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On the Rosen–Edelstein model and the theoretical foundation of nonholonomic mechanics*

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A new model in nonholonomic mechanics, the Rosen–Edelstein model, has been studied. We prove that the new model is a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made stationary. The theoretical basis of nonholonomic mechanics is investigated and discussed. Finally, we give the range of practical applications of the Rosen–Edelstein model.

Keywords: Rosen–Edelstein model, variational principles, nonholonomic constraints
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1. Introduction

German scholar Hertz divided constraints systems into holonomic and nonholonomic in 1894.[1] From then on, nonholonomic mechanics was established.[2−6] There are two different familiar theories for nonholonomic mechanics systems. One is traditional nonholonomic model, namely, Gauss–Appell–Chetaev model, which is a perfect and popular theory based on the differential variational principles. The other is Vacco model presented by Kozlov[7] in 1983, which is a new model to study unintegrable dynamics. Kozlov's method is using variation with one fixed end. The motion satisfies constraint equations, and the equations of motion always have the form of Hamilton principle under this variation. Scholars who support this model think that there should be many different models either in mechanics with limited degrees of freedom or in continuum mechanics. The two different models have aroused furious disputes.

Israeli scholars Rosen and Edelstein presented a new model[8] which is different from the Gauss–Appell–Chetaev model and the Vacco model. This is a new equation obtained by using the generalized Hamilton principle and generalized Lagrangian. German scientist Hagedorn thought that the Rosen–Edelstein equation can be applied only to holonomic systems, not to nonholonomic constraints.[9]

It is beyond doubt that the Rosen–Edelstein model can be used for holonomic systems, but there are still many questions to be solved. In this paper, the Rosen–Edelstein model is studied. We will prove that the new model is a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made stationary. Furthermore, we will investigate the theoretical basis of nonholonomic mechanics. Finally, we discuss the practical application of the Rosen–Edelstein model.

2. Rosen–Edelstein model and its theoretical basis

A mechanical system with n generalized coordinates $q_s(s = 1, \cdots, n)$ to determine its position is subjected to g nonholonomic constraints

$$f_\beta(q_s, \dot{q_s}, t) \quad (\beta = 1, \cdots, g; s = 1, \cdots, n). \tag{1}$$

Lagrangian of the system is $L(q_s, \dot{q_s}, t)$; and the adjustable constraint is

$$\Phi_\beta = \omega \dot{f}_\beta + \varepsilon f_\beta, \tag{2}$$

where ω and ε are constants.

Construct the generalized Lagrangian

$$L_\mathrm{d}(q_s, \dot{q_s}, t) = T - V - \lambda \Phi_\beta, \tag{3}$$

where T is the kinetic energy of the system, $T = T_2 + T_1 + T_0$, V is potential energy, and λ is an undetermined multiplier. According to the generalized

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Hamilton principle, we have

$$\int_{t_0}^{t_1} \delta L \mathrm{d}t = 0. \tag{4}$$

Substituting L in Eq.(4) by L_d, we obtain

$$\int_{t_0}^{t_1} \delta(L - \lambda\Phi_\beta)\mathrm{d}t = 0. \tag{5}$$

Exchanging the order between the variation and integration yields

$$\delta \int_{t_0}^{t_1} (L - \lambda\Phi_\beta)\mathrm{d}t = 0. \tag{6}$$

Using the step-by-step method of integration and the exchange relationship presented by Hölder, we have

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L_\mathrm{d}}{\partial \ddot{q}_s}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\mathrm{d}}{\partial \dot{q}_s}\right) - \frac{\partial L_\mathrm{d}}{\partial q_s} = Q_\mathrm{ext}$$

or

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial \Phi_\beta}{\partial \ddot{q}_s}\lambda_\beta\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \Phi_\beta}{\partial \dot{q}_s}\lambda_\beta\right)\\ &- \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} + \frac{\partial \Phi_\beta}{\partial q_s}\lambda_\beta = Q_\mathrm{ext}.\end{aligned} \tag{7}$$

Equation (7) is a new model different from the traditional Gauss–Appell–Chetaev model and also different from the Vacco model.

