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New exact solutions to some difference differential equations*

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In this paper, we use our method to solve the extended Lotka–Volterra equation and discrete KdV equation. With the help of Maple, we obtain a number of exact solutions to the two equations including soliton solutions presented by hyperbolic functions of sinh and cosh, periodic solutions presented by trigonometric functions of sin and cos, and rational solutions. This method can be used to solve some other nonlinear difference–differential equations.

Keywords: difference differential equation, soliton solutions, Lotka–Volterra equation, discrete KdV equation

PACC: 0340K, 0290, 0230

1. Introduction

Discrete solitons in nonlinear lattices have received considerable attention from diverse branches of science.[1] Discrete solitons are possible in several fields of physics, such as biological systems,[2] atomic chains,[3] solid state physics,[4] electrical lattices[5] and Bose–Einstein condensates.[6] Recently, the existence of discrete solitons in photonic structures (in arrays of coupled nonlinear optical waveguides[7,8] and in a nonlinear photonic crystal structure[9]) was announced that has attracted considerable attention from the scientific community.

There is a lot of research work about nonlinear difference differential equations (NDDEs), including the investigation of integrability criteria, the computation of densities, Bäcklund transformation, recursion operators[10,11] and the discrete variational principle in the Hamiltonian formalism.[12] More recently, Baldwin *et al*[13] have presented an algorithm to find exact travelling wave solutions of NDDE in terms of tanh function. Some computational algorithms[14,15] are also presented. In this paper, we introduce the projective Ricatti equations, which were used to solve partial differential equations in Refs.[16-19], to solve integrable or non-integrable difference–differential equation(s).

2. Summary of method

Consider a nonlinear difference–differential equation

$$\begin{aligned} &H(U_{n+n_1}(X),\ldots,U_n(X),\ldots,\\ &U_{n+n_2}(X),U'_{n+n_1}(X),\ldots,\\ &U'_n(X),\ldots,U^r_{n+i}(X),\ldots)=0, \end{aligned} \tag{1}$$

where U denotes the dependent variable (u_1, u_2, $u_3,\ldots$), X the independent variable (x_1, x_2, $x_3,\ldots$), and $U^r_i(X)$ the set of all rth-order partial derivatives of U_i with respect to X, and introduce a transformation

$$\xi_n = a_{1n} + a_{2n} + \cdots + c_1x_1 + c_2x_2 + c_3x_3 + \cdots, \tag{2}$$

where a_{in} $(i = 0, 1, 2, \ldots)$ are all arbitrary functions with respect to n, and Eq.(1) will reduce to

$$\begin{aligned} &H(U_{n+n_1}(\xi_{n+n_1}),\ldots,U_n(\xi_n),\ldots,\\ &U_{n+n_2}(\xi_{n+n_2}),U'_{n+n_1}(\xi_{n+n_1}),\ldots,\\ &U^r_{n+i}(\xi_{n+i}),\ldots)=0. \end{aligned} \tag{3}$$

We search the solutions of Eq.(3) in the following form:

$$U_n(\xi_n) = A_0 + \sum_{i=1}^{N_n} \left(A_i f(\xi_n) + B_i g(\xi_n)\right) f^{i-1}(\xi_n). \tag{4}$$

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As in the continuous case,[20] we can determine N_n by balancing the highest nonlinear terms and the highest-order derivative term in Eq.(1), and then give the formal solutions.

For an arbitrary integer n, $f(\xi_n)$ and $g(\xi_n)$ satisfy the projective Ricatti equations

$$f'(\xi_n) = pf(\xi_n)g(\xi_n), \tag{5a}$$
$$g'(\xi_n) = q + pg^2(\xi_n) - rf(\xi_n). \tag{5b}$$

We can present $f(\xi_{n+k})$ and $g(\xi_{n+k})$ by $f(\xi_n)$ and $g(\xi_n)$. In other words, we have

$$\begin{aligned} f(\xi_{n+k}) &= \Psi(f(\xi_n), g(\xi_n)), \\ g(\xi_{n+k}) &= \Phi(f(\xi_n), g(\xi_n)). \end{aligned} \tag{6}$$

We will give three families of $f(\xi_n), g(\xi_n)$, Ψ and Φ in next section.

Substituting Eqs.(4)–(6) into Eq.(3) and setting all coefficients of $f(\xi_n)^i g(\xi_n)^j$ $(j = 0, 1, i = 0, 1, \ldots)$ to be zero, we obtain an overdetermined nonlinear algebraic system with respect to $A_0, A_1, \ldots, B_1, B_2, \ldots, c_1, c_2, \ldots$.

