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Generation of four-atom cluster states in thermal cavity and implementing remote controlled not gate*

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We propose an experimentally feasible scheme for preparing a four-atom cluster state in a thermal cavity. In the scheme, the cavity field is only virtually excited and the photon-number-dependent part in the effective Hamiltonian is cancelled so that the system is insensitive to the cavity decay and the thermal field. At the same time, the scheme can be generalized to prepare n-atom cluster states with the success probability 100%. In addition, using the four-atom cluster state, we also propose a simpler scheme for implementing a remote–controlled not gate (CNOT) without the Bell states measurement.

Keywords: atomic cluster states, two-level atom, thermal cavity, remote CNOT gate

PACC: 4250

Entanglement is one of the most important features of quantum mechanics. It not only provides the possibility for testing quantum mechanics against a local hidden variable theory,[1−3] but also is useful in quantum information processing such as quantum teleportation,[4−7] dense coding,[8] quantum cryptography.[9,10] Recently, multi-particle entangled states, an important ingredient of entangled states, have attracted much attention since they also play an important role in quantum communication and quantum computing. As well as GHZ states and W states,[11] there are still several classes of multi-particles entangled states. Briegle and Ruassendorf[12] proposed a class of entangled states called cluster states and showed that cluster states with more than three qubits can be regarded as a resource for GHZ states and are more immune to decoherence.[2,3] An example of four-qubit cluster states is

$$|\phi\rangle_4 = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle\right)_{1234}. \quad (1)$$

Using cluster states, Raussendorf and Briegl[13] proposed the one-way quantum computer which is based on entangled states and single-qubit measurements instead of a large series of logic gates. Some schemes for preparing four-qubit cluster states[14−16] have been proposed in optical systems, but the success probability of them is in all cases less than 1/4. In this paper, we propose a scheme to generate the four-atom cluster state in a thermal cavity and generalize the scheme to prepare an n-atom cluster state, then we implement a remote-controlled not gate via the four-atom cluster state.

We consider two identical two-level atoms simultaneously interacting with a single-mode cavity and driven by a classical field. In the rotating-wave approximation, the Hamiltonian is ($\hbar = 1$)[8,17−19]

$$H = \omega_0 S_z + \omega_c a^+ a + \sum_{j=1}^{2}\left[g(a^+ S_j^- + a S_j^+) + \Omega(S_j^+ \mathrm{e}^{-\mathrm{i}\omega t} + S_j^- \mathrm{e}^{\mathrm{i}\omega t})\right], \quad (2)$$

where $S_z = \frac{1}{2}\sum_{j=1,2}(|e\rangle_{jj}\langle e| - |g\rangle_{jj}\langle g|)$, $S_j^+ = |e\rangle_{jj}\langle g| S_j^- = |g\rangle_{jj}\langle e|$; $|e\rangle_j$ ($|g\rangle_j$) is the excited (ground) state of the jth atom; $a^+(a)$ denotes the creation (annihilation) operation of the cavity mode; $g(\Omega)$ represents the atom-cavity coupling strength (the Rabi frequency of the classical field); ω_0, ω_c and ω denote the frequencies of the atomic transition, cavity mode and the classical field respectively. Assuming $\omega_0 = \omega$, the interaction Hamiltonian in the interaction

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picture reads

$$H_I = \Omega \sum_{j=1}^{2} (S_j^+ + S_j^-) + g \sum_{j=1}^{2} (\mathrm{e}^{-\mathrm{i}\delta t} a^+ S_j^+ + \mathrm{e}^{\mathrm{i}\delta t} a S_j^-), \quad (3)$$

where $\delta = \omega_0 - \omega_c$.

