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Existence of horseshoe maps in voltage-mode controlled buck converters*

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In this paper, chaos in a voltage-mode controlled buck converter is studied. The existence of chaos is proven theoretically in this system. The proof consists of showing that the dynamics of the system is semiconjugate to that of a one-sided shift map, which implies positive entropy of the system and hence chaotic behaviour. The essential tool is the horseshoe hypotheses proposed by Kennedy and Yorke, which will be reviewed prior to the discussion of the main finding.

Keywords: power electronics, proof of chaos, horseshoe map
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1. Introduction

Switching power converters have been shown to exhibit a variety of complex behaviour such as bifurcation and chaos.[1−9] Although chaotic behaviour has been widely recognized in switching power converters, no rigorous proof of chaos in such systems has been reported. Also, the mechanism of chaos in such systems has rarely been investigated since previous works were limited to studying the chaotic attractor by means of numerical simulations and experiments. Hence, the main purpose of this paper is to provide a rigorous proof of chaos in a voltage-mode controlled buck converter, which is a typical model of dc/dc converters frequently used in the previous works. This is realized by proving the existence of a horseshoe in the describing Poincaré map. The proof is based on the work of Kennedy and Yorke,[10] which gives a set of sufficient conditions for a continuous map to exhibit chaotic behaviour.

2. The system

The system studied here is a voltage-mode controlled buck converter as shown in Fig.1, which has been found to be chaos prone by simulations and experimental studies.[11−13] The operation of this buck converter can be briefly described as follows. The output voltage error with respect to a reference voltage is amplified to give a control voltage, $v_{\rm con}$, i.e.

$$v_{\rm con} = a(v - V_{\rm ref}), \tag{1}$$

where v is the output voltage, a is the feedback amplifier gain and $V_{\rm ref}$ is the reference voltage. Then, switch S is controlled by comparing the control voltage, $v_{\rm con}$, with a ramp signal, $V_{\rm ramp}$. The ramp signal is given by

$$V_{\rm ramp} = V_{\rm l} + (V_{\rm u} - V_{\rm l})\left(\frac{t}{T} \bmod 1\right), \tag{2}$$

where $V_{\rm l}$ and $V_{\rm u}$ are the lower and upper thresholds of the ramp, respectively. The comparator output, u, gives the pulse-width-modulated signal necessary for driving the switch, and is described by

$$u(v_{\rm con}, t) = \begin{cases} 0, \text{ if } \ v_{\rm con} \geq V_{\rm ramp}, \\ 1, \text{ if } \ v_{\rm con} < V_{\rm ramp}. \end{cases} \tag{3}$$

When $u = 0$, switch S is turned off, and when $u = 1$, switch S is turned on, as illustrated in Fig.1(b). For simplicity, we consider continuous conduction mode (CCM), in which the inductor current never falls to zero. Thus, the system toggles between only two linear circuit topologies.

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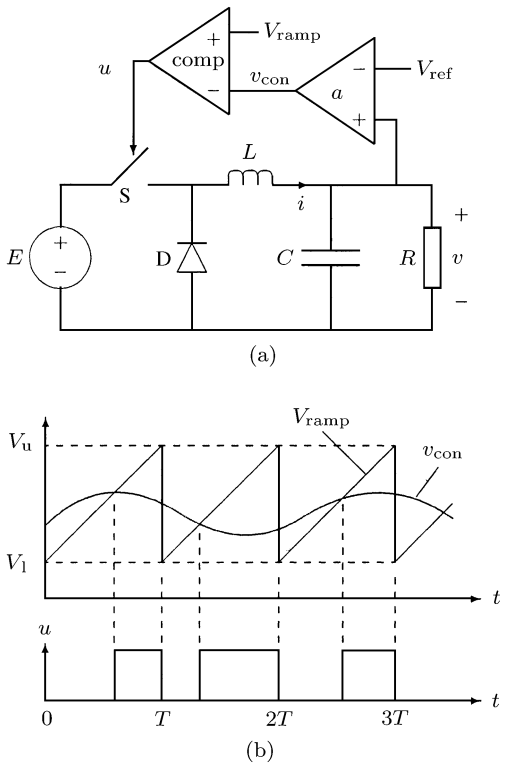


Fig.1. Voltage-mode controlled buck converter. (a) circuit; (b) typical waveforms illustrating the operation.