Solving the obtained overdetermined nonlinear algebraic system, we can find some solutions to the difference–differential equation(s) in the form of Eq.(4).

3. Analysis of the projective Ricatti equations

Now analyse the solutions of projective Ricatti equations. From Eq.(5a), we have

$$g(\xi_n) = \frac{f'(\xi_n)}{pf(\xi_n)}. \tag{7}$$

Substituting Eq.(7) into Eq.(5b) leads to

$$f''f - 2f'^2 - pqf^2 + prf^3 = 0. \tag{8}$$

In order to solve Eq.(8), we introduce the following transformation:

$$f = \frac{1}{w}, \tag{9}$$

then

$$\frac{f'}{f} = -\frac{w'}{w}, \qquad g = -\frac{w'}{pw}, \tag{10}$$

$$w'' + pqw - pr = 0. \tag{11}$$

For the solutions to Eq.(11), the following three basic cases must be considered.

Case A where $pq < 0$

The general solution to Eq.(11) is

$$w = \frac{1}{q}(r + s\cosh(\sqrt{-pq}\xi_n) + h\sinh(\sqrt{-pq}\xi_n)), \tag{12}$$

where s and h are integral constants.

Suppose $p = -1$ and $q = 1$, and we will have

$$f(\xi_n) = \frac{1}{r + s\cosh(\xi_n) + h\sinh(\xi_n)}, \tag{13a}$$

$$g(\xi_n) = \frac{s\sinh(\xi_n) + h\cosh(\xi_n)}{r + s\cosh(\xi_n) + h\sinh(\xi_n)}. \tag{13b}$$

There exists a relationship between $f(\xi_n)$ and $g(\xi_n)$

$$g^2(\xi_n) = 1 - 2rf(\xi_n) + (r^2 + h^2 - s^2)f^2(\xi_n). \tag{14}$$

From Eqs.(13a) and (13b), we have

$$\cosh(\xi_n) = \frac{-f(\xi_n)rs - hg(\xi_n) + s}{(-h^2 + s^2)f(\xi_n)}, \tag{15}$$

$$\sinh(\xi_n) = \frac{g(\xi_n)s + hf(\xi_n)r - h}{(-h^2 + s^2)f(\xi_n)}. \tag{16}$$

Since

$$\sinh(\xi + \omega) = \sinh(\xi)\cosh(\omega) + \cosh(\xi)\sinh(\omega), \tag{17}$$
$$\cosh(\xi + \omega) = \cosh(\xi)\cosh(\omega) + \sinh(\xi)\sinh(\omega), \tag{18}$$

we have

$$\begin{aligned} f(\xi_{n+k}) =& f(\xi_n + \xi_{n+k} - \xi_n) \equiv f(\xi_n + \omega_k) \\ =& \frac{-f(\xi_n)}{-\sinh(\omega_k)g(\xi_n) - rf(\xi_n) + \cosh(\omega_k)f(\xi_n)r - \cosh(\omega_k)}, \end{aligned} \tag{19}$$

$$\begin{aligned} g(\xi_{n+k}) =& g(\xi_n + \xi_{n+k} - \xi_n) \equiv g(\xi_n + \omega_k) \\ =& \frac{-\sinh(\omega_k) - \cosh(\omega_k)g(\xi_n) + \sinh(\omega_k)f(\xi_n)r}{-rf(\xi_n) - \sinh(\omega_k)g(\xi_n) + \cosh(\omega_k)f(\xi_n)r - \cosh(\omega_k)}. \end{aligned} \tag{20}$$

Case B where $pq > 0$

The general solution to Eq.(11) is

$$w = \frac{1}{q}(r + s\cos(\sqrt{pq}\xi_n) + h\sin(\sqrt{pq}\xi_n)), \tag{21}$$

where s and h are integral constants.

