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The pth-order periodic solutions for a family of N-coupled nonlinear Schrödinger equations*

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By using the solutions of an auxiliary Lamé equation, a direct algebraic method is proposed to construct the exact solutions of N-coupled nonlinear Schrödinger equations. The abundant higher-order exact periodic solutions of a family of N-coupled nonlinear Schrödinger equations are explicitly obtained with the aid of symbolic computation and they include corresponding envelope solitary and shock wave solutions.

Keywords: Lamé equation, N-coupled nonlinear Schrödinger equations, higher-order periodic solution, symbolic computation

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1. Introduction

Propagation of optical soliton in a single mode fibre is governed by the well-known nonlinear Schrödinger equation (NLSE) of the form

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2u=0, \tag{1}$$

where $u(t,x)$ is the slowly varying envelope of the axial field. The soliton solutions and abundant periodic solutions of Eq.(1) have been presented in Ref.[1–5].

For handing more channels it is necessary to propagate more fields simultaneously. In 1974, Manakov[6] derived the coupled nonlinear Schrödinger equations (CNLSEs) from the NLSE by considering that the total field was composed of two fields. In the same work he presented the linear eigenvalue problem associated with the CNLSEs and the soliton solutions using the inverse scattering transform. The Painlevé analysis of the CNLSEs was carried out by Sahadevan *et al.*[7] Soliton solutions using the Hirota bilinear method for CNLSEs were presented by Radhakrishnan and Lakshmanan.[8] Porsezian and Nakkeeran have generated the soliton solutions for CNLSEs using the Bäcklund transformation method.[9] In Ref.[10] we presented abundant Jacobi elliptic function solutions for CNLSEs using the auxiliary elliptic equation method. When we consider the simultaneous propagation of N nonlinear waves in a fibre, the wave dynamics of the system will be governed by N-CNLSEs of the form

$$\mathrm{i}\frac{\partial u_n}{\partial t}+\frac{\partial^2 u_n}{\partial x^2}+2\Big(\sum_{m=1}^{N}|u_m|^2\Big)u_n=0,\ \ n=1,2,\ldots,N, \tag{2}$$

where the $|u_m|$ denotes the modulus of the u_m.

For $N=1$, Eqs.(2) are just Eq.(1).

For $N=2$, Eqs.(2) give CNLSEs of the form:[5–9]

$$\mathrm{i}\frac{\partial u_1}{\partial t}+\frac{\partial^2 u_1}{\partial x^2}+2(|u_1|^2+|u_2|^2)u_1=0,$$

$$\mathrm{i}\frac{\partial u_2}{\partial t}+\frac{\partial^2 u_2}{\partial x^2}+2(|u_1|^2+|u_2|^2)u_2=0. \tag{3}$$

The Painlevé analysis of N-CNLSEs has been carried out in Ref.[7]. The Lax pairs for N-CNLSEs have been presented by Fordy and Kulish.[11] Solitary wave and periodic solutions for N-CNLSEs have been studied partly in Refs.[12–14]. In this paper, we propose a new method to construct the exact periodic solutions of N-CNLSEs based on an auxiliary Lamé equation. The abundant exact periodic solutions of N-CNLSEs are explicitly obtained. Particularly, some of the third-order Jacobi elliptic function solutions are obtained that have not ever been presented before. In a limiting case, they give corresponding envelope solitary and shock wave solutions.

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2. Auxiliary Lamé equation

For solving Eqs.(2), we use a form of Lamé equation with a kernel function, expressed in terms of $\mathrm{dn}^2\xi$,

$$\frac{\mathrm{d}^2\Lambda_n^{(p)}(\xi)}{\mathrm{d}\xi^2}+[p(p+1)\mathrm{dn}^2(\xi)-B_n^{(p)}]\Lambda_n^{(p)}(\xi)=0,$$
$$p=1,2,\ldots,\quad n=1,2,\ldots,2p+1. \tag{4}$$

Here $\Lambda_n^{(p)}(\xi)$ is the nth-order eigenfunction of Lamé equation (4), corresponding to the eigenvalue $B_n^{(p)}$. For p=1,2, and 3, a full set of eigenfunctions $\Lambda_n^{(p)}(\xi)$ and eigenvalues $B_n^{(p)}$ of Eqs.(4) is given in Table 1, where $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ are Jacobi elliptic sine function, cosine function and Jacobi elliptic function of the third kind with the modulus $k(0<k<1)$, respectively.

