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Construction of doubly-periodic solutions to nonlinear partial differential equations using improved Jacobi elliptic function expansion method and symbolic computation

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Some doubly-periodic solutions of the Zakharov–Kuznetsov equation are presented. Our approach is to introduce an auxiliary ordinary differential equation and use its Jacobi elliptic function solutions to construct doubly-periodic solutions of the Zakharov–Kuznetsov equation, which has been derived by Gottwald as a two-dimensional model for nonlinear Rossby waves. When the modulus $k \to 1$, these solutions reduce to the solitary wave solutions of the equation.

Keywords: Jacobi elliptic function method, doubly-periodic solutions, Zakharov–Kuznetsov equation
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1. Introduction

With the development of soliton theory, there have been a great number of methods of exactly solving nonlinear evolution equations, such as the inverse scattering method,[1,2] Bäcklund transformation method,[3] Hirota's bilinear method,[4] and Darboux transformation method.[5] In recent years, various direct methods have been developed such as the homogeneous balance method,[6,7] hyperbolic function expansion method,[8] rational expansion method,[9] sine–cosine method,[10] tanh method,[11−13] the generalized Riccati equation method,[14,15] and the generalized projective Riccati equation method.[16] One of most effective and straightforward methods to construct exact solutions of partial differential equations is the Jacobi elliptic function method.[17−20] when the modulus $m \to 1$ or 0, the Jacobi elliptic function solutions reduce to solitonic solutions and trigonometric function solutions. Therefore, seeking the Jacobi elliptic function solutions of nonlinear wave equations is significant.

In this paper, through careful computation, we fortunately find that the ordinary differential equation

$$\phi'^2 = a + b\phi^2 + c\phi^4, \tag{1}$$

where $\phi' = \dfrac{\mathrm{d}\phi}{\mathrm{d}\xi}$, and $\xi = \xi(x, y, z, t)$, which has various solutions in terms of rational formal Jacobi elliptic function. So we can take it as an auxiliary equation and use its Jacobi elliptic function solutions to construct doubly-periodic solutions of nonlinear partial differential equations.

The rest of this paper is organized as follows. In Section 2, we present the further improved Jacobi elliptic function method and its algorithm. In Section 3, we apply the further improved method to Zakharov–Kuznetsov equation and bring out many solutions. Finally, some conclusions are given in Section 4.

2. The further improved Jacobi elliptic function expansion algorithm

In the following we outline the main steps of our further improved method.

Step 1 For a given nonlinear evolution equation in two independent variables x and t,

$$F(u, u_t, u_x, u_{xt}, u_{xx}, u_{tt}, ...) = 0, \tag{2}$$

by using the wave transformation

$$u(x,t) = U(\xi), \qquad \xi = x + \lambda t, \tag{3}$$

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where λ is a constant to be determined later, the nonlinear partial differential equation (2) reduces to a nonlinear ordinary differential equation(ODE):

$$G(U, U', U'', U''', ...) = 0. \tag{4}$$

We assume that the solutions of Eq.(4) can be expressed in the form

$$U(\xi) = a_0 + \sum_{i=1}^{n} a_i \phi^i + \frac{b_i}{\phi^i} + c_i \phi^{i-1} \phi' + \frac{d_i \phi'}{\phi^i}, \tag{5}$$

where ϕ satisfies Eq.(1), n can be determined by comparing the highest order partial derivative term and nonlinear term in Eq.(4) or Eq.(2), a_0, a_i, b_i, c_i and $d_i (i = 1, ..., n)$ are constants to be determined later.

Step 2 The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly-periodic solutions to exhibit the doubly-periodic solutions is that different effects that can change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either individually or jointly are able to balance out. We define the degree of $u(\xi)$ as $D[u(\xi)] = n$, which gives rise to the degrees of other expressions as

$$D[u^{(\alpha)}] = n + \alpha \quad \text{and} \quad D[(u^{(\alpha)})^s] = (n + \alpha)s.$$

Step 3 With the aid of Maple, substituting Eq.(5) into Eq.(4) along with Eq.(1), collecting the coefficients of the same power $\phi^i(\sqrt{a + b\phi^2 + c\phi^4})^j$ $(j = 0, 1, i = 0, 1, 2, ...)$, setting each of the obtained coefficients to zero yield a set of over-determined algebraic equations.