The buck converter under study can be regarded as a second-order non-autonomous continuous-time dynamical system, which can be described by a state equation of the form

$$\dot{x} = f(x,t), \tag{4}$$

where $x = [x_1\ x_2]^{\rm T} = [v\ i]^{\rm T}$ is the state vector (with superscript T denoting transposition) and $f(x,t)$ is the vector field. The system is non-autonomous because the vector field $f(x,t)$ is a function of time t. Moreover, the system is periodic with period T since $f(x,t) = f(x,t+T)$ for any t. Specifically, the vector field $f(x,t)$ can be defined as

$$f(x,t) = \begin{pmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ E/L \end{pmatrix} u(x,t). \tag{5}$$

The input term $u(x,t)$ in Eq.(5) is essentially equivalent to $u(v_{\rm con},t)$ due to the affine relation between x_1 and $v_{\rm con}$, and can be written as

$$u(x,t) = \begin{cases} 0, \text{ if } x_1 \geq v_{\rm ramp}, \\ 1, \text{ if } x_1 < v_{\rm ramp}. \end{cases} \tag{6}$$

Here, $v_{\rm ramp}$ is obtained by an appropriate transformation of (3), which is given by

$$v_{\rm ramp} = V_1 + (V_2 - V_1)\left(\frac{t}{T} \bmod 1\right), \tag{7}$$

where $V_1 = V_{\rm ref} + V_{\rm l}/a$ and $V_2 = V_{\rm ref} + V_{\rm u}/a$.

For the time-periodic non-autonomous system with minimum period T described by Eq.(4), the Poincaré map is

$$P : x \mapsto P(x) = \phi_{t_0+T}(x, t_0), \tag{8}$$

where $\phi_t(x,t_0)$ is the solution to Eq.(4) with the initial condition $x(t_0) = x$, and t_0 is conveniently taken as the start of any clock period.

The construction of $P(x)$ involves stacking of the solutions of the individual vector fields given by Eq.(5) in the time interval $[t_0, t_0+T]$. When $u(x,t)$ is equal to 0 or 1, the solution of the corresponding linear differential equation is readily found. Here, we denote the solutions by $\varphi_0(x_0,t)$ for $u(x,t) = 0$ (i.e., when the switch is off) and $\varphi_1(x_0,t)$ for $u(x,t) = 1$ (i.e., when the switch is on) with the initial condition $\varphi_0(x_0,0) = x_0$ and $\varphi_1(x_0,0) = x_0$, respectively. Note that $\varphi_0(x_0,t), \varphi_1(x_0,t) \in \mathcal{C}^\infty$ with respect to time t. Thus, $P(x)$ can be found in the form of a composition of φ_0 and φ_1. The detailed procedure for deriving such Poincaré map can also be found in the Refs.[14] and [15].

Now, we give a simple example to illustrate the derivation of the Poincaré map. Suppose the buck converter operates with the switch first off and then on in one switching period $[t_0, t_0+T]$. The corresponding solution of (4) is expressed as

$$x(t_0+T) = \varphi_1(\varphi_0(x(t_0), t_1), T - t_1), \tag{9}$$

where t_1 is the time duration for switch being off. Thus, the Poincaré map is

$$P(x) = \varphi_1(\varphi_0(x, t_1), T - t_1). \tag{10}$$

For more complicated combination of switch states, the Poincaré map can be obtained in a similar way. However, it is worth pointing out that the Poincaré map cannot be written in closed form. It is due to the difficulty of obtaining the time duration for each switch state within a switching period (e.g. t_1 above). Thus, we have to resort to numerical methods to complete the Poincaré map.[2,14,15] Olivar *et al*[14] have shown that the resulting Poincaré map is continuous, but piecewise smooth in the phase plane.

To illustrate the dynamical behaviour of this system, we consider the following set of parameters: $E = 33$ V, $L = 0.02$ H, $C = 47$ μF, $R = 22$ Ω, $T = 400$ μs, $a = 8.4$, $V_{\rm ref} = 11.3$ V, $V_{\rm l} = 3.8$ V, $V_{\rm u} = 8.2$ V. With this set of parameters, the converter is observed to operate chaotically. Figure 2 shows a chaotic attractor from the Poincaré map. In the following of this paper, we will try to theoretically prove the chaos existing in this system by showing that the Poincaré map is essentially a horseshoe map.

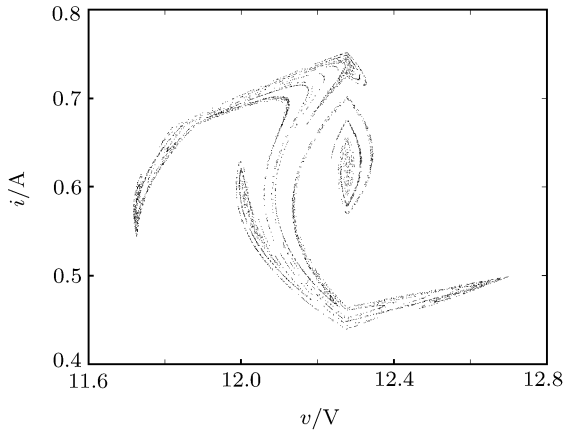


Fig.2. A chaotic attractor of voltage-mode controlled buck converter.