The following designations are used here:

$$\gamma_1^{(2)}=\gamma_5^{(2)}=\frac{k^2-2\pm\sqrt{1-k^2+k^4}}{3},$$

$$\gamma_1^{(3)}=\gamma_5^{(3)}=\frac{2k^2-3\pm\sqrt{4-7k^2+4k^4}}{5},$$

$$\gamma_2^{(3)}=\gamma_6^{(3)}=\frac{k^2-3\pm\sqrt{4-k^2+k^4}}{5},$$

$$\gamma_3^{(3)}=\gamma_7^{(3)}=\frac{2k^2-4\pm\sqrt{1-k^2+4k^4}}{5}, \tag{5}$$

where $k(0<k<1)$ denotes the modulus of Jacobi elliptic functions.

On Jacobi elliptic functions, we have

$$\mathrm{cn}^2(\xi)=1-\mathrm{sn}^2(\xi),\quad \mathrm{dn}^2(\xi)=1-k^2\mathrm{sn}^2(\xi), \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}(\xi)=\mathrm{cn}(\xi)\mathrm{dn}(\xi),\quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}(\xi)=-\mathrm{sn}(\xi)\mathrm{dn}(\xi),$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}(\xi)=-k^2\mathrm{sn}(\xi)\mathrm{cn}(\xi). \tag{7}$$

Table 1. A full set of the eigenfunctions $\Lambda_n^{(p)}(\xi)$ and the eigenvalues $B_n^{(p)}$ of pth-order Lamé equation for p=1, 2, 3.

p	n	$\Lambda_n^{(p)}(\xi)$	$B_n^{(p)}$
1	3	$\mathrm{dn}(\xi)$	$2-k^2$
1	2	$\mathrm{cn}(\xi)$	1
1	1	$\mathrm{sn}(\xi)$	$1-k^2$
2	5	$\mathrm{dn}^2(\xi)+\gamma_5^{(2)}$	$2(k^2-1)/\gamma_5^{(2)}$
2	4	$\mathrm{cn}(\xi)\mathrm{dn}(\xi)$	$5-k^2$
2	3	$\mathrm{sn}(\xi)\mathrm{dn}(\xi)$	$5-4k^2$
2	2	$\mathrm{sn}(\xi)\mathrm{cn}(\xi)$	$2-k^2$
2	1	$\mathrm{dn}^2(\xi)+\gamma_1^{(2)}$	$2(k^2-1)/\gamma_1^{(2)}$
3	7	$\mathrm{dn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_7^{(3)}]$	$5(2-k^2)+2\sqrt{1-k^2+4k^4}$
3	6	$\mathrm{cn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_6^{(3)}]$	$7-2k^2+2\sqrt{4-k^2+k^4}$
3	5	$\mathrm{sn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_5^{(3)}]$	$7-5k^2+2\sqrt{4-7k^2+4k^4}$
3	4	$\mathrm{sn}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)$	$4(2-k^2)$
3	3	$\mathrm{dn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_3^{(3)}]$	$5(2-k^2)-2\sqrt{1-k^2+4k^4}$
3	2	$\mathrm{cn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_2^{(3)}]$	$7-2k^2-2\sqrt{4-k^2+k^4}$
3	1	$\mathrm{sn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_1^{(3)}]$	$7-5k^2-2\sqrt{4-7k^2+4k^4}$

The modulus $k(0<k<1)$ substantially affects solutions of Eqs.(4). When $k\to 0$ or $k\to 1$, Jacobi elliptic functions asymptotically go into trigonometric or hyperbolic ones:

$$\mathrm{sn}(\xi)\to\sin(\xi),\ \mathrm{cn}(\xi)\to\cos(\xi),\ \text{and}$$

$$\mathrm{dn}(\xi)\to 1,\quad \text{when}\quad k\to 0;$$

$$\mathrm{sn}(\xi)\to\tanh(\xi),\ \mathrm{cn}(\xi)\to\mathrm{sech}(\xi),\ \text{and}$$

$$\mathrm{dn}(\xi)\to\mathrm{sech}(\xi),\quad \text{when}\quad k\to 1. \tag{8}$$

When $k\to 0$ the Lamé equation goes into the equation of harmonic oscillators. With k increasing, the Lamé equation becomes the equation for Legendre-associated polynomials, periods of elliptic functions tend to infinity, and all periodical solutions asymptotically turn to spatially localized ones. The detailed information about Jacobi elliptic functions and Lamé equations can be found in Refs.[15–18].