Assuming $\Omega \gg \delta, g$ and $\delta \gg g$, there is no energy exchange between the atomic system and the cavity. We can rewrite the interaction Hamiltonian (3) as

$$H_I = H_0 + H_{\mathrm{eff}}, \quad (4)$$

where $H_0 = \Omega \sum_{j=1}^{2} (S_j^+ + S_j^-)$; H_{eff} is the effective interaction Hamiltonian in the interaction picture and can be given by

$$H_{\mathrm{eff}} = \lambda \left[\frac{1}{2} \sum_{j=1}^{2} (|\mathrm{e}\rangle_{jj}\langle e| + |g\rangle_{jj}\langle g|) + \sum_{j,k=1, j\neq k}^{2} (S_j^+ S_k^+ + S_j^+ S_k^- + \mathrm{H.C.}) \right], \quad (5)$$

where $\lambda = \dfrac{g^2}{2\delta}$. Due to

$$\lfloor H_0, H_{\mathrm{eff}} \rfloor = 0, \quad (6)$$

the evolution operation of the systems is

$$U(t) = \mathrm{e}^{-\mathrm{i}H_0 t} \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}} t}. \quad (7)$$

Suppose that there are two atoms prepared in the state $|gg\rangle$, $|ee\rangle$, $|eg\rangle$ or $|ge\rangle$. Under the application of the unitary operation (7), the atomic states will evolve to the following states:

$$|gg\rangle \xrightarrow{U(t)} \mathrm{e}^{-\mathrm{i}\lambda t} \{\cos \lambda t \lfloor \cos^2 \Omega t |gg\rangle - \sin^2 \Omega t |ee\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|ge\rangle + |eg\rangle) \rfloor - \mathrm{i} \sin \lambda t [\cos^2 \Omega t |ee\rangle - \sin^2 \Omega t |gg\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|ge\rangle + |eg\rangle)] \}, \quad (8)$$

$$|ee\rangle \xrightarrow{U(t)} \mathrm{e}^{-\mathrm{i}\lambda t} \{\cos \lambda t \lfloor \cos^2 \Omega t |ee\rangle - \sin^2 \Omega t |gg\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|ge\rangle + |eg\rangle) \rfloor - \mathrm{i} \sin \lambda t [\cos^2 \Omega t |gg\rangle - \sin^2 \Omega t |ee\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|ge\rangle + |eg\rangle)] \}, \quad (9)$$

$$|ge\rangle \xrightarrow{U(t)} \mathrm{e}^{-\mathrm{i}\lambda t} \{\cos \lambda t \lfloor \cos^2 \Omega t |ge\rangle - \sin^2 \Omega t |eg\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|gg\rangle + |ee\rangle) \rfloor - \mathrm{i} \sin \lambda t [\cos^2 \Omega t |eg\rangle - \sin^2 \Omega t |ge\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|gg\rangle + |ee\rangle)] \}, \quad (10)$$

$$|eg\rangle \xrightarrow{U(t)} \mathrm{e}^{-\mathrm{i}\lambda t} \{\cos \lambda t \lfloor \cos^2 \Omega t |eg\rangle - \sin^2 \Omega t |ge\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|gg\rangle + |ee\rangle) \rfloor - \mathrm{i} \sin \lambda t [\cos^2 \Omega t |ge\rangle - \sin^2 \Omega t |eg\rangle - \mathrm{i} \sin \Omega t \cos \Omega t (|gg\rangle + |ee\rangle)] \}, \quad (11)$$

According to Eqs.(8)–(11), if we choose the interaction time t and Rabi frequency Ω appropriately to satisfy

$$\lambda t = \frac{\pi}{4}, \ \Omega t = \left(k + \frac{3}{4}\right) \pi, \ k = 0, 1, 2, \cdots \quad (12)$$

we will obtain

$$|+\rangle|+\rangle \rightarrow |+\rangle|+\rangle, \quad (13)$$

$$|+\rangle|-\rangle \rightarrow |+\rangle|-\rangle, \quad (14)$$

$$|-\rangle|+\rangle \rightarrow |-\rangle|+\rangle, \quad (15)$$

$$|-\rangle|-\rangle \rightarrow -|-\rangle|-\rangle, \quad (16)$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|e\rangle \pm |g\rangle)$.

In this way, we can implement a controlled phase gate, in which the phase changes only if the two atoms are in the state $|-\rangle|-\rangle$.

Now we propose a scheme of preparing the four-qubit cluster state with thermal cavities (see Fig.1). The operation processing can be separated into the following four steps.