3. Review of the horseshoe hypotheses

In this section, we begin with a review of a fundamental result regarding the horseshoe dynamics,[16] which is essential to proving the existence of a horseshoe in the voltage-mode controlled buck converter.

Let $S = \{1, 2, 3, ..., N\}$ be a collection of symbols. Then, a bi-infinite sequence can be constructed with all elements from S. Let Σ^N be the space of all bi-infinite sequences using the symbol set S. Any point s in Σ^N can be written as

$$s = \{\cdots s_{-n} \cdots s_{-1} s_0 s_1 \cdots s_n \cdots\}, \qquad (11)$$

where $s_i \in S$ for any i. In order to describe limit processes in Σ^N, it is convenient to define a metric on Σ^N. Consider another point $\bar{s} = \{\cdots \bar{s}_{-n} \cdots \bar{s}_{-1} \bar{s}_0 \bar{s}_1 \cdots \bar{s}_n \cdots\}$ in Σ^N. The distance between s and $\bar{s}$ is defined as

$$d(s, \bar{s}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{d_i(s_i, \bar{s}_i)}{1 + d_i(s_i, \bar{s}_i)}, \qquad (12)$$

where

$$d_i(s_i, \bar{s}_i) = \begin{cases} 1 & \text{if } s_i \neq \bar{s}_i, \\ 0 & \text{if } s_i = \bar{s}_i. \end{cases} \qquad (13)$$

With the distance defined above, the space Σ^N is a *metric space* with the metric $d(\cdot, \cdot)$. In the following, we will state some results concerning the structure of Σ^N without proofs. Readers may refer to Wiggins for detailed proofs.[16]

Proposition 1 The space Σ^N equipped with the metric defined by Eq.(12) is compact, totally disconnected, and perfect.

The above proposition essentially states the properties that define the structure of the metric space Σ^N. It may be worth noting that compactness, total disconnectedness and perfectness are often taken as the defining properties of a *Cantor set*, which is used in the characterization of complex structures of invariant sets in a chaotic dynamical system.[16]

Proposition 2 Consider the shift map $\sigma : \Sigma^N \to \Sigma^N$, which is defined as

$$\sigma(s) = \{\cdots s_{-n} \cdots s_{-1} s_0 s_1 \cdots s_n \cdots\} \qquad (14)$$

for $s = \{\cdots s_{-n} \cdots s_{-1} s_0 s_1 \cdots s_n \cdots\} \in \Sigma^N$. The following statements regarding this shift map are true:

1) The metric space Σ^N is invariant over σ, i.e., $\sigma(\Sigma^N) = \Sigma^N$.

2) σ is continuous.

3) σ has

(a) a countable infinity of periodic orbits consisting of orbits of all periods;

(b) an uncountable infinity of nonperiodic orbits; and

(c) a dense orbit.

4) Σ^N is a chaotic, compact invariant set for σ.

We remark here that the afore-defined shift map can be equivalently represented by $[\sigma(s)]_i = s_{i+1}$.

Next, we will recall some aspects of the topological horseshoe theory proposed by Kennedy and Yorke,[10] which are relevant to the main result to be presented in this paper. Essentially, Kennedy and Yorke found a set of sufficient conditions for a continuous map f to exhibit chaos. These conditions are summarized in what has been called the horseshoe hypotheses, which include the following:

1) X is a separable metric space;

2) $Q \subset X$ is locally connected and compact;

3) The map $f : Q \to X$ is continuous;

4) The sets $\mathbf{end}_0 \subset Q$ and $\mathbf{end}_1 \subset Q$ are disjoint and compact, and each component of Q intersects both $\mathbf{end}_0$ and $\mathbf{end}_1$;

5) Q has a crossing number $N \geq 2$.

The crossing number N in the above hypotheses can be reasoned as follows. A connection Γ is a compact connected subset of Q that intersects both sets $\mathbf{end}_0$ and $\mathbf{end}_1$. A preconnection γ is a compact connected subset of Q for which $f(\gamma)$ is a connection. Then, the crossing number N is defined as the largest number such that every connection contains at least N mutually disjoint preconnections.