Step 4 solving the above overdetermined algebraic equations by use of Maple, to obtain $\lambda, a_0, a_i, b_i, c_i$ and $d_i (i = 1, ..., n)$.

Step 5 Substituting the obtained conclusions in Step 3 into Eq.(5) along with the solutions of Eq.(1) gives the doubly-periodic solutions of Eq.(2).

In the following we illustrate our method by considering the Zakharov–Kuznetsov equation.

3. Exact solutions of the Zakharov–Kuznetsov equation

$$u_t + \delta u_x + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = 0. \tag{6}$$

Gottwald derived the nonlinear dispersive Zakharov–Kuznetsov (ZK) equation from the quasi-geostrophic barotropic vorticity equation,[21] where weakly nonlinear and long wave multiple scale analysis was applied to the two-dimensional case, and the Zakharov–Kuznetsov equation for nonlinear Rossby waves as well. The ZK equation is a very attractive model equation for the study of vortices in geophysical flows. Actually, there are more multiple structures in the ZK equations,[22] which will be shown later. All these studies may help to describe two-dimensional coherent structures such as atmospheric blocking events, long-lived eddies in the ocean or coherent structures in the Jovian atmosphere such as the Great Red Spot.

We seek its travelling wave solutions in the following frame:

$$u = U(\xi), \qquad \xi = mx + ly - \omega t, \tag{7}$$

where ω is angular frequency, m and l are wave number in x and y directions, respectively. Substituting Eq.(7) into Eq.(6) yields

$$(m\delta - \omega) U' + m\alpha U U' + \left(\beta m^3 + \gamma l^2 m\right) U''' = 0. \tag{8}$$

By comparing U''' and UU', we suppose that Eq.(8) has the following formal solution:

$$\begin{aligned} U =& a_0 + a_1\phi + a_2\phi^2 + \frac{b_1}{\phi} + \frac{b_2}{\phi^2} \\ &+ c_1\phi' + c_2\phi\phi' + \frac{d_1\phi'}{\phi} + \frac{d_2\phi'}{\phi^2}, \end{aligned} \tag{9}$$

where $\phi(\xi)$ satisfies Eq.(1), a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2 are constants to be determined later.

With the aid of Maple, substituting Eq.(9) into Eq.(8) along with Eq.(1), collecting coefficients of ϕ^i and $\phi^i(\sqrt{a + b\phi^2 + c\phi^4})^j$ $(j = 0, 1, i = 0, 1, 2, ...)$, and setting them to be zero, we obtain a set of overdetermined algebraic equations with respect to a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, m, l and ω. For simplicity, we omit them.

By use of the Maple soft package 'Charsets' developed by Wang Dong-Ming, which is based on the Wu-elimination method,[23] to solve the overdetermined algebraic equations, we obtain the following results.

Case 1

$$\begin{aligned} & b_1 = d_1 = d_2 = c_1 = c_2 = a_1 = 0, \quad m = m, \\ & a_0 = a_0, \ l = l, \quad a_2 = -\frac{12c\left(\gamma l^2 + \beta k^2\right)}{\alpha}, \\ & b_2 = -\frac{12a\left(\gamma l^2 + \beta k^2\right)}{\alpha}, \end{aligned}$$

and

$$\omega = 4\, b\beta\, m^3 + 4\, bm\gamma\, l^2 + \delta\, m + \alpha\, m a_0.$$