Suppose $p = 1$ and $q = 1$, and we will have

$$f(\xi_n) = \frac{1}{r + s\cos(\xi_n) + h\sin(\xi_n)}, \tag{22a}$$

$$g(\xi_n) = \frac{s\sin(\xi_n) - h\cos(\xi_n)}{r + s\cos(\xi_n) + h\sin(\xi_n)}. \tag{22b}$$

There exists a relationship between $f(\xi_n)$ and $g(\xi_n)$

$$g^2(\xi_n) = -1 + 2rf(\xi_n) - (r^2 - h^2 - s^2)f^2(\xi_n). \tag{23}$$

Then we have

$$f(\xi_{n+k}) = -\frac{f(\xi_n)}{\sin(\omega_k)\,g(\xi_n) - rf(\xi_n) + \cos(\omega_k)\,f(\xi_n)r - \cos(\omega_k)}, \tag{24}$$

$$g(\xi_{n+k}) = \frac{-\sin(\omega_k) - \cos(\omega_k)\,g(\xi_n) + \sin(\omega_k)\,f(\xi_n)r}{\sin(\omega_k)\,g(\xi_n) - rf(\xi_n) + \cos(\omega_k)\,f(\xi_n)r - \cos(\omega_k)}. \tag{25}$$

Case C where $q = 0$

Then Eq.(11) can be rewritten as

$$w'' - pr = 0. \tag{26}$$

Its general solution is

$$w = \frac{1}{2}(pr\xi_n^2 + C_1\xi_n - C_2), \tag{27}$$

where C_1 and C_2 are integral constants.

From Eq.(26), we have

$$f(\xi_n) = \frac{2}{pr\xi_n^2 + C_1\xi_n - C_2}, \tag{28a}$$

$$g(\xi_n) = -\frac{2\,pr\xi_n + C_1}{\left(pr{\xi_n}^2 + C_1\xi_n - C_2\right)p}, \tag{28b}$$

where C_1 and C_2 are arbitrary constants.

There exists a relationship between $f(\xi_n)$ and $g(\xi_n)$

$$g^2(\xi_n) = \frac{2r}{p}f(\xi_n) + \frac{C_1^2 + 4C_2pr}{4p^2}f^2(\xi_n). \tag{29}$$

From Eqs.(28a) and (28b), we have

$$f(\xi_{n+k}) = \frac{2f(\xi_n)}{2 + pr{\omega_k}^2 f(\xi_n) - 2\,\omega_k pg(\xi_n)}, \tag{30}$$

$$g(\xi_{n+k}) = \frac{2\,g(\xi_n) - 2\,r\omega_k f(\xi_n)}{2 + pr{\omega_k}^2 f(\xi_n) - 2\,\omega_k pg(\xi_n)}. \tag{31}$$

4. Applications of our method

4.1. Lotka–Volterra equation

Itoh [21] has studied this extended version of the Lotka–Volterra equation

$$u'_{nt} = \sum_{r=1}^{k-1}(u_{n+r} - u_{n-r})u_n, \tag{32}$$

and Hereman *et al*[22] have researched its conserved densities, symmetries and the associate fluxes.

For $k = 2$, it reduces to the Volterra equation

$$u'_{nt} = u_n(u_{n+1} - u_{n-1}). \tag{33}$$

Taking transformation $\xi_n = dn + ct + \xi_0$, we have

$$cu'_{n\xi_n} = u_n(u_{n+1} - u_{n-1}). \tag{34}$$

The balancing procedure allows us to give the following formal solutions to Eq.(32):

$$u_n(\xi_n) = A_0 + A_1 f(\xi_n) + B_1 g(\xi_n), \tag{35}$$

where $f(\xi_n)$ and $g(\xi_n)$ satisfy the projective Ricatti equations, i.e. Eqs.(5a) and (5b).

For simplicity, let $\omega_1 = -\omega_{-1} \equiv d$, in other words, $a(n+1) - a(n) = a(n) - a(n-1)$, i.e. $a(n) = dn + c_0$ is an arithmetic progression.

Corresponding to Case A, with the aid of Maple, substituting Eqs.(5a), (5b), (19), (20) and Eq.(35) into Eq.(34) yields an algebraic equation for $f(\xi_n)^i g(\xi_n)^j$ ($i = 0, 1, 2, 3, \ldots$, $j = 0, 1$). Setting all coefficients of these terms to be zero yields a set of overdetermined algebraic equations with respect to A_0, A_1, B_1, c, d, r, s and h.

