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Hawking radiation of a uniformly accelerating black hole*

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In this paper, we study the Hawking radiation via tunnelling from a uniformly accelerating black hole. Although the Bekenstein–Hawking entropy is proportional also to the area of the event horizon, the radius of it, $r_{\rm H}$, is a function of θ, which leads to the difficulties in the calculation of the emission rate. In order to overcome the mathematical difficulties, we propose a new technique to calculate the emission rate and the result obtained is reasonable.

Keywords: tunnelling effect, Arnowitt–Deser–Misner mass, event horizon
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1. Introduction

In the 1970s, Hawking[1] discovered that the basic principles of quantum field theory lead to the emission of thermal radiation from a classical black hole, which gives rise to the information loss paradox of black hole physics.

Recently, Parikh and Wilczek proposed a method in order to solve the paradox above.[2−4] Hawking radiation is described as a tunnelling process and self-interaction effect is taken into account in their method. Finally, they obtained a leading correction to the semiclassical emission rate.

The research work concerned gained much progress, such as the works of Hemming,[5] Medved[6] and Zhang.[7−9] But no one of them has studied the tunnelling effect from the uniformly accelerating black hole so far. The main difficulty is that the event horizon of the uniformly accelerating black hole, $r_{\rm H}$, is a function of θ. As a result, the area of the event horizon, $A_{\rm H}=\int_{r_{\rm H}}\sqrt{-g}{\rm d}\theta{\rm d}\varphi=\int r_{\rm H}^2(\theta)\sin\theta{\rm d}\theta{\rm d}\varphi$, could not be worked out. In addition, in order to describe the tunnelling process across the event horizon, we should choose a suitable coordinate system in which all the components of the metric are well behaved at the event horizon. Although none of the components are singular when we adopt the advanced Eddington coordinates, the event horizon and its time-like limit surface are not located at the same place. But Hawking radiation occurs at the event horizon, not at the time-like limit surface. So, we have to find a coordinate system in which the event horizon is identical to its time-like limit surface first. It is also tenable that the entropy of the black hole is proportional to the area of the event horizon.[10] The entropy, however, is only written as $S_{\rm BH}=\frac{1}{4}A_{\rm H}=\frac{\pi}{2}\int_0^{\pi}r_{\rm H}^2(\theta)\sin\theta{\rm d}\theta$. In order to overcome the above difficulty, we could divide the event horizon into innumerable small parts and add up the imaginary part of the action in every small part. Then, we obtain the emission rate, $\Gamma\sim\exp(-2{\rm Im}S)=\exp(\Delta S_{\rm BH})$, which is in the same form as that in Ref.[2]–[4] Throughout this paper, the units ($G=C=\hbar=1$) are used.

2. Suitable coordinates

The line elements of the uniformly accelerating black hole in the advanced Eddington coordinates is written as[10,11]

$$\begin{aligned}{\rm d}s^2=&-\left(1-2ar\cos\theta-r^2a^2\sin\theta-\frac{2m}{r}\right){\rm d}v^2\\&+2{\rm d}v{\rm d}r-2r^2a\sin\theta{\rm d}v{\rm d}\theta\\&+r^2\left({\rm d}\theta^2+\sin^2\theta{\rm d}\varphi^2\right),\end{aligned}\tag{1}$$

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where a is the value of acceleration. The determinant of the metric and the area of the event horizon are, respectively,

$$\begin{aligned} g =& -r^4 \sin^2\theta, \\ A_{\rm H} =& \int r_{\rm H}^2(\theta)\sin\theta {\rm d}\theta {\rm d}\varphi \\ =& 2\pi \int_0^\pi r_{\rm H}^2(\theta)\sin\theta {\rm d}\theta. \end{aligned} \tag{2}$$