Case 2

$$b_2 = b_1 = d_1 = d_2 = c_2 = a_1 = 0, \quad m = m, \quad l = l,$$
$$a_0 = a_0, \quad c_1 = 6\,\frac{\sqrt{c}\left(\gamma\, l^2 + \beta\, m^2\right)}{\alpha},$$
$$a_2 = -6\,\frac{c\left(\gamma\, l^2 + \beta\, k^2\right)}{\alpha},$$

and

$$\omega = \delta\, m + \alpha\, m a_0 + bm\gamma\, l^2 + b\beta\, m^3.$$

Case 3

$$a_2 = 0,\ b_1 = 0,\ d_1 = 0,\ c_1 = 0,\ c_2 = 0,$$
$$a_1 = 0,\ m = m,\ l = l,\ a_0 = a_0,$$
$$d_2 = -6\,\frac{\sqrt{a}\left(\gamma\, l^2 + \beta\, m^2\right)}{\alpha},$$
$$\omega = \delta\, m + \alpha\, m a_0 + bm\gamma\, l^2 + b\beta\, m^3,$$

and

$$b_2 = -6\,\frac{a\left(\gamma\, l^2 + \beta\, m^2\right)}{\alpha}.$$

Case 4

$$b_1 = 0,\ l = l,\ a_0 = a_0,\ d_1 = 0,$$
$$c_2 = 0,\ a_1 = 0,\ c_1 = c_1, d_2 = \frac{c_1\sqrt{ac}}{c},$$
$$b_2 = -\frac{ac_1\sqrt{c}}{c}, a_2 = -c_1\sqrt{c_1},$$
$$m = \sqrt{\frac{\alpha c_1\sqrt{c}}{6c\beta} - \frac{\gamma l^2}{\beta}},$$

and

$$\omega = \frac{\sqrt{\dfrac{\alpha c_1\sqrt{c}}{6c\beta} - \dfrac{\gamma l^2}{\beta}}\left(6\delta c + 6\alpha a_0 c + b\alpha c_1\sqrt{c} - 6c\sqrt{a}\right)}{6c}.$$

Equation (1) has many kinds of solutions. Next we will show some solutions in rational forms expressed in terms of different elliptic functions.

1) If $a = 1,\ b = 2 - 4\,k^2$ and $c = 1$, then the solution to Eq.(1) is

$$\phi_1 = \frac{\mathrm{sn}\,(\xi, k)\,\mathrm{dn}\,(\xi, k)}{\mathrm{cn}\,(\xi, k)},$$

where $\mathrm{sn}\,(\xi, k)$ and $\mathrm{cn}\,(\xi, k)$ are Jacobi elliptic sine and cosine functions respectively, $\mathrm{dn}\,(\xi, k)$ is Jacobi elliptic function of the third kind, $k(0 < k < 1)$ denotes the modulus of the Jacobi elliptic functions. Detailed explanations about Jacobi elliptic functions can be found in Refs.[24, 25]. This solution to Eq.(1) is new and has not been reported in literature. So, based on the new solution, we can derive a new solution to Eq.(6):

$$\begin{aligned} u_1 =& a_0 - c_1\left(1 - 2\mathrm{dn}^2\,(\xi)\right) \\ & - \frac{\left(c_1\sqrt{c} + c_1\alpha\right)\mathrm{sn}^2\,(\xi)\,\mathrm{dn}^2\,(\xi)}{\mathrm{cn}^2\,(\xi)} \\ & - \frac{ac_1\sqrt{c}\mathrm{cn}^2\,(\xi)}{c\mathrm{sn}^2\,(\xi)\,\mathrm{dn}^2\,(\xi)} \\ & - \frac{c_1\sqrt{ac}\left(k^2\mathrm{sn}^4\,(\xi) - 2\,\mathrm{sn}^2\,(\xi)\,k^2 + 1\right)}{c\mathrm{sn}\,(\xi)\left(k^2\mathrm{sn}^2\,(\xi) - 1\right)}. \end{aligned}$$