By use of Maple or Mathematica, we obtain

$$\begin{aligned} &A_1 = -2\,A_0\,r + 2\,A_0\,\cosh(d)r,\\ &h = \pm\sqrt{s^2 - \frac{2r^2}{1 + \cosh d}},\\ &c = 2\,A_0\,\sinh(d), \quad B_1 = 0. \end{aligned} \tag{36}$$

According to Eqs.(13a), (13b) and (35), we obtain the following soliton solutions of Eq.(33):

$$u_n(t)=A_0+\frac{-2\,A_0\,r+2\,A_0\,\cosh(d)r}{r+s\cosh(\xi_n)\pm\sqrt{s^2-\frac{2r^2}{1+\cosh d}}\sinh(\xi_n)}, \tag{37}$$

where $\xi_n=dn+2\,A_0\,\sinh(d)t+\xi_0$, A_0, d, r, s and ξ_0 are all arbitrary constants.

Corresponding to Case B, substituting Eqs.(5a), (5b), (24), (25) and (35) into Eq.(34), yields an algebraic equations for $f(\xi_n)^i g(\xi_n)^j$ $(i=0,1,2,3,\ldots,$ $j=0,1)$. Setting the coefficients of these terms to be zero, we obtain a set of overdetermined algebraic equations with respect to A_0, A_1, B_1, c, d, r, s and h.

By use of Maple or Mathematica, we obtain $B_1=0$,

$$\begin{aligned}
&c=\frac{\pm4rA_0\sqrt{s^2+h^2-r^2}}{s^2+h^2},\\
&A_1=\frac{4rA_0(r^2-s^2-h^2)}{s^2+h^2},\\
&d=\arctan\left(\frac{\pm2r\sqrt{h^2+s^2-r^2}}{s^2+h^2},\frac{2\,r^2-h^2-s^2}{s^2+h^2}\right).
\end{aligned} \tag{38}$$

According to Eqs.(22a), (22b) and (35), we obtain the following trigonal periodic function solutions of Eq.(33)

$$u_n(t)=A_0+\frac{-4rA_0(s^2+h^2-r^2)}{(s^2+h^2)(r+s\cos(\xi_n)+h\sin(\xi_n))}, \tag{39}$$

where

$$\begin{aligned}
\xi_n=&\arctan\left(\frac{\pm2r\sqrt{h^2+s^2-r^2}}{s^2+h^2},\frac{2\,r^2-h^2-s^2}{s^2+h^2}\right)n\\
&+\frac{4rA_0\sqrt{s^2+h^2-r^2}}{s^2+h^2}t+\xi_0.
\end{aligned}$$

Corresponding to Case C, taking a procedure similar to that in either of the above two cases, we obtain

$$\begin{aligned}
&C_2=\frac{p^2r^2{\omega_1}^2-{C_1}^2}{4rp},\qquad B_1=0,\\
&A_1=-A_0{\omega_1}^2pr,\qquad c=2\,A_0\,\omega_1,
\end{aligned} \tag{40}$$

and the corresponding rational solution to Eq.(33)

$$u_n(t)=A_0-\frac{2A_0{\omega_1}^2pr}{pr(\omega_1n+2A_0\omega_1t+\xi_0)^2+C_1(\omega_1n+2A_0\omega_1t+\xi_0)-\dfrac{p^2r^2{\omega_1}^2-{C_1}^2}{4rp}}. \tag{41}$$

For $k=3$, Eq.(32) reduces to

$$u'_{nt}=u_n(u_{n+1}-u_{n-1})+u_n(u_{n+2}-u_{n-2}). \tag{42}$$

Taking the same procedure as that in the case of $k=2$, we obtain the following soliton solution and periodic solution:

Corresponding to Case A, we have

$$\begin{aligned}
&A_1=4A_0r\cosh^2(d)-2A_0\cosh(d)r-2A_0r,\\
&c=2A_0\sinh(d)+4A_0\sinh(d)\cosh(d),\\
&B_1=0,\qquad h=\pm\sqrt{s^2-\frac{2r^2}{1+\cosh d}},
\end{aligned} \tag{43}$$

and the soliton solution

$$u_n(t)=A_0+\frac{4A_0r\cosh^2(d)-2A_0\cosh(d)r-2A_0r}{r+s\cosh(\xi_n)\pm\sqrt{s^2-\dfrac{2r^2}{1+\cosh d}}\sinh(\xi_n)}, \tag{44}$$

where $\xi_n=dn+(2A_0\sinh(d)+4A_0\sinh(d)\cosh(d))t+\xi_0$, r, s, d, A_0 and ξ_0 are all constants.