From the null surface equation

$$g^{\mu\nu}\frac{\partial Y}{\partial x^\mu}\frac{\partial Y}{\partial x^\nu}=0, \tag{3}$$

we obtain the equation of the event horizon

$$1-2ar_{\rm H}\cos\theta-\frac{2m}{r_{\rm H}}-(2a\sin\theta)\,r'_{\rm H}+\frac{r'^2_{\rm H}}{r^2_{\rm H}}=0, \tag{4}$$

where $r'_{\rm H}\equiv\dfrac{{\rm d}r_{\rm H}}{{\rm d}\theta}$. From the above equation, we find that the radius of the event horizon, $r_{\rm H}$, is a function of θ. There are many papers which prove that the relations between the entropy and the area of the event horizon, $S_{\rm BH}=\dfrac{1}{4}A_{\rm H}$, are also tenable,[10,12,13] and obtain the entropy

$$S_{\rm BH}=\frac{\pi}{2}\int_0^\pi r_{\rm H}^2(\theta)\sin\theta {\rm d}\theta. \tag{5}$$

If $r_{\rm H}$ is a function of θ, it manifests that the entropy $S_{\rm BH}$ could only be written as an integral of θ, not as the usual form, $S_{\rm BH}=\pi r_{\rm H}^2$, thus causing difficulties in calculating the emission rate.

From Eqs.(1) and (4), although none of the components of the metric are singular at the event horizon, it is shown that the event horizon and the time-like limit surface are not located at the same place. So, we have to introduce a new coordinate

$$R=r-r_{\rm H}(\theta). \tag{6}$$

Substituting Eq.(6) into Eq.(1), the line element is rewritten as

$$\begin{aligned} {\rm d}s^2 =& -\left(1-2ar\cos\theta-r^2a^2\sin\theta-\frac{2m}{r}\right){\rm d}v^2 \\ &+2{\rm d}v{\rm d}R-2\left(r^2a\sin\theta-r'_{\rm H}\right){\rm d}v{\rm d}\theta \\ &+r^2\left({\rm d}\theta^2+\sin^2\theta{\rm d}\varphi^2\right) \\ =& g'_{00}{\rm d}v^2+2{\rm d}v{\rm d}R+2g'_{02}{\rm d}v{\rm d}\theta \\ &+g'_{22}{\rm d}\theta^2+g'_{33}{\rm d}\varphi^2 \\ =& \left(g'_{00}-\frac{g'^2_{02}}{g'_{22}}\right){\rm d}v^2+2{\rm d}v{\rm d}R \\ &+\frac{g'^2_{02}}{g'_{22}}\left(\frac{g'_{22}}{g'_{02}}{\rm d}\theta+{\rm d}v\right)^2+g'_{33}{\rm d}\varphi^2. \end{aligned} \tag{7}$$

Adopting the dragging coordinate system, $\dfrac{{\rm d}\theta}{{\rm d}v}=-\dfrac{g'_{02}}{g'_{22}}$, the line element (7) becomes

$$\begin{aligned} {\rm d}s^2 =& \left(g'_{00}-\frac{g'^2_{02}}{g'_{22}}\right){\rm d}v^2+2{\rm d}v{\rm d}R+g'_{33}{\rm d}\varphi^2 \\ =& \hat{g}_{00}{\rm d}v^2+2{\rm d}v{\rm d}R+\hat{g}_{33}{\rm d}\varphi^2. \end{aligned} \tag{8}$$

From $\hat{g}_{00}=0$ in Eq.(8), we obtain the equation of the event horizon. Letting ${\rm d}s^2=0$ in Eq.(8), we obtain the radial null geodesics

$$\begin{aligned} \dot{R}\equiv\frac{{\rm d}R}{{\rm d}v} =& \frac{1}{2}\Bigg(1-2ar\cos\theta-\frac{2m}{r} \\ &-(2a\sin\theta)\,r'_{\rm H}+\frac{r'^2_{\rm H}}{r^2}\Bigg). \end{aligned} \tag{9}$$

In our discussion, we consider the particle as a shell (a non-spherically symmetric shell) of energy ω. If the particle's self-gravitation is taken into account, Eqs.(8) and (9) should be modified. We let the total mass of the space–time be fixed and the hole mass fluctuate. Then, we should replace m with $m-\omega$ in the metric (8) and the geodesic Eq.(9) to describe the motion of the shell.