The main result from Kennedy and Yorke can be summarized in the following theorem.[10]

Kennedy–Yorke Theorem Consider a map f. Suppose the horseshoe hypotheses are satisfied. Then, there is a closed invariant set $\Lambda \subset Q$ for which $f|\Lambda$ is semiconjugate to the one-sided shift map on N symbols.[1)]

The main consequence of this result is that the entropy of f is not less than that of σ, which implies that f has positive entropy and is therefore chaotic. Since the theorem provides a weaker set of conditions for proving chaos, it is more readily applied to specific practical systems.[17,18]

4. Horseshoe in voltage-mode controlled buck converter

In this section, the existence of a horseshoe map in a specific buck converter will be proven theoretically by using the Kennedy–Yorke theorem. The parameters used here are the same as those given in Section 2.

Here, we will consider five iterates of the Poincaré map f and denote it by P, i.e., $P = f^5$.[2)] Consider the parallelogram $ABCD$ with the coordinates of four endpoints defined as $A = (12.12, 0.52)$, $B = (12.19, 0.5)$, $C = (12.19, 0.46)$ and $D = (12.12, 0.48)$.

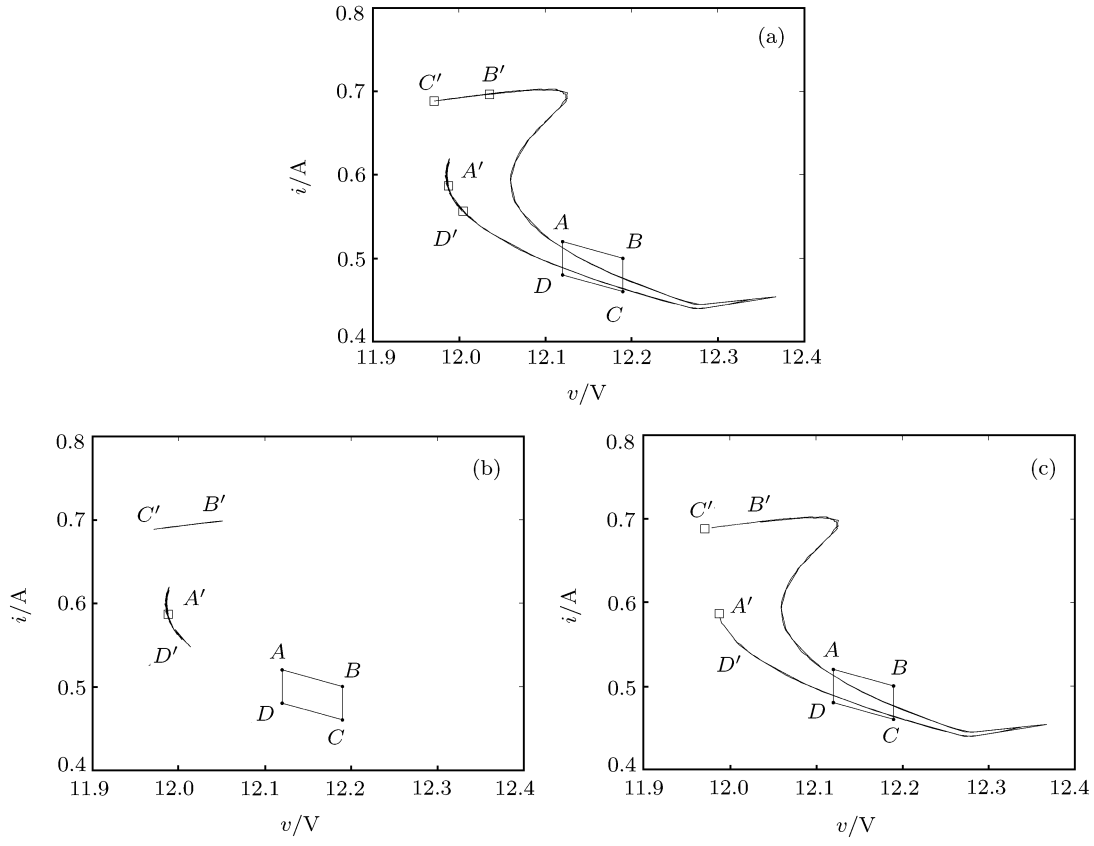


Fig.3. Parallelogram $ABCD$ and its image $A'B'C'D'$ under P. (a) All images; (b) only images $A'D'$ and $B'C'$; (c) only images $A'B'$ and $D'C'$ are shown. (A (B, C, D) and A' (B', C', D') are denoted by • and □, respectively.).

[1)] A shift map on N symbols is denoted by $\sigma|\Sigma^N$. If $f|\Lambda$ is semiconjugate to $\sigma|\Sigma^N$, there exists a continuous and onto map $\phi : \Lambda \to \Sigma^N$ such that $\phi \circ f = \sigma \circ \phi$.