Corresponding to Case B, we have

$$B_1=0,\quad A_1=\frac{4A_0r(h^4+2h^2s^2-5r^2h^2-5r^2s^2+4r^4+s^4)}{h^4+s^4+2\,h^2s^2},$$

$$c=\pm\frac{4A_0r\sqrt{s^2+h^2-r^2}(s^2+h^2-4r^2)}{h^4+s^4+2h^2s^2},\quad d=\arctan\left(\pm\frac{2r\sqrt{s^2+h^2-r^2}}{s^2+h^2},\frac{2r^2-s^2-h^2}{s^2+h^2}\right), \tag{45}$$

and the trigonal periodic solution

$$u_n(t) = A_0 + \frac{4A_0 r(h^4 + 2h^2s^2 - 5r^2h^2 - 5r^2s^2 + 4r^4 + s^4)}{(h^4 + s^4 + 2\,h^2s^2)(r + s\cos(\xi_n) + h\sin(\xi_n))}, \tag{46}$$

where $\xi_n = dn + 2A_0^2\sin(d)t + \xi_0$,$r, s, h, d, A_0$ and ξ_0 are all constants.

4.2. Discrete KdV equation

The discrete KdV equation[23] can be written as

$$u'_{nt} = u_n^2(u_{n+1} - u_{n-1}). \tag{47}$$

Taking a transformation $\xi_n = dn + ct + \xi_0$, we have

$$cu'_{n\xi_n} = u_n^2(u_{n+1} - u_{n-1}). \tag{48}$$

We suppose that Eq.(47) has the following formal solution:

$$\begin{aligned} u_n(\xi_n) =& A_0 + A_1 f(\xi_n) + B_1 g(\xi_n) \\ &+ A_2 f^2(\xi_n) + B_2 f(\xi_n)g(\xi_n), \end{aligned} \tag{49}$$

where $f(\xi_n)$ and $g(\xi_n)$ satisfy the projective Ricatti equations i.e. Eqs.(5a) and (5b).

Taking the same procedure as that for solving the Lotka–Volterra equation, we can obtain a number of soliton solutions, periodic solutions and rational solutions.

Corresponding to Case A, we have

$$\begin{aligned} &h = \pm\sqrt{s^2 - r^2}, \quad A_1 = -A_0\, r + A_0\, \cosh(d)r, \\ &c = 2\,{A_0}^2\sinh(d), \\ &B_1 = A_2 = B_2 = 0; \end{aligned} \tag{50}$$

$$\begin{aligned} &A_2 = A_0(1 - \cosh^2(d))(h^2 - s^2), \\ &c = 2\,{A_0}^2\sinh(d)\cosh(d), \\ &A_1 = B_1 = B_2 = r = 0; \end{aligned} \tag{51}$$

$$\begin{aligned} &A_2 = -2\,(\cosh(d) - 1)r^2A_0, \quad h = \pm s, \\ &c = 2\,{A_0}^2\sinh(d), \\ &A_1 = 2\,(\cosh(d) - 1)rA_0, \quad B_1 = B_2 = 0, \end{aligned} \tag{52}$$

and the following soliton solutions:

$$u_n(t) = A_0 + \frac{-A_0\, r + A_0\cosh(d)r}{r + s\cosh(\xi_n) \pm \sqrt{s^2 - r^2}\sinh(\xi_n)}, \tag{53}$$

where $\xi_n = dn + 2\,{A_0}^2\sinh(d)t + \xi_0$, A_0, d, r, s and ξ_0 are all arbitrary constants;

$$u_n(t) = A_0 + \frac{A_0(1 - \cosh^2(d))(h^2 - s^2)}{(s\cosh(\xi_n) + h\sinh(\xi_n))^2}, \tag{54}$$

where $\xi_n = dn + 2\,{A_0}^2\sinh(d)\cosh(d)t + \xi_0$, A_0, d, s, h and ξ_0 are all arbitrary constants;

$$\begin{aligned} u_n(t) =& A_0 + \frac{2\,(\cosh(d) - 1)rA_0}{r + s\cosh(\xi_n) \pm s\sinh(\xi_n)} \\ &- \frac{2\,(\cosh(d) - 1)r^2A_0}{(r + s\cosh(\xi_n) \pm s\sinh(\xi_n))^2}, \end{aligned} \tag{55}$$

where $\xi_n = dn + 2\,{A_0}^2\sinh(d)t + \xi_0$, A_0, d, r, s and ξ_0 are all arbitrary constants.