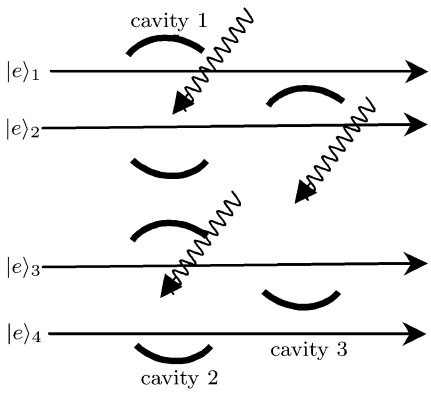


Fig.1. The circuit of generation of a four-atom cluster state. A curve arrow represents a classical field and a line arrow denotes a direction of the moving atom.

Firstly, we prepare the four atoms in the initial state

$$|\phi(0)\rangle_4 = |eeee\rangle_{1234}. \tag{17}$$

Secondly, atoms 1 and 2 (3 and 4) are simultaneously injected into the single-mode cavity 1 (2) and at the same time the two atoms are driven by a classical field. Choosing the interaction time t_1 and the Rabi frequency Ω to satisfy Eq.(12), the system state of Eq.(17) evolves to

$$|\phi(t_1)\rangle_4 = \frac{1}{2}(|e\rangle|+\rangle + |g\rangle|-\rangle)_{12} \otimes(|e\rangle|+\rangle + |g\rangle|-\rangle)_{34}. \tag{18}$$

Thirdly, after all the atoms leave the cavities, we send atoms 2 and 3 into the cavity 3 simultaneously and drive them by a classical field to implement a controlled phase gate described by Eqs.(13)–(16). Then the system state of Eq.(18) becomes

$$|\phi(t_2)\rangle_4 = \frac{1}{2}(|e\rangle|+\rangle|+\rangle|e\rangle + |e\rangle|+\rangle|-\rangle|g\rangle + |g\rangle|-\rangle|+\rangle|e\rangle - |g\rangle|-\rangle|-\rangle|g\rangle)_{1234}. \tag{19}$$

Fourthly, after the two atoms 2 and 3 exit from the cavity 3, the atoms are drilled through a classical field, separately, to implement a Hadamard gate operation

$$|e\rangle \to |+\rangle,\ \ \|g\rangle \to |-\rangle, \tag{20}$$

then the system will be transformed to

$$|\phi\rangle_4 = \frac{1}{2}(|e\rangle|e\rangle|e\rangle|e\rangle + |e\rangle|e\rangle|g\rangle|g\rangle + |g\rangle|g\rangle|e\rangle|e\rangle - |g\rangle|g\rangle|g\rangle|g\rangle)_{1234}, \tag{21}$$

which is obviously a four-atom cluster state described by Eq.(1) where we assume that $|e\rangle = |0\rangle$ and $|g\rangle = |1\rangle$.

The scheme can also be generalized to prepare an n-atom cluster state. We prepare the n atoms initially in the state $|ee\cdots e\rangle_{12\cdots n}(n = 2N \text{ or } 2N+1, N > 2)$. Let every two atoms (marked as atoms 1 and 2, 3 and 4,$\cdots$, $2N-1$ and $2N$) interact simultaneously with a corresponding single-mode cavity (marked as cavities 1, 2, $\cdots$, $N-1$ and N) and at the same time the two atoms in every cavity are driven by a classical field, the atomic state will evolve to the state which is similar to the state of Eq.(18). After all the atoms leave the cavities, the atoms 2 and 3 (4 and 5, ..., $2N-2$ and $2N-1$ or $2N$ and $2N+1$) are simultaneously driven into the corresponding cavity $N+1(N+2, \cdots, 2N-1$ or $2N)$ to implement the controlled phase gate. It is noted that if the atom number is $2N$, the $2N$th atom need not be driven into the cavity. In other words, there are $2N-1$ cavities needed if $n = 2N$. After that, a Hadamard gate operation is performed on the atoms 2, 3, 5, 6, ..., n. Then, we can obtain an n-atom cluster state with a simple formula[5]

$$|\phi\rangle_n = \frac{1}{\sqrt{2}}(|\phi\rangle_{n-1}|0\rangle_n + |\eta\rangle_{n-1}|1\rangle_n), n \geq 5 \tag{22}$$

where $|\phi\rangle_4$ is a four-atom cluster state expressed by Eq.(1) and

$$|\eta\rangle_4 = \frac{1}{2}(|e\rangle|e\rangle|e\rangle|g\rangle + |e\rangle|e\rangle|g\rangle|e\rangle + |g\rangle|g\rangle|e\rangle|g\rangle - |g\rangle|g\rangle|g\rangle|e\rangle)_{1234}, \tag{23}$$

$$|\eta\rangle_n = \frac{1}{\sqrt{2}}(|\phi\rangle_{n-1}|0\rangle_n - |\eta\rangle_{n-1}|1\rangle_n). \quad (24)$$