2) If $a = 1,\ b = 2$ and $c = k^4$, then the solution to Eq.(1) is

$$\phi_2 = \frac{\mathrm{sn}\,(\xi, k)\,\mathrm{cn}\,(\xi, k)}{\mathrm{dn}\,(\xi, k)},$$

and the new solution to Eq.(6) is

$$\begin{aligned} u_2 =& a_0 - \frac{c_1\sqrt{c}\mathrm{sn}^2\,(\xi)\,\mathrm{cn}^2\,(\xi)}{\mathrm{dn}^2\,(\xi)} \\ & - \frac{ac_1\sqrt{c}\mathrm{dn}^2\,(\xi)}{c\mathrm{sn}^2\,(\xi)\,\mathrm{cn}^2\,(\xi)} \\ & + \frac{c_1\left(\mathrm{cn}^2\,(\xi) - \mathrm{sn}^2\,(\xi)\,\mathrm{dn}^2\,(\xi)\right)}{\mathrm{dn}^2\,(\xi)} \\ & - \frac{c_1\sqrt{ac}\left(k^2\mathrm{sn}^4\,(\xi) - 2\,\mathrm{sn}^2\,(\xi) + 1\right)}{c\mathrm{sn}^2\,(\xi)\left(\mathrm{sn}^2\,(\xi) - 1\right)}. \end{aligned}$$

3) If $a = -2\,k^2 + k^4 + 1,\ b = k^2 + 2$ and $c = 1$, then the solution to Eq.(1) is

$$\phi_3 = \frac{\mathrm{dn}\,(\xi, k)\,\mathrm{cn}\,(\xi, k)}{\mathrm{sn}\,(\xi, k)},$$

and the new solution to Eq.(6) is

$$\begin{aligned} u_3 =& a_0 - \frac{c_1\sqrt{c}\mathrm{cn}^2\,(\xi)\,\mathrm{dn}^2\,(\xi)}{\mathrm{sn}^2\,(\xi)} \\ & - \frac{ac_1\sqrt{c}\mathrm{sn}^2\,(\xi)}{c\mathrm{cn}^2\,(\xi)\,\mathrm{dn}^2\,(\xi)} \\ & - \frac{c_1\left(\mathrm{dn}^2\,(\xi) + k^2\mathrm{sn}^2\,(\xi)\,\mathrm{cn}^2\,(\xi)\right)}{\mathrm{sn}^2\,(\xi)} \\ & + \frac{c_1\sqrt{ac}\left(\mathrm{sn}^4\,(\xi) - 1\right)}{c\left(1 - \left(k^2 + 1\right)\mathrm{sn}^2\,(\xi) + k^2\mathrm{sn}^4\,(\xi)\right)}. \end{aligned}$$

4) If $a = \dfrac{k^2 - 2\,k + 1}{4A^2},\ b = \dfrac{1 + k^2}{2} + 3\,k$ and $c = \dfrac{A^2(k-1)^2}{4}$, then the solution to Eq.(1) is

$$\phi_4 = \frac{\mathrm{dn}\,(\xi, k)\,\mathrm{cn}\,(\xi, k)}{A\left(1 \pm \mathrm{sn}\,(\xi, k)\right)\left(1 \pm k\mathrm{sn}\,(\xi, k)\right)},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_4 =&a_0-\frac{c_1\sqrt{c}\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}{A^2\left(1+\mathrm{sn}(\xi)\right)^2\left(1+k\mathrm{sn}(\xi)\right)^2}\\&-\frac{ac_1\sqrt{c}A^2\left(1+\mathrm{sn}(\xi)\right)^2\left(1+k\mathrm{sn}(\xi)\right)^2}{c\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}\\&-\frac{c_1(k+1)\left(k\left(\mathrm{sn}(\xi)\right)^2-1\right)}{A\left(1+\mathrm{sn}(\xi)\right)\left(1+k\mathrm{sn}(\xi)\right)}\\&-\frac{c_1\sqrt{ac}A(k+1)\left(k\mathrm{sn}^2(\xi)-1\right)}{c\left(\mathrm{sn}(\xi)-1\right)\left(k\mathrm{sn}(\xi)-1\right)}.\end{aligned}$$