The theoretical basis of the new equation (7) is the integral principle of nonholonomic mechanics—the generalized Hamilton principle. By exchanging the order between variation and integration, we can obtain Eq.(6) from Eq.(5). Therefore, the Rosen–Edelstein model is such a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made a stationary one among all the paths, and this model is an Euler equation obtained by using the variational approach. So the new model and the Vacco model both belong to the same type of equations. In the new model, the inserted term in the Hamilton principle (4) is the adjustable constraint (2). On the other hand, in the Vacco model, the inserted term is the constraint equations (1). So the two models are not equivalent.

Theorem 1 The Rosen–Edelstein model is a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made a stationary one among all the paths. Rosen–Edelstein model is an Euler equation obtained by using the variational approach.

Usually, there exist the undetermined multipliers $\ddot{\lambda}_\beta$ and $\dot{\lambda}_\beta$ in the new equation (7), so other initial conditions should be given:

$$\begin{aligned}\lambda_\beta(t_0) &= \lambda_\beta^0,\\ \dot{\lambda}_\beta(t_0) &= \dot{\lambda}_\beta^0.\end{aligned} \tag{8}$$

Even if λ_β can be eliminated, the order of equations become high, except for some special cases. Thus other initial conditions also should be given:

$$\begin{aligned}\ddot{q}_{s\beta}(t_0) &= \ddot{q}_{s\beta}^0,\\ \dddot{q}_{s\beta}(t_0) &= \dddot{q}_{s\beta}^0.\end{aligned} \tag{9}$$

The solutions obtained from the Gauss–Appell–Chetaev model can satisfy the determinate conditions of classical mechanics, namely, the states of motion of the systems can be determined by the states of motion of a point on the orbit. On the other hand, the solutions obtained from the Rosen–Edelstein model can only satisfy weak deterministic conditions, namely, the states of motion of the systems can be determined by the states of motion in some intervals of time.

3. The theoretical basis of nonholonomic mechanics

The variational principle of mechanics can be divided into two types: (1) differential variational principle; (2) integral variational principle. It is proved that all the differential variational principles can be used both in holonomic and nonholonomic constraints.[10] However, it is still not clear if integral variational principle, such as the Hamilton principle, can be used in nonholonomic systems .

3.1. Differential variational principle

Differential variational principles mainly include:

A d'Alembert–Lagrange principle

$$\sum_{i=1}^{N} (F_i - m_i\ddot{r}_i)\cdot \delta r_i = 0. \tag{10}$$

B Jourdain principle

$$\sum_{i=1}^{N} (F_i - m_i\ddot{r}_i)\cdot \delta \dot{r}_i = 0. \tag{11}$$

C Gauss principle

$$\sum_{i=1}^{N} (F_i - m_i\ddot{r}_i)\cdot \delta \ddot{r}_i = 0. \tag{12}$$

For the d'Alembert–Lagrange principle, we must define the restriction of constraints to virtual displacement δq_s, then the well known definition of Chetaev is given by

$$\sum_{i=1}^{N}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0. \tag{13}$$

By using the d'Alembert–Lagrange principle (10) and the definition of Chetaev (13) and Lagrange multiplier, we can obtain the traditional equation of nonholonomic systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_{\beta=1}^{r}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s = 1, \cdots, n). \tag{14}$$

By using the d'Alembert–Lagrange principle (10) and Niu's definition[11]

$$\sum_{i=1}^{N}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta \dot{q}_s = 0, \tag{15}$$

Eq.(14) can also be obtained.

Take derivative of Eq.(1) with respect to t, we have

$$\frac{\partial f_\beta}{\partial q_s}\delta \dot{q}_s + \frac{\partial f_\beta}{\partial \dot{q}_s}\delta \ddot{q}_s + \frac{\partial f_\beta}{\partial t} = 0. \tag{16}$$

By using Eq.(16) and Gauss variation

$$\sum_{i=1}^{N}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta \ddot{q}_s = 0, \tag{17}$$

and Lagrange multiplier, we can also obtain Eq.(14).