3. Construction of higher-order solutions of equations (2)

Since N-CNLSEs (2) describe a family of nonlinear wave packets, we can seek their envelope wave solutions in the following form:

$$u_n = \rho_n(\xi)\mathrm{e}^{\mathrm{i}(\lambda_n x-\omega_n t)}, \quad \xi = x - 2\lambda_n t, \\ n = 1, 2, \ldots, N, \tag{9}$$

where $\rho_n(\xi)(n = 1, 2, \ldots, N)$ is real amplitude of the nth component in the family, and λ_n and ω_n denote its wave number and circular frequency respectively.

Substituting the expression (9) into Eqs.(2) leads to a set of ordinary differential equations

$$\frac{\mathrm{d}^2\rho_n}{\mathrm{d}\xi^2} + (\omega_n - \lambda_n^2)\rho_n + 2\rho_n \sum_{m=1}^{N} \rho_m^2 = 0, \\ n = 1, 2, \ldots, N. \tag{10}$$

Now suppose that the solutions of Eqs.(10) are proportional to eigenfunctions of the pth-order auxiliary equation (4) with unknown amplitudes $A_n^{(p)}$, i.e.

$$\rho_n^{(p)}(\xi) = A_n^{(p)} \Lambda_n^{(p)}(\alpha\xi), \quad n = 1, 2, \ldots, 2p+1, \tag{11}$$

where α is a scale factor, and $p = 1, 2, \ldots$.

Expanding $[\Lambda_n^{(p)}(\alpha\xi)]^2$ in a power series of $\mathrm{dn}^2(\alpha\xi)$ yields

$$[\Lambda_n^{(p)}(\alpha\xi)]^2 = \sum_{m=0}^{p} a_{nm}^{(p)}(k)\mathrm{dn}^{2m}(\alpha\xi), \\ p = 1, 2, \ldots, \quad n = 1, 2, \ldots, 2p+1. \tag{12}$$

According to relationships (6), $a_{nm}^{(p)}(k)$ can be easily determined for fixed p and n.

Substituting the expressions (11) and (12) into Eqs.(10), comparing the equations obtained with Lamé equation (4), and equating the coefficients at the same power of $\mathrm{dn}^2(\alpha\xi)$, we obtain $N + p$ independent linear equations in $[A_n^{(p)}]^2$:

$$\alpha^2 B_n^{(p)} + \omega_n - \lambda_n^2 + 2\sum_{m=1}^{N} [A_m^{(p)}]^2 a_{m0}^{(p)}(k) = 0, \\ n = 1, 2, \ldots, N, \tag{13}$$

$$\sum_{m=1}^{N} [A_m^{(p)}]^2 a_{m1}^{(p)}(k) = \frac{p(p+1)\alpha^2}{2}, \tag{14}$$

$$\sum_{m=1}^{N} [A_m^{(p)}]^2 a_{mj}^{(p)}(k) = 0, \quad j = 2, 3, \ldots, p. \tag{15}$$

It is easy to see that the first N equations (13) set a limit on the ranges of ω_n and λ_n, and the next p equations (14) and (15) determine $[A_n^{(p)}]^2$.

For each p value, one can construct no more than

$$C_{2p+1}^{N} = \frac{(2p+1)\times(2p)\times(2p-N+2)}{1\times 2\times\cdots N}$$

linearly independent solutions of Eqs.(10). Since Eqs.(14) and (15) are linear in $[A_n^{(p)}]^2$ and contain p independent equations, nonzero squared amplitudes $[A_n^{(p)}]^2$ are determined ambiguously for $N > p$. When $N = p$, nonzero squared amplitudes $[A_n^{(p)}]^2$ are determined uniquely. It means that there are C_{2p+1}^{p} self-consistent p-component pth-order solutions, which differ from each other by a choice of the eigenfunctions $\Lambda_n^{(p)}$ of Lamé equation (4). When $N < p$ Eqs.(14) and (15) are over-determined and have no solutions.

