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Dynamics of quintessential inflation*

Zhai Xiang-Hua(翟向华)† and Zhao Yi-Bin(赵一斌)

Shanghai United Center for Astrophysics, Shanghai Normal University, Shanghai 200234, *China*

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In this paper, we study a realistic model of quintessential inflation with radiation and matter. By the analysis of the dynamical system and numerical work about the evolution of the equation of state and cosmic density parameter, we show that this model is a good match for the current astronomical observation. The conclusion we obtain is in favour of the model where the modular part of the complex field plays the role of the inflaton whereas the argument part is the quintessence field. Numerical calculation shows that a heteroclinic orbit (solution of the dynamical system) is interpolated between early-time de Sitter phase (an unstable critical point) and a late-time de Sitter attractor.

Keywords: dark energy, inflation, attractor
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1. Introduction

Astronomical observations on the cosmic microwave background (CMB) anisotropy,[1] supernova type Ia (SNIa)[2] and SLOAN digital sky survey (SDSS)[3] concur that our Universe is spatially flat, with about 70 percent of the total density resulting from dark energy that has an equation of state $w < -1/3$ and drives the accelerating expansion of the Universe which began at a redshift of an order of one-half. The origin of the dark energy remains elusive from the point of view of general relativity and standard particle physics. Several candidates representing dark energy have been suggested and confronted with observations: cosmological constant, quintessence with a single field[4,5] or with N coupled field,[6−12] phantom field with canonical[13] or Born–Infield type Lagrangian,[14−18] k-essence[19] and generalized Chaplygin gas (GCG).[20−22] Among these models, the most typical ones are the cosmological constant and quintessence which has attracted much attention ever since its invention.

The idea of inflation is legitimately regarded as a great advance of modern cosmology: it solves the horizon, flatness and monopole problems, and also provides a mechanism for the generation of density perturbations needed to seed the formation of structures in the universe.[23,24] In standard inflationary models,[25] the physics lies in the inflation potential. The underlying dynamics is simply that of a single scalar field rolling in its potential. This scenario is generally referred to as chaotic inflation in reference to its choice of initial conditions. This picture is widely favoured because of its simplicity and has received by far the most attention to date. The properties of inflationary models are also tightly constrained by the recent result from the observation. The standard inflationary lambda-cold dark matter (ΛCDM) model provides a good fit to the observed CMB anisotropy. Peeples and Vilenkin[26] proposed and quantitatively analysed the intriguing idea that a substantial fraction of the present cosmic energy density could reside in the vacuum potential energy of the scalar field responsible for inflation (quintessential inflation). After that, there were some models presented in succession. However, most of these models involve non-renormalizable, special potentials which usually do not have a local minimum, making the conventional reheating process inoperative.

On the other hand, the Peccei–Quinn (PQ) symmetry[27] is the most elegant solution to the strong CP problem of quantum chromodynamics (QCD). The global $U(1)_{\rm PQ}$ is a spontaneous symmetry breaking (SSB). Weinberg[28] and Wilczek[29] pointed out that there is a Nambu–Goldstone boson, 'the axion', associated with SSB. Such a global symmetry often arises in supersymmetric[30] and superstring-inspired models.[31] A priori PQ symmetry breaking scale is

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†E-mail: kychz@shnu.edu.cn

arbitrary, which can take a value between 10^2 and 10^{19}GeV. In most axion models PQ symmetry breaking happens at a temperature $T \sim f$, when a complex scalar field Φ, which carries $U(1)_{\rm PQ}$ charge, develops a vacuum expectation value. At temperatures below $T \sim f$ the renormalizable potential for Φ is

$$V_1(\eta) = \frac{\lambda}{4}(\eta^2 - f^2)^2, \tag{1}$$

where $\eta = (\Phi^* \Phi)^{1/2}$. However, the argument of $\langle\Phi\rangle$ is left undetermined when $U(1)_{\rm PQ}$ has been broken because $V_1(\eta)$ is independent of $\arg\langle\Phi\rangle$. The massless ξ degree of freedom is the axion: $\xi = (f/\widetilde{N})\arg\langle\Phi\rangle$, where $\widetilde{N}$ is the colour anomaly of the PQ symmetry.[32] The axion is massless at $T \gg$ QCD scale, but it develops a mass due to instanton effects at low temperatures. In the $\widetilde{N} = 1$ case, the axion mass is easy to concretize by using a potential

$$V_2(\xi) = M^4[1 - \cos(\xi/f)], \tag{2}$$

where $M^4 = m_\xi^2 f^2$, and the axion develops a mass m_ξ. The conception that the modular part of a complex field with PQ symmetry may drive inflation is also not new.[33,34] Another version of a complex field was proposed as a realistic candidate for inflation[35,36] and, separately, for quintessence[37,38] in a natural fashion. The two-axion model has also been suggested for quintessence[39] or inflation.[40] The axion model can also be used for quintessence[41] and, separately, for inflation[42,43] from extra dimensions.

