system, which contains the original Lorenz system as special case. The two-wing complex butterfly chaotic attractors in the $X - Y - Z$ three dimensional space are firstly observed from this novel system and can continuously transit to the original Lorenz attractor, and there are abundant and complex dynamical behaviours about these interesting compound structures of the complex butterfly chaotic attractors, which may contribute to a better understanding of the essence of Lorenz system and its generalization. The results of this article clearly show that the classical Lorenz system needed to been further explored from a novel point of view. It is believed that the classical chaotic Lorenz system still has many complex and interesting dynamics to be further discovered and studied. We validate our proposals via numerical simulations and the forming procedure of the new butterfly attractors are explored in somewhat detail. More detailed analysis is under way and will be reported in the near future.

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