5) If $a=\dfrac{(k+1)^2}{4A^2}, b=\dfrac{1+k^2}{2}-3k$ and $c=\dfrac{A^2(1+k)^2}{4}$, then the solution to Eq.(1) is

$$\phi_5=\frac{\mathrm{dn}(\xi,k)\,\mathrm{cn}(\xi,k)}{A\left(1\pm\mathrm{sn}(\xi,k)\right)\left(1\mp k\mathrm{sn}(\xi,k)\right)},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_5 =&a_0-\frac{c_1\sqrt{c}\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}{A^2\left(1+\mathrm{sn}(\xi)\right)^2\left(1-k\mathrm{sn}(\xi)\right)^2}\\&-\frac{ac_1\sqrt{c}A^2\left(1+\mathrm{sn}(\xi)\right)^2\left(1-k\mathrm{sn}(\xi)\right)^2}{c\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}\\&-\frac{c_1(k-1)\left(k\left(\mathrm{sn}(\xi)\right)^2+1\right)}{A\left(1+\mathrm{sn}(\xi)\right)\left(k\mathrm{sn}(\xi)-1\right)}\\&-\frac{c_1\sqrt{ac}A(k-1)\left(k\left(\mathrm{sn}(\xi)\right)^2+1\right)}{c\left(\mathrm{sn}(\xi)-1\right)\left(1+k\mathrm{sn}(\xi)\right)}.\end{aligned}$$

6) If $a=-2k^3+k^4+k^2,\ b=6k-k^2-1$ and $c=-\dfrac{4}{k}$, then the solution to Eq.(1) is

$$\phi_6=\frac{k\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)}{k\mathrm{sn}^2(\xi,k)+1},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_6 =&a_0-\frac{c_1\sqrt{c}k^2\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}{\left(k\mathrm{sn}^2(\xi)+1\right)^2}\\&-\frac{ac_1\sqrt{c}\left(k\ \mathrm{sn}^2(\xi)+1\right)^2}{ck^2\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}\\&-\frac{c_1k\mathrm{sn}(\xi)(k+1)^2\left(k\mathrm{sn}^2(\xi)-1\right)}{\left(k\mathrm{sn}^2(\xi)+1\right)^2}\\&-\frac{Ac_1\sqrt{ac}(k+1)\left(k\mathrm{sn}^2(\xi)-1\right)}{ac\left(\mathrm{sn}(\xi)-1\right)\left(k\mathrm{sn}(\xi)-1\right)}.\end{aligned}$$

7) If $a=2k^3+k^4+k^2,\ b=-6k-k^2-1$ and $c=\dfrac{4}{k}$, then the solution to Eq.(1) is

$$\phi_7=\frac{k\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)}{k\mathrm{sn}^2(\xi,k)-1},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_7 =&a_0-\frac{c_1\sqrt{c}k^2\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}{\left(k\mathrm{sn}^2(\xi)+1\right)^2}-\frac{ac_1\sqrt{c}\left(k\ \mathrm{sn}^2(\xi)+1\right)^2}{ck^2\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}\\&-\frac{c_1k\mathrm{sn}(\xi)(k-1)^2\left(k\mathrm{sn}^2(\xi)+1\right)}{\left(k\mathrm{sn}^2(\xi)-1\right)^2}-\frac{c_1(k-1)^2\sqrt{ac}\mathrm{sn}(\xi)\left(k\mathrm{sn}^2(\xi)+1\right)}{kc\left(\mathrm{sn}(\xi)-1\right)\left(1+\mathrm{sn}(\xi)\right)\left(k\mathrm{sn}(\xi)-1\right)\left(1+k\mathrm{sn}(\xi)\right)}.\end{aligned}$$