Recently, a fascinating model for quintessential inflation has been proposed by Rosenfeld and Frieman (RF)[44] with aid of the axion theory. In their model, a renormalizable complex scalar field Φ described by a Lagrangian with a $U(1)$ symmetry spontaneously broken at a high energy scale $f \sim 10^{19}$ GeV and explicitly broken by instanton effects at a much lower energy can account for both the early inflationary phase and the recent accelerated expansion of the Universe. The modular and argument parts of the field Φ were identified with the inflaton and the quintessence fields respectively.

In this paper, we put the emphasis on the study of the dynamics of RF model with the composition of radiation and matter. According to the phase space analysis and numerical calculation, we will show how the modular part of the complex field plays the role of the inflaton, whereas the argument part is the quintessence field. Phase space methods are particularly useful when the equations of motion are hard to solve analytically for the presence of radiation and dust matter. In fact, the numerical solutions under random initial conditions are not a satisfactory alternative because these may not reveal all the important properties. Therefore, combining the information from the critical points analysis with numerical solutions, one is able to give the complete classification of solutions according to their early-time and late-time behaviours. There are stable and unstable critical points for RF model with the composition of radiation and dust matter. Critical points are always exact constant solutions in the context of autonomous dynamical systems. These points are often the extreme points of the orbits and therefore can describe the asymptotic behaviours. In the present case, the unstable critical points correspond to a de Sitter behaviour at very first moments of the Universe. The stable critical points correspond to the second stage of accelerated expansion which began at a redshift $z \sim 1.5$ and is still operative, thus alleviates the fine tuning problem. The equation of state w varies with the cosmic evolution and approaches -1 asymptotically, showing the existence of a cosmological constant at late times. If the solutions are interpolated between the critical points, they can be divided into heteroclinic and homoclinic orbits. In our case, the heteroclinic orbit connects the unstable and stable critical points. Numerical calculations show that the Universe is brought back to the usual Friedman-Robertson-Walker expansion, then the second stage of accelerated expansion begins at $z \sim 1.5$.

2. Phase space of $U(1)$ quintessential inflation

For convenience, we investigate firstly the global structure of the dynamical system without radiation and matter. The equations of motion for RF model[44] are

$$\begin{aligned} &\ddot{\eta} + 3H\dot{\eta} - \frac{\dot{\xi}^2}{f^2}\eta + V_1'(\eta) = 0, \\ &\ddot{\xi} + \left(3H + 2\frac{\dot{\eta}}{\eta}\right)\dot{\xi} + \frac{f^2}{\eta^2}V_2'(\xi) = 0, \end{aligned} \tag{3}$$

and the Einstein equation is

$$H^2 = \frac{\kappa^2}{3}(\rho_\eta + \rho_\xi), \tag{4}$$

where $H = \dot{a}/a$ is Hubble parameter and $\kappa^2 = 8\pi G$. Here we define the energy densities:

$$\rho_\eta = \frac{1}{2}\dot{\eta}^2 + V_1(\eta),$$
$$\rho_\xi = \frac{1}{2}\frac{\eta^2}{f^2}\dot{\xi}^2 + V_2(\xi). \tag{5}$$

To gain more insight into the dynamical system, we introduce the new variables

$$u = \eta, \quad x = \xi,$$
$$v = \dot{\eta}, \quad y = \dot{\xi}, \tag{6}$$

then the equations of motion are reduced to

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= v,\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -3Hv + \frac{uy^2}{f^2} - V_1'(u),\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\left(3H + \frac{2v}{u}\right)y - \frac{f^2}{u^2}V_2'(x).
\end{aligned} \tag{7}$$

And H could be rewritten as

$$H = \left[\frac{\kappa^2}{3}\left(\frac{1}{2}v^2 + V_1(u) + \frac{1}{2}\frac{u^2}{f^2}y^2 + V_2(x)\right)\right]^{1/2}. \tag{8}$$

Here we have $V_1(u) = \dfrac{\lambda}{4}(u^2 - f^2)^2$ and $V_2(x) = M^4[1 - \cos(x/f)]$. Before we carry out the numerical study, we would analyse the system of equations qualitatively. The critical point of the autonomous system (7) is $(u_\mathrm{c}, 0, x_\mathrm{c}, 0)$, where $u_\mathrm{c}, x_\mathrm{c}$ are given by $V_1'(u_\mathrm{c}) = 0, V_2'(x_\mathrm{c}) = 0$ respectively. So one can obtain the critical points

$$(u_\mathrm{c}, v_\mathrm{c}, x_\mathrm{c}, y_\mathrm{c}) = (\pm f, 0, kf\pi, 0), \tag{9}$$

where k is any integer.