Finally, substituting $\rho_n^{(p)}(\xi)$ into Eqs.(9), one can obtain the pth-order periodic solutions of N-coupled nonlinear Schrödinger equations (2). That means that the pth-order solutions contain the terms of product of p Jacobi elliptic functions.

For short, N-component periodic waves of Eqs.(2), composed of the n_1th-, n_2th-, …, n_Nth-order eigenfunctions of the pth-order auxiliary Lamé equation (4), are called further the pth-order solution of $n_1 n_2 \ldots n_N$-type. It is easy to check that a full number of possible one-component first-order solutions might be equal to 3, a full number of possible two-component second-order solutions might be equal to 10, and a full number of possible three-component third-order solutions might be equal to 35.

4. Some examples

4.1. The first-order periodic solutions of NLSE

For NLSE (1), $N = 1$. According to the analysis above, when $p = 1$, the equation has a unique solution. From the expression (12), we have

$$a_{10}^{(1)}(k) = \frac{1}{k^2}, \qquad a_{11}^{(1)}(k) = -\frac{1}{k^2}; \tag{16}$$

$$a_{20}^{(1)}(k) = 1 - \frac{1}{k^2}, \qquad a_{21}^{(1)}(k) = \frac{1}{k^2}; \tag{17}$$

$$a_{30}^{(1)}(k) = 0, \qquad a_{31}^{(1)}(k) = 1. \tag{18}$$

Substituting the expressions (16)–(18) into Eqs.(14) and (15), using Table 1, we obtain

$$[A_1^{(1)}]^2 = -\alpha^2 k^2, \quad [A_2^{(1)}]^2 = \alpha^2 k^2, \quad [A_3^{(1)}]^2 = \alpha^2. \tag{19}$$

Since both α and k are real parameters and $A_1^{(1)}$ is a real function, from the expression (19), $A_1^{(1)}$ does not exist. Therefore, NLSE (1) has only the 1st-order solutions of 2- and 3-type.

Substituting the expression (19) into (11), then into the expression (9) and using Table 1 , the exact periodic solutions of NLSE (1) are obtained as follows:

$$u_{12}(x,t) = k\alpha \mathrm{cn}[\alpha(x-2\lambda t)]\ \mathrm{e}^{\mathrm{i}(\lambda x-\omega t)},$$
$$\alpha = \pm\sqrt{\frac{\omega-\lambda^2}{1-2k^2}}$$
$$\text{for}\ \ \omega>\lambda^2,\ k^2<\frac{1}{2}\ \ \text{or}\ \ \omega<\lambda^2,\ k^2>\frac{1}{2}; \tag{20}$$

$$u_{13}(x,t) = \alpha \mathrm{dn}[\alpha(x-2\lambda t)]\ \mathrm{e}^{\mathrm{i}(\lambda x-\omega t)},$$
$$\alpha = \pm\sqrt{\frac{\omega-\lambda^2}{2-k^2}}$$
$$\text{for}\ \ \omega>\lambda^2. \tag{21}$$

Notice that when $k\to 1$, $\mathrm{cn}(\xi)\to\mathrm{sech}(\xi)$, and $\mathrm{dn}(\xi)\to\mathrm{sech}(\xi)$, the solutions u_{12} and u_{13} degenerate into the corresponding envelope solitary solutions.

4.2. The second-order periodic solutions of CNLSEs

For CNLSEs (3), $N=2$. According to the analysis above, when $p=2$, the equation has a unique solution. A full number of possible two-component second-order solutions might be equal to 10. They are 54-, 53-, 52-, 51-, 43-, 42-, 41-, 32-, 31- and 21-type solutions respectively.

