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Heisenberg algebra for noncommutative Landau problem*

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The Landau problem on non-commutative quantum mechanics is studied, where the Heisenberg algebra and the Landau energy levels as well as the non-commutative angular momentum are constructed in detail in non-commutative space and non-commutative phase space respectively.

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1. Introduction

In the investigation of physics in a non-commutative (NC) space[1−4] motivated by arguments of string theory, apart from field theory, non-commutative quantum mechanics (NCQM) has recently attracted much attention;[5−10] this is because not only is the NC space necessary to the study of the low energy effective theory of D-brane with B field background, but also because of the occurrence of the effect of non-commutativity of both space–space and momentum–momentum on a very tiny string scale or in a very high energy situation. Many interesting topics of NC quantum theory have been extensively investigated, from the Aharonov–Bohm effect to the quantum Hall effect.[11−16]

In the NC space, coordinate and momentum operators satisfy the commutation relations

$$[\widehat{x}_i, \widehat{x}_j] = \mathrm{i}\theta_{ij}, \quad [\widehat{x}_i, \widehat{p}_j] = \mathrm{i}\hbar\delta_{ij},$$
$$[\widehat{p}_i, \widehat{p}_j] = 0, \tag{1}$$

where θ_{ij} is an anti-symmetric matrix with very small elements representing the non-commutativity property of space in the NC space. In a general commutative space the operators $\hat{x}_i$ and $\hat{p}_i$ in Eq.(1) have the following representation forms:

$$\widehat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij}p_j,$$
$$\widehat{p}_i = p_i. \tag{2}$$

The case in which both space–space and momentum–momentum are non-commuting[17] is different from the case where only the space–space is non-commuting, which is called the non-commutative phase space. In the NC phase space the momentum operators satisfy the commutation relation

$$[\widehat{p}_i, \widehat{p}_j] = \mathrm{i}\overline{\theta}_{ij}, \tag{3}$$

where $\overline{\theta}_{ij}$ is an anti-symmetric matrix with very small elements representing the non-commutativity property of momentum in the NC phase space. The $\widehat{x}_i$ and $\widehat{p}_i$ have the following representations:

$$\widehat{x}_i = \alpha x_i - \frac{1}{2\alpha\hbar}\theta_{ij}p_j,$$
$$\widehat{p}_i = \alpha p_i + \frac{1}{2\alpha\hbar}\overline{\theta}_{ij}x_j. \tag{4}$$

Here α is a scaling constant related to the non-commutativity of phase space. When $\overline{\theta} = 0$, we can obtain $\alpha = 1$,[18] which is the case that is extensively studied in the literature, where the space–space is non-commuting while momentum–momentum is commuting.

2. Landau problem in NC space

Consider a charged particle with electric charge q and mass m moving in a two-dimensional plane under a uniform magnetic field B perpendicular to the

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plane, and the magnetic vector potential will have the following form:

$$A_1 = -\frac{1}{2}Bx_2, \quad A_2 = \frac{1}{2}Bx_1. \tag{5}$$

The Hamiltonian of the system is

$$\begin{aligned} H =& \frac{1}{2m}\left[\left(p_1 + \frac{qB}{2c}x_2\right)^2 + \left(p_2 - \frac{qB}{2c}x_1\right)^2\right] \\ =& \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2) - \omega L, \end{aligned} \tag{6}$$

where $\omega = \dfrac{qB}{2mc}$, and L is the angular momentum and determined by

$$L = x_1 p_2 - x_2 p_1.$$

Now consider the Landau problem in a two-dimensional NC space. In the NC space the Schrödinger equation can be written as

$$H(\widehat{x}, \widehat{p})\psi = H\left(x_i - \frac{1}{2\hbar}\theta\varepsilon_{ij}p_j, p_i\right)\psi = E\psi. \tag{7}$$

And the Hamiltonian for the Landau problem takes the form

$$\begin{aligned} H(\widehat{x}, \widehat{p}) =& \frac{1}{2m}\left[\left(p_1 + \frac{qB}{2c}\widehat{x}_2\right)^2 + \left(p_2 - \frac{qB}{2c}\widehat{x}_1\right)^2\right] \\ =& \frac{1}{2m}\left\{\left[\left(1 + \frac{qB}{4\hbar c}\theta\right)p_1 + \frac{qB}{2c}x_2\right]^2 \right. \\ & \left. + \left[\left(1 + \frac{qB}{4\hbar c}\theta\right)p_2 - \frac{qB}{2c}x_1\right)^2\right]\right\} \qquad (8) \\ =& \frac{1}{2\widetilde{m}}(p_1^2 + p_2^2) + \frac{\widetilde{m}\widetilde{\omega}^2}{2}(x_1^2 + x_2^2) - \widetilde{\omega}L, \qquad (9) \end{aligned}$$

where

$$\widetilde{m} = \frac{m}{\left(1 + \dfrac{qB}{4\hbar c}\theta\right)^2}, \quad \widetilde{\omega} = \frac{qB}{2\widetilde{m}c\left(1 + \dfrac{qB}{4\hbar c}\theta\right)}. \tag{10}$$