In order to investigate the stability of these critical points, we linearize Eqs.(7) about the critical points

$$\dot{\bar{y}} = A\bar{y}. \tag{10}$$

Here $\bar{y} = (u, v, x, y)^\mathrm{T}$. Then we have the stability matrix A expressed as

$$\begin{pmatrix}
0 & 1 & 0 & 0\\
-V_1''(u_\mathrm{c}) & -\sqrt{3\kappa^2 V_2(x_\mathrm{c})} & 0 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & -V_2''(x_\mathrm{c}) & -\sqrt{3\kappa^2 V_2(x_\mathrm{c})}
\end{pmatrix}. \tag{11}$$

The eigenvalues of the stability matrix are

$$\begin{bmatrix}
\frac{1}{2}(-\sqrt{3\kappa^2 V_2(x_\mathrm{c})} - \sqrt{3\kappa^2 V_2(x_\mathrm{c}) - 4V_1''(u_\mathrm{c})},\\
\frac{1}{2}(-\sqrt{3\kappa^2 V_2(x_\mathrm{c})} + \sqrt{3\kappa^2 V_2(x_\mathrm{c}) - 4V_1''(u_\mathrm{c})},\\
\frac{1}{2}(-\sqrt{3\kappa^2 V_2(x_\mathrm{c})} - \sqrt{3\kappa^2 V_2(x_\mathrm{c}) - 4V_2''(x_\mathrm{c})},\\
\frac{1}{2}(-\sqrt{3\kappa^2 V_2(x_\mathrm{c})} + \sqrt{3\kappa^2 V_2(x_\mathrm{c}) - 4V_2''(x_\mathrm{c})}
\end{bmatrix}. \tag{12}$$

It is clear that the critical points are always stable when k is even because the real part of all the eigenvalues is negative. So these critical points correspond to late-time attractor solutions. When k is odd, the fourth eigenvalue in expression (12) is always larger than zero and the corresponding critical points are unstable, which are exact constant solution with de Sitter behaviour at early-time of the Universe. That is to say, the dynamical system has stable critical points at the minimum of the potential $V_1(\eta)$ and the minimum of the potential $V_2(\xi)$. Next, we find the physical implications when the system is in the critical point regime. The cosmic parameters are

$$\Omega_\eta = \frac{\rho_\eta}{\rho_\mathrm{c}}, \quad \Omega_\xi = \frac{\rho_\xi}{\rho_\mathrm{c}}, \tag{13}$$

and the equation of state of the complex field is

$$w = \frac{v^2 + \dfrac{u^2}{f^2}y^2 - 2V_1(u) - 2V_2(x)}{2(\rho_\eta + \rho_\xi)}. \tag{14}$$

Clearly, from Eqs.(13) and (14), one can find that $w = -1$, $\Omega_\eta = 0$ and $\Omega_\xi = 1$ at the late-time attractor.

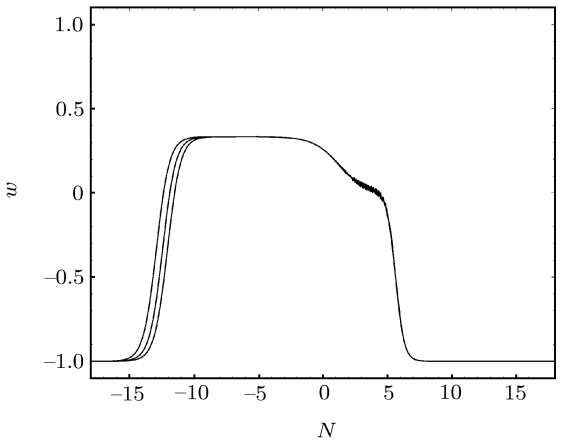


Fig.1. Evolution of the equation of state in the presence of radiation and matter. The curves are obtained under different initial conditions.

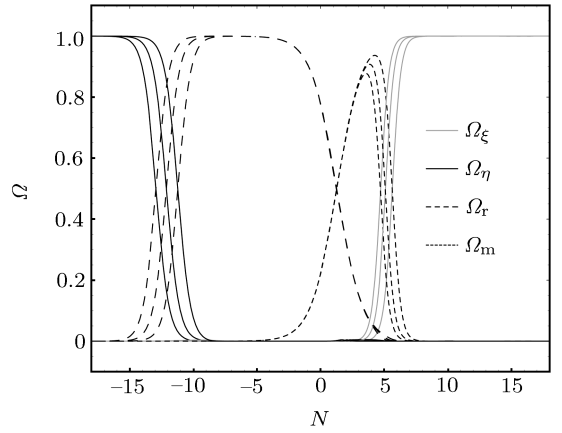


Fig.2. Evolution of cosmic density parameter of quintessence energy density Ω_ξ, inflaton energy density Ω_η, radiation energy density Ω_r and matter energy density Ω_m.The curves are obtained under different initial conditions.