From the expression (12), we have

$$a_{10}^{(2)}(k)=[\gamma_1^{(2)}]^2,\quad a_{11}^{(2)}(k)=2\gamma_1^{(2)},$$
$$a_{12}^{(2)}(k)=1; \tag{22}$$

$$a_{20}^{(2)}(k)=\frac{1}{k^2}\left(1-\frac{1}{k^2}\right),$$
$$a_{21}^{(2)}(k)=-\frac{1}{k^2}\left(1-\frac{2}{k^2}\right),$$
$$a_{22}^{(2)}(k)=-\frac{1}{k^4}; \tag{23}$$

$$a_{30}^{(2)}(k)=0,\quad a_{31}^{(2)}(k)=\frac{1}{k^2},$$
$$a_{32}^{(2)}(k)=-\frac{1}{k^2}; \tag{24}$$

$$a_{40}^{(2)}(k)=0,$$
$$a_{41}^{(2)}(k)=1-\frac{1}{k^2},$$
$$a_{42}^{(2)}(k)=\frac{1}{k^2}; \tag{25}$$

$$a_{50}^{(2)}(k)=[\gamma_5^{(2)}]^2,$$
$$a_{51}^{(2)}(k)=2\gamma_5^{(2)},$$
$$a_{52}^{(2)}(k)=1. \tag{26}$$

Substituting any two expressions among the expressions (22)–(26) into Eqs.(14) and (15), then into the expressions (9) and using Table 1 and the expression (11), then the 2nd-order periodic solutions of CNLSEs (2) can be obtained. For example, the 2nd-order solutions of 43-type are

$$u_{23}(x,t)=\pm\sqrt{3}\alpha \mathrm{sn}[\alpha(x-2\lambda_1 t)]\mathrm{dn}[\alpha(x-2\lambda_1 t)]$$
$$\times\,\mathrm{e}^{\mathrm{i}(\lambda_1 x-\omega_1 t)},$$
$$\text{for}\ \ \omega_1-\lambda_1^2=-\alpha^2(5-4k^2); \tag{27}$$

$$u_{24}(x,t)=\pm\sqrt{3}\alpha \mathrm{cn}[\alpha(x-2\lambda_2 t)]\mathrm{dn}[\alpha(x-2\lambda_2 t)]$$
$$\times\,\mathrm{e}^{\mathrm{i}(\lambda_2 x-\omega_2 t)},$$
$$\text{for}\ \ \omega_2-\lambda_2^2=-\alpha^2(5-k^2). \tag{28}$$

Notice that when $k\to 1$, $\mathrm{sn}(\xi)\to\tanh(\xi)$, $\mathrm{cn}(\xi)\to\mathrm{sech}(\xi)$, and $\mathrm{dn}(\xi)\to\mathrm{sech}(\xi)$, the solutions u_{21} and u_{22} degenerate into the corresponding envelope solitary or shock wave solutions.

4.3. The third-order periodic solutions of 3-CNLSEs

For 3-CNLSEs, $N=3$. According to the analysis above, when $p=3$, the equation has a unique solution. A full number of possible three-component third-order solutions might be equal to 35. They are 765-, 764-, 763-, …, 421- and 321-type solutions respectively.

From the expression (12), we have

$$a_{10}^{(3)}(k)=\frac{[\gamma_1^{(3)}]^2}{k^2},\quad a_{11}^{(3)}(k)=\frac{2\gamma_1^{(3)}-[\gamma_1^{(3)}]^2}{k^2},\quad a_{12}^{(3)}(k)=\frac{1-2\gamma_1^{(3)}}{k^2},\quad a_{13}^{(3)}(k)=-\frac{1}{k^2}; \tag{29}$$

$$a_{20}^{(3)}(k)=\left(1-\frac{1}{k^2}\right)[\gamma_2^{(3)}]^2,\ a_{21}^{(3)}(k)=\left(2-\frac{2}{k^2}\right)\gamma_2^{(3)}+\frac{1}{k^2}[\gamma_2^{(3)}]^2,\ a_{22}^{(3)}(k)=1-\frac{1}{k^2}+\frac{2}{k^2}\gamma_2^{(3)},\ a_{23}^{(3)}(k)=\frac{1}{k^2}; \tag{30}$$

$$a_{30}^{(3)}(k)=0,\qquad a_{31}^{(3)}(k)=[\gamma_3^{(3)}]^2,\qquad a_{32}^{(3)}(k)=2\gamma_3^{(3)},\qquad a_{33}^{(3)}(k)=1; \tag{31}$$