Now, let us give a brief discussion on the experiments. In our scheme, the required interaction time between the two atoms and the cavity field is in units of $T \approx 10^{-4}$s, while the radiation time of the Rydberg atoms with principal quantum numbers 49, 50, 51 and the coupling constant are $T_{\rm r} = 3 \times 10^{-2}$s and $g = 2\pi \times 24$kHz respectively. Thus, we have enough time to complete the whole procedure. Meanwhile, it is noted that the atomic state evolution is independent of the cavity field state, thus the proposed scheme might be realizable based on the current cavity QED techniques.

In the following, we propose a simpler scheme for implementing a remote CNOT gate via the four-atom cluster state. Compared with the schemes in Ref.[15,20], our scheme does not need the measurement of Bell states.

Suppose that Alice and Bob share the four-atom cluster state of Eq.(1) in which atoms 1 and 3 belong to Alice and atoms 2 and 4 belong to Bob. Alice implements the CNOT gate between the atoms 2 and 4 with the help of Bob, and they make some operation on the atoms as follows.

Alice injects atom 1 (3) into a classical field $1'(2')$. Choosing appropriate phase of the classical field and interaction time, Alice can implement the following transformations:

$$\begin{aligned}
&|0\rangle_1 \to \alpha|0\rangle_1 + \beta|1\rangle_1;\\
&|1\rangle_1 \to \beta|0\rangle_1 - \alpha|1\rangle_1;\\
&|0\rangle_3 \to \frac{\gamma+\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta-\gamma}{\sqrt{2}}|1\rangle_3;\\
&|1\rangle_3 \to \frac{\gamma-\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta+\gamma}{\sqrt{2}}|1\rangle_3, \qquad (25)
\end{aligned}$$

where $|\alpha\rangle^2 + |\beta\rangle^2 = 1$, $|\gamma\rangle^2 + |\delta\rangle^2 = 1$, $\alpha, \beta, \gamma, \delta$ are all real.

$$\begin{aligned}
|\varphi\rangle = \ & \frac{1}{2}\left[(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \left(\frac{\gamma+\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta-\gamma}{\sqrt{2}}|1\rangle_3\right)|00\rangle_{24}\right.\\
& +(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \left(\frac{\gamma-\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta+\gamma}{\sqrt{2}}|1\rangle_3\right)|01\rangle_{24}\\
& +(\beta|0\rangle_1 - \alpha|1\rangle_1) \otimes \left(\frac{\gamma+\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta-\gamma}{\sqrt{2}}|1\rangle_3\right)|10\rangle_{24}\\
& \left.- (\beta|0\rangle_1 - \alpha|1\rangle_1) \otimes \left(\frac{\gamma-\delta}{\sqrt{2}}|0\rangle_3 + \frac{\delta+\gamma}{\sqrt{2}}|1\rangle_3\right)|11\rangle_{24}\right], \qquad (26)
\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}
|\varphi\rangle = \ & \frac{1}{2\sqrt{2}}\{[\alpha(\gamma+\delta)|00\rangle + \alpha(\gamma-\delta)|01\rangle + \beta(\gamma+\delta)|10\rangle - \beta(\gamma-\delta)|11\rangle]_{24}|00\rangle_{13}\\
& +[\alpha(\delta-\gamma)|00\rangle + \alpha(\gamma+\delta)|01\rangle + \beta(\delta-\gamma)|10\rangle - \beta(\delta+\gamma)|11\rangle]_{24}|01\rangle_{13}\\
& +[\beta(\gamma+\delta)|00\rangle + \beta(\gamma-\delta)|01\rangle - \alpha(\gamma+\delta)|10\rangle + \alpha(\gamma-\delta)|11\rangle]_{24}|10\rangle_{13}\\
& + [\beta(\delta-\gamma)|00\rangle + \beta(\delta+\gamma)|01\rangle - \alpha(\delta-\gamma)|10\rangle + \alpha(\delta+\gamma)|11\rangle]_{24}|11\rangle_{13}\}. \qquad (27)
\end{aligned}$$