[2)] The map P can also be regarded as the Poincaré map with stroboscopic sampling period $5T$ which is five times that of f.

Figure 3(a) shows the parallelogram $ABCD$ and its image $A'B'C'D'$ under P. Here, $A' = P(A)$, $B' = P(B)$, $C' = P(C)$ and $D' = P(D)$. The image $A'B'C'D'$ looks like a curve but is actually a thin band. To clearly illustrate this fact, Fig.3(b) and Fig.3(c) are also given to present the images separately.

Now, we choose the line AD and BC as the two end subsets required in the horseshoe hypotheses. To ease the construction of the preconnections, we look at $ABCD$ and $A'B'C'D'$ again. From Fig.3(a), it is straightforward to find that $A'B'C'D'$ intersects the line AD twice. We denote these two intersections by L_1' and L_4', respectively. Namely, $A'B'C'D' \cap AD = \{L_1', L_4'\}$. Similarly, $A'B'C'D'$ also intersects the line BC twice, and we denote the intersections by L_2' and L_3', respectively. Namely, $A'B'C'D' \cap BC = \{L_2', L_3'\}$. By numerical computation, the corresponding preimages of the lines L_1', L_2', L_3' and L_4' can also be obtained, i.e., $P(L_1) = L_1'$, $P(L_2) = L_2'$, $P(L_3) = L_3'$ and $P(L_4) = L_4'$, as shown in Fig.4. It should be noted that, in Fig.4, the zone encompassed by L_1 and L_2 within $ABCD$ (denoted by Ω_1) is disjoint to that by L_3 and L_4 (denoted by Ω_2). It can also be concluded that any curve across $\Omega_1(\Omega_2)$ with two ends located on $L_1(L_3)$ and $L_2(L_4)$ will be a preconnection. This will be useful for constructing the proof of the following theorem, which summarizes our main findings.

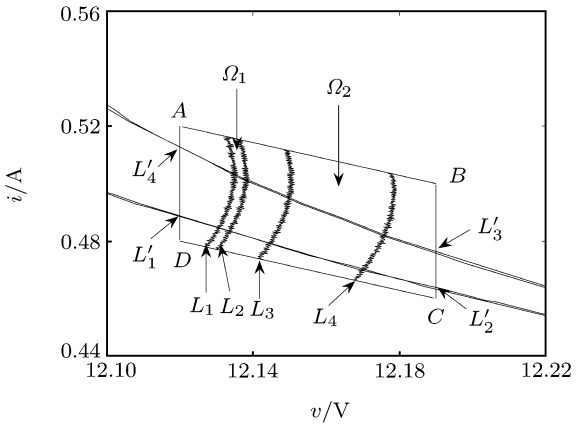


Fig.4. Intersections of $A'B'C'D'$ with two ends (AD and BC) and preimages corresponding to these intersections. ($P(L_i) = L_i'$, i=1, 2, 3, 4).

Main Theorem For the Poincaré map P describing the dynamics of the voltage-mode controlled buck converter, there exists a closed invariant set $\Lambda \subset Q$, where $Q \subset R^2$, such that $P|\Lambda$ is semiconjugate to a one-sided shift map.

Proof Recall the horseshoe hypotheses. First, R^2 (which is X in the horseshoe hypotheses) is separable. Consider $Q = ABCD$ in the afore-described construction. Clearly, $Q \subset X$ is locally connected and compact. The Poincaré map $P : Q \to X$ is continuous. Let $\mathbf{end}_0 = AD$ and $\mathbf{end}_1 = BC$. Then, they are disjoint and compact. The crossing number N can be derived as follows. Referring to the afore-presented construction, given any connection Γ, it will cross both Ω_1 and Ω_2. Moreover, $\Omega_1 \cap \Gamma$ and $\Omega_2 \cap \Gamma$ are two mutually disjoint preconnections according to the previous statement. Thus, the crossing number $N \geq 2$. Hence, the horseshoe hypotheses are satisfied. From the Kennedy–Yorke theorem, there exists a closed invariant set $\Lambda \subset ABCD$ such that $P|\Lambda$ is semiconjugate to a one-sided shift map.

5. Conclusion

In this paper, we have studied the voltage-mode controlled buck converter and presented a rigorous proof for the existence of a horseshoe in the describing Poincaré map with the assistance of numerical computation. The proof is based on the result given by Kennedy and Yorke,[10] which provides a set of sufficient conditions for guaranteeing semiconjugation of a continuous map to the shift map, which implies chaos in the continuous map.

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