$$a_{40}^{(3)}(k)=0,\quad a_{41}^{(3)}(k)=\frac{1}{k^2}-\frac{1}{k^4},\quad a_{42}^{(3)}(k)=\frac{2}{k^4}-\frac{1}{k^2},\quad a_{43}^{(3)}(k)=-\frac{1}{k^4}; \tag{32}$$

$$a_{50}^{(3)}(k)=\frac{[\gamma_5^{(3)}]^2}{k^2},\quad a_{51}^{(3)}(k)=\frac{2\gamma_5^{(3)}-[\gamma_5^{(3)}]^2}{k^2},\quad a_{52}^{(3)}(k)=\frac{1-2\gamma_5^{(3)}}{k^2},\quad a_{53}^{(3)}(k)=-\frac{1}{k^2}; \tag{33}$$

$$a_{60}^{(3)}(k)=\left(1-\frac{1}{k^2}\right)[\gamma_6^{(3)}]^2,\ a_{61}^{(3)}(k)=\left(2-\frac{2}{k^2}\right)\gamma_6^{(3)}+\frac{1}{k^2}[\gamma_6^{(3)}]^2,\ a_{62}^{(3)}(k)=1-\frac{1}{k^2}+\frac{2}{k^2}\gamma_6^{(3)},\ a_{63}^{(3)}(k)=\frac{1}{k^2}; \tag{34}$$

$$a_{70}^{(3)}(k)=0,\qquad a_{71}^{(3)}(k)=[\gamma_7^{(3)}]^2,\qquad a_{72}^{(3)}(k)=2\gamma_7^{(3)},\qquad a_{73}^{(3)}(k)=1. \tag{35}$$

Substituting any three expressions among the expressions (29)–(35) into Eqs.(14) and (15) yields unknown $A_n^{(p)}$, which in turn determine the functions $\rho_n^{(p)}(\xi)$ and u_n by the expressions (11) and (9), where the eigenfunctions $\Lambda_n^{(p)}(\xi)$ are given in Table 1. So, the 3rd-order periodic solutions u_n of 3-CNLSEs are obtained.

For example, substituting the expressions (31), (32) and (35) into Eqs.(14) and (15), the 3rd-order solutions of 743-type are obtained as follows:

$$u_{33}(x,t)=A_3\mathrm{dn}[\alpha(x-2\lambda_1 t)][\mathrm{dn}^2[\alpha(x-2\lambda_1 t)]+\gamma_3^{(3)}]\ \mathrm{e}^{\mathrm{i}(\lambda_1 x-\omega_1 t)},$$

$$\text{for}\ \ \omega_1-\lambda_1^2=-\alpha^2[5(2-k^2)-2\sqrt{1-k^2+4k^4}]; \tag{36}$$

$$u_{34}(x,t)=A_4\mathrm{sn}[\alpha(x-2\lambda_2 t)]\mathrm{cn}[\alpha(x-2\lambda_2 t)]\mathrm{dn}[\alpha(x-2\lambda_2 t)]\mathrm{e}^{\mathrm{i}(\lambda_2 x-\omega_2 t)},$$

$$\text{for}\ \ \omega_2-\lambda_2^2=-4\alpha^2(2-k^2); \tag{37}$$

$$u_{37}(x,t)=A_7\mathrm{dn}[\alpha(x-2\lambda_3 t)][\mathrm{dn}^2[\alpha(x-2\lambda_3 t)]+\gamma_7^{(3)}]\mathrm{e}^{\mathrm{i}(\lambda_3 x-\omega_3 t)},$$

$$\text{for}\ \ \omega_3-\lambda_3^2=-\alpha^2[5(2-k^2)+2\sqrt{1-k^2+4k^4}], \tag{38}$$

where

$$A_3=\pm\frac{\sqrt{15}\alpha}{2k^2}\sqrt{\frac{2-k^2+2\sqrt{1-k^2+4k^4}}{\sqrt{1-k^2+4k^4}}},$$