Subsequently, Alice detects atoms 1 and 3 in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and informs Bob of the measurement results via a classical channel. Then Bob performs a unitary operation $U_{24} = U^{(2)} \otimes U^{(4)}$ on atoms 2 and 4 based on the measurement result. Relation of atoms 1 and 3's measurement result (R_{13}), the states of the atoms 2 and 4 ($|\chi\rangle_{24}$) and the unitary operation (U_{24}) is given in Table 1, where $I^{(\rm i)}$ is the identical operator; $\sigma_x^{(\rm i)}, \sigma_y^{(\rm i)}, \sigma_z^{(\rm i)}$ are Pauli operators; $H^{(i)}$ is Hadamard gate. The superscript (i) stands for the operation on the ith atom.

After that, Alice will always obtain the same output state

$$\alpha|0\rangle_2(\gamma|0\rangle + \delta|1\rangle)_4 + \beta|1\rangle_2(\gamma|1\rangle + \delta|0\rangle)_4. \quad (28)$$

This is exactly the state that is obtained by applying the CNOT gate to the target qubit $(\gamma|0\rangle + \gamma|1\rangle)_4$ and the control qubit $(\alpha|0\rangle + \beta|1\rangle)_2$.

Since the technology of a single atom interacting with a classical field and the single atomic state measurement are well developed, this scheme may be easily demonstrated in current technology after the four-atom cluster state has been prepared.

Table 1. Relation of atoms 1 and 3's measurement result (R_{13}), the states of the atoms 2 and 4 ($|\chi\rangle_{24}$) and the unitary operation (U_{24}).

R_{13}	$\|\chi\rangle_{24}$	U_{24}
$\|00\rangle_{13}$	$[\alpha(\gamma+\delta)\|00\rangle+\alpha(\gamma-\delta)\|01\rangle+\beta(\gamma+\delta)\|10\rangle-\beta(\gamma-\delta)\|11\rangle]_{24}$	$I^{(2)}\otimes H^{(4)}$
$\|01\rangle_{13}$	$[\alpha(\delta-\gamma)\|00\rangle+\alpha(\gamma+\delta)\|01\rangle+\beta(\delta-\gamma)\|10\rangle-\beta(\delta+\gamma)\|11\rangle]_{24}$	$\sigma_z^{(2)}\otimes\sigma_x^{(4)}\sigma_z^{(4)}H^{(4)}$
$\|10\rangle_{13}$	$[\beta(\gamma+\delta)\|00\rangle+\beta(\gamma-\delta)\|01\rangle-\alpha(\gamma+\delta)\|10\rangle+\alpha(\gamma-\delta)\|11\rangle]_{24}$	$\sigma_x^{(2)}\sigma_z^{(2)}\otimes\sigma_x^{(4)}H^{(4)}$
$\|11\rangle_{13}$	$[\beta(\delta-\gamma)\|00\rangle+\beta(\delta+\gamma)\|01\rangle-\alpha(\delta-\gamma)\|10\rangle+\alpha(\delta+\gamma)\|11\rangle]_{24}$	$\sigma_x^{(2)}\otimes\sigma_z^{(4)}H^{(4)}$

In conclusion, we have proposed a scheme of preparing a four-atom cluster state in thermal cavity and generalized it for preparing an n-atom cluster state. Compared with previous schemes, our scheme has two distinct advantages. First, the success probability is 100% which the optical system can never reach. Second, the cavity field is only virtually excited and the photon-number-dependent part in the effective Hamiltonian is cancelled. Thus it is insensitive to cavity decay and the thermal field. Moreover, we also propose a simpler scheme for implementing the remote CNOT using the four-atom cluster state without the measurement of Bell states. Therefore, the schemes may be realized using the current technology.

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