Corresponding to Case B, we have

$$\begin{aligned} &B_1 = A_2 = B_2 = 0, \quad A_1 = A_0\cos(d)r - A_0 r, \\ &c = 2{A_0}^2\sin(d), \quad h = \pm\sqrt{r^2 - s^2}; \end{aligned} \tag{56}$$

$$\begin{aligned} &A_2 = -A_0\sin^2(d)(s^2 + h^2), \\ &c = 2{A_0}^2\sin(d)\cos(d), \\ &r = A_1 = B_1 = B_2 = 0; \end{aligned} \tag{57}$$

$$\begin{aligned} &A_1 = 2rA_0(\cos(d) - 1), \\ &A_2 = -2r^2A_0(\cos(d) - 1), \\ &c = 2{A_0}^2\sin(d), \quad h = \pm\mathrm{i}s, \\ &B_1 = B_2 = 0; \end{aligned} \tag{58}$$

and three kinds of trigonal periodic solutions

$$u_n(t) = A_0 + \frac{A_0\cos(d)r - A_0 r}{r + s\cos(\xi_n) \pm \sqrt{r^2 - s^2}\sin(\xi_n)}, \tag{59}$$

where $\xi_n = dn + 2A_0^2\sin(d)t + \xi_0$, r, s, d, A_0 and ξ_0 are all constants;

$$u_n(t) = A_0 - \frac{A_0\sin^2(d)(s^2 + h^2)}{(s\cos(\xi_n) + h\sin(\xi_n))^2}, \tag{60}$$

where $\xi_n = dn + 2A_0^2\sin(d)\cos(d)t + \xi_0$, h, s, d, A_0 and ξ_0 are all constants;

$$\begin{aligned} u_n(t) =& A_0 + \frac{2rA_0(\cos(d) - 1)}{r + s\cos(\xi_n) \pm is\sin(\xi_n)} \\ &- \frac{2r^2A_0(\cos(d) - 1)}{(r + s\cos(\xi_n) \pm is\sin(\xi_n))^2}, \end{aligned} \tag{61}$$

where $\xi_n = dn + 2A_0^2 \sin(d)t + \xi_0$, $i = \sqrt{-1}$, r, s, d, A_0 and ξ_0 are all constants;

Corresponding to Case C, we have

$$C_2 = -\frac{-16\,{B_2}^2 + {A_0}^2{\omega_1}^4 p^2 {C_1}^2}{4{A_0}^2{\omega_1}^4 p^3 r},$$

$$A_1 = -\frac{A_0\,{\omega_1}^2 pr}{2}, \quad A_2 = -\frac{2{B_2}^2}{A_0\,p^2{\omega_1}^2},$$
$$c = 2\,{A_0}^2\omega_1, \quad B_1 = 0. \tag{62}$$

and the rational solution

$$u_n(t) = A_0 - \frac{A_0\,{\omega_1}^2 pr}{pr\left(\omega_1 n + 2\,{A_0}^2\omega_1 t + \xi_0\right)^2 + C_1\left(\omega_1 n + 2\,{A_0}^2\omega_1 t + \xi_0\right) + \dfrac{-16\,{B_2}^2 + {A_0}^2{\omega_1}^4 p^2 {C_1}^2}{4{A_0}^2{\omega_1}^4 p^3 r}}$$
$$- \frac{8\,{B_2}^2 {A_0}^{-1} p^{-2} {\omega_1}^{-2} + 2\,B_2\left(2pr\left(\omega_1 n + 2\,{A_0}^2\omega_1 t + \xi_0\right) + C_1\right) p^{-1}}{\left(pr\left(\omega_1 n + 2\,{A_0}^2\omega_1 t + \xi_0\right)^2 + C_1\left(\omega_1 n + 2\,{A_0}^2\omega_1 t + \xi_0\right) + \dfrac{-16\,{B_2}^2 + {A_0}^2{\omega_1}^4 p^2 {C_1}^2}{4{A_0}^2{\omega_1}^4 p^3 r}\right)^2}, \tag{63}$$

where ω_1, C_1, A_0, B_2, p, r and ξ_0 are all arbitrary constants.

5. Conclusions

Introducing the projective Ricatti equations in solving the difference–differential equation, we obtain a number of solutions to the discrete difference–differential equation(s) including soliton solutions presented by hyperbolic functions of sinh and cosh, periodic solutions presented trigonal functions of sin and cos, and rational solutions. This method can also be used to solve some other nonlinear difference–differential equations.

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