3. Tunnelling effect

Because in the new coordinate system (v,R,θ,φ), the event horizon and the time-like surface are located at the same place, the infinite blue shift occurs near the horizon. Hence the geometrical optics limit becomes an especially reliable approximation and the Wentzel–Kramers–Brillouin formula could be used. According to the quantum theory, the emission rate Γ could be written as

$$\Gamma\sim\exp\left(-2\,{\rm Im}\,S\right). \tag{10}$$

We first discuss the Schwarzschild black hole in order to illuminate our method more explicitly. We could divide the shell into innumerable small parts and calculate the imaginary part of the action in every small part ${\rm Im}{\rm d}S$. Then adding up all the ${\rm Im}{\rm d}S$ in the small parts, we could work out the total ${\rm Im}S$. The imaginary part of the action in a small part, the region of $\theta\to\theta+{\rm d}\theta$, $\varphi\to\varphi+{\rm d}\varphi$, for an outgoing positive energy particle which crosses the horizon outwards from $r_{\rm in}$ to $r_{\rm out}$, can be expressed as

$$\begin{aligned} {\rm Im}{\rm d}S =& {\rm Im}\int_{r_{\rm in}}^{r_{\rm out}} p_r{\rm d}r\left(\sin\theta\right)\frac{{\rm d}\theta{\rm d}\varphi}{4\pi} \\ =& {\rm Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_r}{\rm d}p'_r{\rm d}r\left(\sin\theta\right)\frac{{\rm d}\theta{\rm d}\varphi}{4\pi}, \end{aligned} \tag{11}$$

where p_r is the canonical momentum conjugate to r, $r_{\rm in} = r_{\rm H}$ is the initial radius of the black hole, and $r_{\rm out} = \tilde{r}_{\rm H}$ is the final radius of the hole, with $\tilde{r}_{\rm H} = \tilde{r}_{\rm H}(m - \omega')$. And $\sin\theta {\rm d}\theta {\rm d}\varphi$ is the solid angle in the region from θ to $\theta + {\rm d}\theta$ and from φ to $\varphi + {\rm d}\varphi$. If integrating over θ and φ, we obtain

$$\mathrm{Im}S = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_r}\int_0^{\pi}\int_0^{2\pi} {\rm d}p_r' {\rm d}r\,(\sin\theta)\,\frac{{\rm d}\theta {\rm d}\varphi}{4\pi}. \tag{12}$$

Because of the symmetry of the Schwarzschild black hole, ${\rm d}p_r'$ is not a function of θ and φ. So, when we integrate over θ and φ from 0 to π and from 0 to 2π respectively in Eq.(12), we obtain

$$\mathrm{Im}S = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_r} {\rm d}p_r' {\rm d}r, \tag{13}$$

which is in the same form as that in Parikh's paper.[2]

Now let us study the tunnelling effect from the uniformly accelerating black hole. We substitute Hamilton's equation $\dot{R} = \dfrac{{\rm d}H}{{\rm d}p_R}\,|_R$ (as we adopt the new coordinate system v, R, θ, φ) into Eq.(12), change the variable from momentum to energy, and switch the order of integration to obtain

$$\begin{aligned}&\mathrm{Im}\int_m^{m-\omega}\int_{R_{\rm in}}^{R_{\rm out}}\int_0^{\pi}\int_0^{2\pi}\frac{{\rm d}R}{\dot{R}}{\rm d}H\,(\sin\theta)\,\frac{{\rm d}\theta{\rm d}\varphi}{4\pi}\\ =&\mathrm{Im}\int_0^{\omega}\int_{R_{\rm in}}^{R_{\rm out}}\int_0^{\pi}\int_0^{2\pi}\frac{2{\rm d}R}{1-2ar\cos\theta-\dfrac{2(m-\omega')}{r}-(2a\sin\theta)\,r_{\rm H}'+\dfrac{r_{\rm H}'^2}{r^2}}\,(-{\rm d}\omega')\,(\sin\theta)\,\frac{{\rm d}\theta{\rm d}\varphi}{4\pi},\end{aligned} \tag{14}$$