$$A_4=\pm\sqrt{15}\alpha,$$

$$A_7=\pm\frac{\sqrt{15}\alpha}{2k^2}\sqrt{\frac{k^2-2+2\sqrt{1-k^2+4k^4}}{\sqrt{1-k^2+4k^4}}},$$

$$\gamma_3^{(3)}=\frac{2k^2-4-\sqrt{1-k^2+4k^4}}{5},$$

$$\gamma_7^{(3)}=\frac{2k^2-4+\sqrt{1-k^2+4k^4}}{5}.$$

Substituting the expressions (30), (32) and (34) into Eqs.(14) and (15), the 3rd-order solutions of 642-type are obtained as follows:

$$u_{32}(x,t)=A_2\mathrm{cn}[\alpha(x-2\lambda_1 t)][\mathrm{dn}^2[\alpha(x-2\lambda_1 t)]+\gamma_2^{(3)}]\mathrm{e}^{\mathrm{i}(\lambda_1 x-\omega_1 t)},$$

$$\text{for}\ \ \omega_1-\lambda_1^2=\alpha^2(5-10k^2+2\sqrt{4-k^2+k^4}); \tag{39}$$

$$u_{34}(x,t)=A_4\mathrm{sn}[\alpha(x-2\lambda_2 t)]\mathrm{cn}[\alpha(x-2\lambda_2 t)]\mathrm{dn}[\alpha(x-2\lambda_2 t)]\mathrm{e}^{\mathrm{i}(\lambda_2 x-\omega_2 t)},$$

$$\text{for}\ \ \omega_2-\lambda_2^2=4\alpha^2(1-2k^2); \tag{40}$$

$$u_{36}(x,t)=A_6\mathrm{cn}[\alpha(x-2\lambda_3 t)][\mathrm{dn}^2[\alpha(x-2\lambda_3 t)]+\gamma_6^{(3)}]\mathrm{e}^{\mathrm{i}(\lambda_3 x-\omega_3 t)},$$

$$\text{for}\ \ \omega_3-\lambda_3^2=\alpha^2(5-10k^2-2\sqrt{4-k^2+k^4}), \tag{41}$$

where

$$A_2=\pm\frac{\sqrt{15}k\alpha}{2}\sqrt{\frac{-1+2k^2+2\sqrt{4-k^2+k^4}}{\sqrt{4-k^2+k^4}}},$$

$$A_4=\pm\sqrt{15}\alpha k^2,$$

$$A_6=\pm\frac{\sqrt{15}k\alpha}{2}\sqrt{\frac{1-2k^2+2\sqrt{4-k^2+k^4}}{\sqrt{4-k^2+k^4}}},$$

$$\gamma_2^{(3)} = \frac{k^2 - 3 - \sqrt{4 - k^2 + k^4}}{5},$$
$$\gamma_6^{(3)} = \frac{k^2 - 3 + \sqrt{4 - k^2 + k^4}}{5}.$$

To the author's knowledge, these third-periodic solutions of 3-CNLSEs have not ever been presented before. When $k \to 1$, $\mathrm{sn}(\xi) \to \tanh(\xi)$, $\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$, and $\mathrm{dn}(\xi) \to \mathrm{sech}(\xi)$, the solutions degenerate into the corresponding envelope solitary or shock wave solutions.

5. Discussion and conclusions

Now we summarize the overall conclusions of this paper.

1) An auxiliary Lamé equation method is proposed to construct the pth-order periodic solutions for a family of N-coupled nonlinear Schrödinger equations. It is shown that this method can be used to construct abundant pth-order periodical solutions of N-CNLSEs in a unified way. In principle, more higher-order periodic solutions of N-coupled nonlinear Schrödinger equations can be obtained by this method.

2) The pth-order periodic solutions depend on the eigenfunctions of the Lamé equation. Different eigenfunctions have different pth-order solutions. Not all pth-order periodic solutions are realizable. For example, the second-order solutions of 54-type and the third-order solutions of 541-type are never self-consistent.

3) Since Jacobi elliptic functions are more general than the hyperbolic tangent and secant functions, under the limit condition ($m \to 1$), our results can give corresponding envelope solitary or shock wave solutions.

4) This method is also a computational method, which allows us to perform complicated and tedious algebraic calculation with the aid of symbolic computation(using MATHEMATICA) on computer.

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