For convenience, in a non-commutative space we define the annihilation and creation operators as

$$\begin{aligned} & a_i = \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{2\hbar}}\left(x_i + \frac{i}{\widetilde{m}\widetilde{\omega}}p_i\right), \\ & a_i^+ = \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{2\hbar}}\left(x_i - \frac{i}{\widetilde{m}\widetilde{\omega}}p_i\right), \\ & i = 1, 2, \end{aligned} \tag{11}$$

then

$$\begin{aligned} & [a_i, a_j] = 0, [a_i^+, a_j^+] = 0, [a_i, a_j^+] = \delta_{ij}, \\ & L = \mathrm{i}\hbar(a_2^+ a_1 - a_1^+ a_2). \end{aligned} \tag{12}$$

Therefore the angular momentum can be written as

$$\begin{aligned} H(\widehat{x}, p) =& \hbar\widetilde{\omega}(a_1^+ a_1 + a_2^+ a_2 + 1) + \mathrm{i}\hbar\widetilde{\omega}(a_1^+ a_2 - a_2^+ a_1) \\ =& \hbar\widetilde{\omega}(\widehat{N}_1 + \widehat{N}_2 + 1) - \widetilde{\omega}L. \end{aligned} \tag{13}$$

According to this function we can determine the deformed creation and annihilation operators as follows:

$$\begin{aligned} b_1 =& \frac{1}{\sqrt{2}}(a_1 + \mathrm{i}a_2) \\ =& \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{4\hbar}}(x_1 + \mathrm{i}x_2) + \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}\widetilde{\omega}}}(p_1 + \mathrm{i}p_2), \\ b_1^+ =& \frac{1}{\sqrt{2}}(a_1^+ - \mathrm{i}a_2^+) \\ =& \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{4\hbar}}(x_1 - \mathrm{i}x_2) - \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}\widetilde{\omega}}}(p_1 - \mathrm{i}p_2), \\ b_2 =& \frac{1}{\sqrt{2}}(-a_1 + \mathrm{i}a_2) \\ =& \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{4\hbar}}(-x_1 + \mathrm{i}x_2) - \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}\widetilde{\omega}}}(p_1 - \mathrm{i}p_2), \\ b_2^+ =& \frac{1}{\sqrt{2}}(-a_1^+ - \mathrm{i}a_2^+) \\ =& \sqrt{\frac{\widetilde{m}\widetilde{\omega}}{4\hbar}}(-x_1 - \mathrm{i}x_2) + \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}\widetilde{\omega}}}(p_1 + \mathrm{i}p_2), \end{aligned} \tag{14}$$

which satisfy Heisenberg algebraic relations

$$\begin{aligned} & [b_i, b_j^+] = \delta_{ij}, \\ & [b_i, b_j] = [b_i^+, b_j^+] = 0. \end{aligned} \tag{15}$$

From the expressions (14), we can have angular momentum

$$L = \hbar(b_2^+ b_2 - b_1^+ b_1), \tag{16}$$

and the Hamiltonian (9) now takes the following form:

$$H(\widehat{x}_i, p_i) = \hbar\widetilde{\omega}(2b_1^+ b_1 + 1). \tag{17}$$

The relevant eigenvectors are

$$|n\rangle = \frac{(b_1^+)^n}{\sqrt{n}}|0\rangle, \tag{18}$$

and the eigenvalues of $\widehat{H}$

$$\begin{aligned} E =& \hbar\widetilde{\omega}(2n + 1) \\ =& \hbar\widetilde{\omega}(N + 1) \quad N = 0, 2, 4, ..., \end{aligned} \tag{19}$$

The self-energies correspond to $\dfrac{N}{2}+1$ quantum states, and the energy levels are called Landau levels. It is obvious that all results return to the space–space commuting case while $\theta = 0$. One can easily check that the energy shift caused by the space–space noncommutativity can be given by

$$\Delta E = \hbar(\widetilde{\omega} - \omega)(N + 1) = (N - 1)\frac{q^2 B^2 \theta}{8mc^2}. \tag{20}$$