where $R_{\rm in} = r\,|_{r_{\rm H}} - r_{\rm H} = 0$ is the initial radius of the black hole, and $R_{\rm out} = \tilde{r}_{\rm H} - r_{\rm H} = -\delta$ is the final radius of the hole, with $\tilde{r}_{\rm H} = r_{\rm H}(m - \omega)$. We have used the modified Eq.(9) and H is the Arnowitt–Deser–Misner (ADM) energy of the space–time.[3] The minus sign appears because $H = m - \omega$. As $r_{\rm H}$ is a function of θ in the uniformly accelerating space–time, we could not calculate the imaginary part of the action, ImS, as in the calculation in other kind of space–time, such as Schwarzschild,[2] Kerr[7] and de Sitter space–time.[6] We integrate over φ first in the calculation of the emission rate because of the symmetry ($r_{\rm H}$ is not a function of φ) and obtain

$$\begin{aligned}\mathrm{Im}S =&\mathrm{Im}\int_0^{\omega}\int_{R_{\rm in}}^{R_{\rm out}}\int_0^{\pi}\int_0^{2\pi}\frac{2{\rm d}R}{1-2ar\cos\theta-\dfrac{2m}{r}-(2a\sin\theta)\,r_{\rm H}'+\dfrac{r_{\rm H}'^2}{r^2}}\,(-{\rm d}\omega')\,\frac{\sin\theta{\rm d}\theta{\rm d}\varphi}{4\pi}\\ =&\mathrm{Im}\int_0^{\omega}\int_{R_{\rm in}}^{R_{\rm out}}\int_0^{\pi}\frac{2{\rm d}R}{1-2ar\cos\theta-\dfrac{2m}{r}-(2a\sin\theta)\,r_{\rm H}'+\dfrac{r_{\rm H}'^2}{r^2}}\,(-{\rm d}\omega')\,\frac{\sin\theta{\rm d}\theta}{2}.\end{aligned} \tag{15}$$

Integrating R and ω', we obtain

$$\begin{aligned}\mathrm{Im}S =& -\frac{\pi}{2}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{\pi}(R + r_{\rm H})\sin\theta{\rm d}R{\rm d}\theta\\ =& -\frac{\pi}{4}\int_0^{\pi}\left(\tilde{r}_{\rm H}^2 - r_{\rm H}^2\right)\sin\theta{\rm d}\theta.\end{aligned} \tag{16}$$

The entropies of the black hole[10] before and after the emission are, respectively,

$$S_{\rm BH} = \frac{\pi}{2}\int_0^{\pi} r_{\rm H}^2(\theta)\sin\theta{\rm d}\theta, \tag{17}$$

$$\tilde{S}_{\rm BH} = \frac{\pi}{2}\int_0^{\pi}\tilde{r}_{\rm H}^2(\theta)\sin\theta{\rm d}\theta. \tag{18}$$

The tunnelling rate is therefore

$$\Gamma \sim \exp(-2\mathrm{Im}S) = \exp(\Delta S_{\rm BH}). \tag{19}$$

4. Conclusion and discussion

There has been no research work about the tunnelling effect from the black holes whose event horizon $r_{\rm H}$ is a function of θ. By dividing the event horizon

into innumerable small parts and adding up all the imaginary parts of the action in the small parts, we obtain the total ImS, overcoming the difficulties that the event horizon $r_{\rm H}$ is a function of θ. Then we work out the emission rate correctly. In our opinion, our method could be used for other kinds of black holes whose event horizon is a function of θ and even φ. Our results are in agreement with the underlying unitary theory.

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