8) If $a=1,\ b=2+6k_1-k^2$ and $c=8+8k_1-4k^2+4$, then the solution to Eq.(1) is

$$\phi_8=\frac{\mathrm{sn}(\xi,k)\,\mathrm{cn}(\xi,k)}{\mathrm{cn}^2(\xi,k)-k_1\mathrm{sn}^2(\xi,k)},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_8 =&a_0-\frac{c_1\sqrt{c}\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{\left(\mathrm{cn}^2(\xi)-k_1\mathrm{sn}^2(\xi)\right)^2}-\frac{ac_1\sqrt{c}\left(\mathrm{cn}^2(\xi)-k_1\mathrm{sn}^2(\xi)\right)^2}{c\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}\\&+\frac{c_1\left(1+(k_1-1)\mathrm{sn}^2(\xi)\right)\mathrm{dn}(\xi)}{\mathrm{cn}^4(\xi)-2k_1\mathrm{cn}^2(\xi)\,\mathrm{sn}^2(\xi)-(k^2-1)\mathrm{sn}^4(\xi)}-\frac{c_1\sqrt{ac}\mathrm{dn}(\xi)\left(1-\mathrm{sn}^2(\xi)+k_1\mathrm{sn}^2(\xi)\right)}{c\mathrm{sn}^2(\xi)\left(\mathrm{sn}^2(\xi)-1\right)}.\end{aligned}$$

9) If $a=1,\ b=2-6k_1-k^2$ and $c=8-8k_1-4k^2+4$, then the solution to Eq.(1) is

$$\phi_9=\frac{\mathrm{sn}(\xi,k)\,\mathrm{cn}(\xi,k)}{\mathrm{cn}^2(\xi,k)+k_1\mathrm{sn}^2(\xi,k)},$$

and the new solution to Eq.(6) is

$$u_9 = a_0 - \frac{ac_1\sqrt{c}\left(\mathrm{cn}^2(\xi)+k_1\mathrm{sn}^2(\xi)\right)^2}{c\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)} - \frac{c_1\sqrt{ac}\,\mathrm{dn}(\xi)\left((k_1+1)\,\mathrm{sn}^2(\xi)-1\right)}{c\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)} - \frac{c_1\sqrt{c}\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{\left(\mathrm{cn}^2(\xi)+k_1\mathrm{sn}^2(\xi)\right)^2} - \frac{c_1\mathrm{dn}(\xi)\left((1+k_1)\,\mathrm{sn}^2(\xi)-1\right)}{2\,(1-k_1)\,\mathrm{sn}^2(\xi)+(2\,k_1+k^2-2)\,\mathrm{sn}^4(\xi)-1}.$$

10) If $a = 2+2k_1-k^2,\ b = 2+6k_1-k^2$ and $c = 4k_1$, then the solution to Eq.(1) is

$$\phi_{10} = \frac{k^2\mathrm{sn}(\xi,k)\,\mathrm{cn}(\xi,k)}{-\mathrm{dn}^2(\xi,k)+k_1},$$

and the new solution to Eq.(6) is

$$u_{10} = a_0 - \frac{ac_1\sqrt{c}\left(-\mathrm{dn}^2(\xi)+k_1\right)^2}{\mathrm{sn}^2(\xi)\,k^4\mathrm{cn}^2(\xi)} - \frac{c_1k^2\mathrm{dn}(\xi)\left(1+\left(k^2-2+2k_1\right)\mathrm{sn}^2(\xi)-k_1\right)}{\left(\mathrm{dn}^2(\xi)-k_1\right)^2} - \frac{c_1\sqrt{c}k^4\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{\left(-\mathrm{dn}^2(\xi)+\sqrt{1-k^2}\right)^2} - \frac{c_1\sqrt{ac}\mathrm{dn}(\xi)\left(1+\left(k^2-2+2k_1\right)\mathrm{sn}^2(\xi)-k_1\right)}{ck^2\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}.$$