3. Evolution in the presence of radiation and matter

If we are to search for a realistic model, we must know exactly how the complex scalar field Φ couples to ordinary matter. For convenience, we assume that the coupling is very weak following traditional ideas. In fact, the most significant feature of all the axion couplings is that they are proportional to $\widetilde{N}/f$: the larger the PQ SSB scale, the weaker the axion coupling. Furthermore, we have taken $f \sim 10^{19}$GeV so that couples can be neglected. Certainly, the ordinary matter still affect the evolution of Φ via their contribution to the general expansion of the Universe.

Now we solve the equations of motion via a numerical calculation in the presence of radiation and matter, and obtain the results that will confirm our qualitative analysis. To do this, we introduce the new variable

$$N = \ln a. \tag{15}$$

Therefore, Eqs.(7) are reduced to

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}N} = \frac{v}{H},\\
&\frac{\mathrm{d}v}{\mathrm{d}N} = -3v + \frac{uy^2}{f^2 H} - \frac{V_1'(u)}{H},\\
&\frac{\mathrm{d}x}{\mathrm{d}N} = \frac{y}{H},\\
&\frac{\mathrm{d}y}{\mathrm{d}N} = -(3 + \frac{2v}{uH})y - \frac{f^2 V_2'(x)}{u^2 H}.
\end{aligned} \tag{16}$$

and

$$H^2 = H_\mathrm{i}^2 E^2(N), \tag{17}$$

where H_i denotes the Hubble parameter at an initial time. $E(N)$ is defined as[45−47]

$$\begin{aligned}
E(N) = &\Big[\frac{\kappa^2}{3H_\mathrm{i}^2}\Big(\frac{1}{2}v^2 + V_1(u) + \frac{1}{2}\frac{u^2}{f^2}y^2 + V_2(x)\Big)\\
&+ \Omega_{\mathrm{r,i}}\mathrm{e}^{-4N} + \Omega_{\mathrm{m,i}}\mathrm{e}^{-3N}\Big]^{1/2}.
\end{aligned} \tag{18}$$

$\Omega_{\mathrm{r,i}}$ and $\Omega_{\mathrm{m,i}}$ are the cosmic density parameters for radiation and dust matter at the initial time respectively. Note that when N goes to be very large, or at late-time, the contribution to H from matter and radiation will become negligible.

Solving these equations will give us some insight into the evolution of the fields and the quantities of interest. The corresponding conclusions of numerical calculation are shown in Figs.1 and 2. This numerical solution describes a heteroclinic orbit, interpolated between an unstable de Sitter critical point and a late-time de Sitter attractor. Therefore, this model is more natural for explaining the two stages of acceleration. A point worth emphasizing is that ordinary matter (radiation and dust) affects the evolution of the scalar field via the contribution to the general expansion of the Universe because the couples can be neglected in the model. Figure 1 shows the behaviour of the equation of state. At the initial time, $w = -1$ corresponds to the de Sitter expansion (inflationary phase), then it increases and becomes positive. After arriving at the value 1/3, the Universe comes to the radiation-dominated epoch and w stays on a broad platform. Next, w drops to zero and stays on a narrow platform. Finally, w drops below zero and approaches -1, which corresponds to the second stage of accelerated expansion. The evolution of cosmic density parameters are

shown in Fig.2. The contribution of argument part Ω_ξ stays at $\Omega = 0$ at the very first moments of the Universe and becomes 1 at late-time, thus playing the role of quintessence field. On the other hand, the contribution of modular part Ω_η plays the role of inflaton. The radiation energy density and the dust matter energy density have their own dominant epochs during the whole evolution. Therefore, we see that the constraints arising from cosmological nucleosynthesis and structure formation are satisfied.

4. Conclusion and discussion

In this paper, we have discussed the cosmological implication of a complex scalar model of quintessence inflation with radiation and matter. Using numerical calculation for this model, we show that the argument contribution is negligible at the early epoch of the Universe while it becomes dominant with the time evolves and the evolution of the modular part being opposite, which is a viable way to unify the description of the inflationary stage and the current accelerated expansion. Analysis of the dynamical evolution of the complex scalar model indicates that it admits a late-time attractor solution, by which the field behaves as a cosmological constant. Obviously, attractor and heteroclinic orbit both alleviate the fine tuning problem. We can also extend our discussion to the Lagrangian with non-Abelian symmetry, which is an interesting extension that we will discuss in another work.

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