3. Landau problem in NC phase space

The Bose–Einstein statistics in NCQM requires both space–space and momentum–momentum non-commutativity. So the momentum–momentum non-commutativity should be considered when dealing with physical problems. Thus in a non-commutative phase space the Schrödinger equation can be written as

$$H(\widehat{x}_i,\widehat{p}_i)\psi = H\bigg(\alpha x_i - \frac{1}{2\alpha\hbar}\theta\varepsilon_{ij}p_j, \alpha p_i + \frac{1}{2\alpha\hbar}\overline{\theta}\varepsilon_{ij}x_j\bigg)\psi = E\psi. \tag{21}$$

From Eqs.(4), in the two-dimensional case we have

$$\begin{aligned}
\widehat{x}_1 &= \alpha x_1 - \frac{1}{2\alpha\hbar}\theta p_2,\\
\widehat{x}_2 &= \alpha x_2 + \frac{1}{2\alpha\hbar}\theta p_1,\\
\widehat{p}_1 &= \alpha p_1 + \frac{1}{2\alpha\hbar}\overline{\theta} x_2,\\
\widehat{p}_2 &= \alpha p_2 - \frac{1}{2\alpha\hbar}\overline{\theta} x_1.
\end{aligned} \tag{22}$$

where θ, $\bar{\theta}$ and α are related by[18]

$$\overline{\theta} = 4\hbar^2\alpha^2(1-\alpha^2)/\theta. \tag{23}$$

The Hamiltonian for the two-dimensional Landau problem in the non-commutative phase space is

$$\begin{aligned}
H(\widehat{x},\widehat{p}) =& \frac{1}{2m}\left[\left(\widehat{p}_1 + \frac{qB}{2c}\widehat{x}_2\right)^2 + \left(\widehat{p}_2 - \frac{qB}{2c}\widehat{x}_1\right)^2\right]\\
=& \frac{1}{2m}\left\{\left[\left(\alpha + \frac{qB}{4\hbar\alpha c}\theta\right)p_1 + \left(\frac{qB}{2c}\alpha + \frac{\overline{\theta}}{2\hbar\alpha}\right)x_2\right]^2 + \left[\left(\alpha + \frac{qB}{4\hbar\alpha c}\theta\right)p_2 - \left(\frac{qB}{2c}\alpha + \frac{\overline{\theta}}{2\hbar\alpha}\right)x_1)^2\right]\right\}\\
=& \frac{1}{2\widetilde{m}'}(p_1^2+p_2^2) + \frac{\widetilde{m}'\widetilde{\omega}'^2}{2}(x_1^2+x_2^2) - \widetilde{\omega}' L,
\end{aligned} \tag{24}$$

where

$$\widetilde{m}' = \frac{m}{\left(\alpha + \frac{qB}{4\alpha\hbar c}\theta\right)^2}, \qquad \widetilde{\omega}' = \frac{\frac{qB}{c}\alpha + \frac{\overline{\theta}}{\hbar\alpha}}{2\widetilde{m}'\left(\alpha + \frac{qB}{4\alpha\hbar c}\theta\right)}. \tag{25}$$

Similarly we can determine the deformed creation and annihilation operators as follows:

$$\begin{aligned}
d_1 &= \frac{1}{\sqrt{2}}(a_1 + \mathrm{i}a_2) = \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{4\hbar}}(x_1 + \mathrm{i}x_2) + \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}'\widetilde{\omega}'}}(p_1 + \mathrm{i}p_2),\\
d_1^+ &= \frac{1}{\sqrt{2}}(a_1^+ - \mathrm{i}a_2^+) = \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{4\hbar}}(x_1 - \mathrm{i}x_2) - \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}'\widetilde{\omega}'}}(p_1 - \mathrm{i}p_2),\\
d_2 &= \frac{1}{\sqrt{2}}(-a_1 + \mathrm{i}a_2) = \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{4\hbar}}(-x_1 + \mathrm{i}x_2) - \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}'\widetilde{\omega}'}}(p_1 - \mathrm{i}p_2),\\
d_2^+ &= \frac{1}{\sqrt{2}}(-a_1^+ - \mathrm{i}a_2^+) = \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{4\hbar}}(-x_1 - \mathrm{i}x_2) + \frac{\mathrm{i}}{\sqrt{4\hbar\widetilde{m}'\widetilde{\omega}'}}(p_1 + \mathrm{i}p_2),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
a_i &= \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{2\hbar}}\left(x_i + \frac{\mathrm{i}}{\widetilde{m}'\widetilde{\omega}'}p_i\right),\\
a_i^+ &= \sqrt{\frac{\widetilde{m}'\widetilde{\omega}'}{2\hbar}}\left(x_i - \frac{\mathrm{i}}{\widetilde{m}'\widetilde{\omega}'}p_i\right).
\end{aligned} \tag{27}$$