11) If $a = 2-2k_1-k^2,\ b = 2-6k_1-k^2$ and $c = -4k_1$, then the solution to Eq.(1) is

$$\phi_{11} = \frac{k^2\mathrm{sn}(\xi,k)\,\mathrm{cn}(\xi,k)}{-\mathrm{dn}^2(\xi,k)-k_1},$$

and the new solution to Eq.(6) is

$$u_{11} = a_0 + \frac{c_1k^2\mathrm{dn}(\xi)\left[\left(2-k^2+2k_1\right)(\mathrm{sn}(\xi))^2-1-k_1\right]}{\left((\mathrm{dn}(\xi))^2+k_1\right)^2} + \frac{c_1\sqrt{c}k^4\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{c\left(\mathrm{dn}^2(\xi)+k_1\right)^2} + \frac{ac_1\sqrt{c}\left(\mathrm{dn}^2(\xi)+k_1\right)^2}{\mathrm{sn}^2(\xi)\,k^4\mathrm{cn}^2(\xi)} + \frac{c_1\sqrt{ac}\mathrm{dn}(\xi)\left[\left(2-k^2+2\,k_1\right)\mathrm{sn}^2(\xi)-1-k_1\right]}{ck^2\mathrm{sn}^2(\xi)\,\mathrm{cn}^2}.$$

12) If $a = \dfrac{2-k^2-k_1}{4},\ b = \dfrac{k^2}{2}-1-3k_1$ and $c = \dfrac{2-\,k^2-2k_1}{4}$, then the solution to Eq.(1) is

$$\phi_{12} = \frac{k^2\mathrm{cn}(\xi,k)\,\mathrm{sn}(\xi,k)}{\mathrm{sn}^2(\xi,k)+(1+k_1)\,\mathrm{dn}(\xi,k)-1-k_1},$$

and the new solution to Eq.(6) is

$$u_{12} = a_0 - \frac{c_1\sqrt{c}k^4\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{\left(\mathrm{sn}^2(\xi)+(1+k_1)\,\mathrm{dn}(\xi)-1-\sqrt{1-k^2}\right)^2} - \frac{ac_1\sqrt{c}\left(\mathrm{sn}^2(\xi)+(1+k_1)\,\mathrm{dn}(\xi)-1-k_1\right)^2}{c\,(\mathrm{sn}(\xi))^2\,k^4\,(\mathrm{cn}(\xi))^2} + \frac{c_1k^4\,(1+k_1)\,\mathrm{sn}^4(\xi)+c_1k^2[(2\,k+1)\,\mathrm{dn}(\xi)-2\,k_1-2]\mathrm{sn}^2(\xi)+c_1k^2[1+k_1-(1+k_1)\,\mathrm{dn}(\xi)]}{-\mathrm{sn}^4(\xi)+[(2+2\,k_1)\,\mathrm{dn}(\xi)+N]\,(\mathrm{sn}(\xi))^2-4-4\,k_1+(4-2\,k^2+4\,k_1)\,\mathrm{dn}(\xi)+2\,k^2} + \frac{c_1\left(\left(k_1k^2+k^2\right)\mathrm{sn}^4(\xi)+(-2\,k_1+\mathrm{dn}(\xi)-2+2\,\mathrm{dn}(\xi)\,k_1)\,\mathrm{sn}^2(\xi)+1+k_1-\mathrm{dn}(\xi)-\mathrm{dn}(\xi)\,k_1\right)}{k^2c\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}.$$

where $N = 2\,k^2+2\,k^2\sqrt{1-k^2}-k^4+2+2\,\sqrt{1-k^2}$.

13) If $a = \dfrac{2-\,k^2+2k_1}{4},\ b = \dfrac{k^2}{2}-1+3k_1$ and $c = \dfrac{2-\,k^2+2k_1}{4}$, then the solution to Eq.(1) is

$$\phi_{13} = \frac{k^2\mathrm{cn}(\xi,k)\,\mathrm{sn}(\xi,k)}{\mathrm{sn}^2(\xi,k)+(k_1-1)\,\mathrm{dn}(\xi,k)-1+k_1},$$