From the analysis above, it is easy to know that the traditional model of nonholonomic system (Gauss–Appell–Chetaev model) is compatible with these differential variational principles. Without question, the theoretical basis of nonholonomic mechanics should be the differential variational principle.

3.2. Integral variational principle

By integral variational principle is mainly meant the Hamilton principle.

Applying generalized coordinates to the d'Alembert–Lagrange principle (10) and integrating with respect to t and considering the boundary conditions, we have

$$\int_{t_1}^{t_2}\left\{\delta L + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_s) - \delta \dot{q}_s\right)\right\}\mathrm{d}t = 0. \tag{18}$$

Equation (18) is the universal form of Hamilton principle. It depends on the exchange relationship if Hamilton principle of nonholonomic systems has the same form as holonomic systems . There are two different exchange relationships. One is given by Suslov

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_\sigma) - \delta \dot{q}_\sigma = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta q_{\varepsilon+\beta}\right) - \delta \dot{q}_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon}T_\sigma^{\varepsilon+\beta}\delta q_\sigma.\end{aligned} \tag{19}$$

Substituting Eq.(19) into Eq.(18), we have

$$\int_{t_1}^{t_2}\left\{\delta L + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}\sum_{\sigma=1}^{\varepsilon}T_\sigma^{\varepsilon+\beta}\delta q_\sigma\right\}\mathrm{d}t = 0. \tag{20}$$

Equation(20) is the Hamilton principle of the form of Suslov which is obviously different from that of holonomic systems.

The other is the exchange relationship of Hölder

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_s) - \delta \dot{q}_s = 0. \tag{21}$$

Substituting Eq.(21) into Eq.(18), we can obtain the Hamilton principle of the form of Hölder

$$\int_{t_1}^{t_2}\delta L\mathrm{d}t = 0. \tag{22}$$

Equation(22) has the same form as the Hamilton principle of holonomic system, but δq_s are not independent of each other.

By using Hamilton principle Eq.(22) and Lagrange multiplier, it is easy to obtain the traditional equation of nonholonomic system (14).

Theorem 2 The theoretical basis of nonholonomic mechanics is the differential variational principle and Hamilton principle.

From the analysis above, we know that the theoretical basis of nonholonomic mechanics can also be the Hamilton principle. What we must pay attention to is that the Hamilton principle of nonholonomic systems Eqs.(20) and (22) cannot express the fact that the action integral $\int_{t_0}^{t_1}L\mathrm{d}t$ is the stationary one among all the paths satisfying the equations of constraints. But it does not mean that the Hamilton principle of nonholonomic systems is not the stationary action under any conditions. The sufficient and necessary condition to make Eq.(22) a stationary action principle is[2,3]

$$\sum_{s=1}^{n}\sum_{\beta=1}^{g}K_\beta\left(\frac{\partial f_\beta}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\delta q_s = 0. \tag{23}$$

The sufficient and necessary condition to make Eq.(20) a stationary action principle is

$$\sum_{\beta=1}^{g}\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}T_\sigma^{\varepsilon+\beta} = 0. \tag{24}$$

What needs to point out is that nonholonomic systems which satisfy conditions Eqs.(23) and (24) are only exceptions, such as the Appel–Hamel constrained system.

4. Range of practical application of the Rosen–Edelstein model

From the analysis above, we know that the Rosen–Edelstein model is a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made a stationary one among all the paths. Therefore, its range of practical application is: (A) holonomic systems; (B) nonholonomic systems which satisfy conditions Eq.(23) or Eq.(24).

5. Conclusions

In this paper the Rosen–Edelstein model of nonholonomic mechanics has been studied. We have analysed the theoretical basis of nonholonomic mechanics and prove that the Rosen–Edelstein model is essentially a Lagrange problem in which the action integral $\int_{t_0}^{t_1} L\mathrm{d}t$ can be made a stationary one among all the paths. We prove that the theoretical basis of nonholonomic mechanics is the differential variational principle and Hamilton principle. Finally, we give the range of practical application of the Rosen–Edelstein model.

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