Similarly, d_i and d_j^+ satisfy the same algebraic relations as those in the expression (15),

$$\begin{aligned}
&[d_i, d_j^+] = \delta_{ij},\\
&[d_i, d_j] = [d_i^+, d_j^+] = 0,
\end{aligned} \tag{28}$$

and

$$L = \hbar(d_2^+ d_2 - d_1^+ d_1). \tag{29}$$

Therefore the Hamiltonian (24) now is in the form

$$H(\widehat{x},\widehat{p}) = \hbar\widetilde{\omega}'(2d_1^+ d_1 + 1). \tag{30}$$

The eigenvalues of $\widehat{H}$ are

$$\begin{aligned}
E =& \hbar\widetilde{\omega}'(2n+1)\\
=& \hbar\widetilde{\omega}'(N+1), \quad N = 0, 2, 4, ...,
\end{aligned} \tag{31}$$

and the relevant eigenvectors are

$$|n\rangle = \frac{(d_1^+)^n}{\sqrt{n}}|0\rangle . \tag{32}$$

The energy shift caused by the non-commutivity of both space–space and momentum–momentum can be given as follows:

$$\begin{aligned}\Delta E =&\hbar(\widetilde{\omega}' - \omega)(N+1)\\ =&(N-1)\frac{q^2B^2\theta + 4c^2\overline{\theta}}{8mc^2}.\end{aligned} \tag{33}$$

From the expression (23), we can easily obtain that when $\alpha = 1$, we can have $\overline{\theta}$=0, so $\widetilde{\omega}' \to \widetilde{\omega}$, and $\widetilde{m}' \to \widetilde{m}$, which corresponds to the case where only the space–space is non-commuting. When θ=0 and $\overline{\theta}$=0, the results return to the case of general quantum mechanics.

From the Heisenberg algebra of the Landau problem in NC quantum mechanics, we can not only solve the problem easily and quickly, but also see the non-commutative effect explicitly: for example, Eqs.(20) and (33) give out the energy shift caused by the non-commutativity of space (or NC phase space). The merit of the method in this paper is that this method is practical in the study of non-commutative effect of the angular momentum. We take NC phase space for example. The angular momentum can be written as

$$\begin{aligned}\hat{L} =&\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\\ =&\left(\alpha x_1 - \frac{\theta}{2\alpha\hbar}p_2\right)\left(\alpha p_2 - \frac{\bar{\theta}}{2\alpha\hbar}x_1\right)\\ &-\left(\alpha x_2 + \frac{\theta}{2\alpha\hbar}p_1\right)\left(\alpha p_1 + \frac{\bar{\theta}}{2\alpha\hbar}x_2\right)\\ =&(x_1p_2 - x_2p_1) + \frac{\bar{\theta}}{2\hbar}(x_1^2 + x_2^2)\\ &-\frac{\theta}{2\hbar}(p_1^2 + p_2^2),\end{aligned} \tag{34}$$

where the expression (23) has been used. The last two terms in Eq.(34) are obviously the contribution of the non-commutativity of the NC phase space. From Eq.(27) and also

$$\begin{aligned}d_1 &= \frac{1}{\sqrt{2}}(a_1 + \mathrm{i}a_2),\\ d_2 &= \frac{1}{\sqrt{2}}(-a_1 + \mathrm{i}a_2),\end{aligned} \tag{35}$$

the angular momentum (34) can be expressed in terms of the Heisenberg algebra of the Landau problem on NC Phase space as

$$L = \hbar(d_2^+d_2 - d_1^+d_1) - \frac{1}{2}\theta\widetilde{m}'\widetilde{\omega}'(d_1d_2 + d_1^+d_2^+). \tag{36}$$

To obtain Eq.(36), the relation between θ and $\overline{\theta}$ has been used (see Ref.[18]).

4. Conclusions

In this paper we have studied the Landau problems in NCQM. We have solved the Landau problem in both NC space and NC phase space separately. We obtain the Heisenberg algebraic construction and also the Landau levels for the model, and finally we obtain the energy shift terms and also the angular momentum shift terms caused by the space-space non-commutivity as well as momentum-momentum non-commutivity. Similarly, this method can also be applied to other NCQM problems, which will be reported in our forthcoming papers.

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