and the new solution to Eq.(6) is

$$u_{13} = a_0 - \frac{6c_1\sqrt{c}k^4\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}{\left(\mathrm{sn}^2(\xi)+(k_1-1)\,\mathrm{dn}(\xi)-1+k_1\right)^2} - \frac{ac_1\sqrt{c}\left(\mathrm{sn}^2(\xi)+(k_1-1)\,\mathrm{dn}(\xi)-1+k_1\right)^2}{c\mathrm{sn}^2(\xi)\,k^4\mathrm{cn}^2(\xi)} + \frac{c_1k^4\,(k_1-1)\,\mathrm{sn}^4(\xi)+c_1[k^2\,(1-2k_1)\,\mathrm{dn}(\xi)+2\,k^2\,(1-k_1)]\mathrm{sn}^2(\xi)+c_1k^2\,(k_1-1)\,(1+\mathrm{dn}(\xi))}{\mathrm{sn}^4(\xi)+[(2\,k_1-2)\,(\mathrm{dn}(\xi)+1)]\mathrm{sn}^2(\xi)+(2-k^2-2\,k_1)\,\mathrm{dn}^2(\xi)+(-4\,k_1-2\,k^2+4)\,\mathrm{dn}(\xi)+2-k^2-2\,k_1} + \frac{c_1\sqrt{ac}\left\{k^2\,(k_1+1)\,\mathrm{sn}^4(\xi)+((1+2\,k_1)\,\mathrm{dn}(\xi)-2\,k_1-2)\,\mathrm{sn}^2(\xi)-(1+k_1)\,\mathrm{dn}(\xi)+k_1+1\right\}}{ck^2\,(\mathrm{sn}(\xi))^2\,\mathrm{cn}^2(\xi)}.$$

14) If $a=\dfrac{k^2-1}{4(-B^2+C^2k^2}),\ b=\dfrac{k^2+1}{2}$ and $c=\dfrac{C^2k^4-\ B^2k^2-\ C^2k^2+\ B^2}{4}$, then the solution to Eq.(1) is

$$\phi_{14}=\frac{\left(\sqrt{\dfrac{B^2-C^2}{B^2-C^2k^2}}+\mathrm{sn}\left(\xi,k\right)\right)}{B\mathrm{cn}\left(\xi,k\right)+C\mathrm{dn}\left(\xi,k\right)},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_{14}=&a_0-\frac{c_1\sqrt{c}\left(\sqrt{B^2-C^2}+\sqrt{B^2-k^2C^2}\mathrm{sn}\left(\xi\right)\right)^2}{\left(B^2-k^2C^2\right)\left(B\mathrm{cn}\left(\xi\right)+C\mathrm{dn}\left(\xi\right)\right)^2}-\frac{ac_1\sqrt{c}\left(B^2-k^2C^2\right)\left(B\mathrm{cn}\left(\xi\right)+C\mathrm{dn}\left(\xi\right)\right)^2}{c\left(\sqrt{B^2-C^2}+\sqrt{B^2-k^2C^2}\mathrm{sn}\left(\xi\right)\right)^2}\\&+\frac{c_1[\sqrt{B^2-k^2C^2}\left(B\mathrm{dn}\left(\xi\right)+C\mathrm{cn}\left(\xi\right)\right)+\sqrt{B^2-C^2}\mathrm{sn}\left(\xi\right)\left(B\mathrm{dn}\left(\xi\right)+Ck^2\mathrm{cn}\left(\xi\right)\right)]}{\left(B^2-C^2k^2\right)\left(B\mathrm{sn}\left(\xi\right)+C\mathrm{dn}\left(\xi\right)\right)^2}\\&+\frac{c_1\sqrt{ac}\left(B^2-k^2C^2\right)[\sqrt{B^2-k^2C^2}\left(B\mathrm{dn}\left(\xi\right)+C\mathrm{cn}\left(\xi\right)\right)+\sqrt{B^2-C^2}\mathrm{sn}\left(\xi\right)\left(B\mathrm{dn}\left(\xi\right)+Ck^2\mathrm{cn}\left(\xi\right)\right)]}{c[\left(B^2-k^2C^2\right)\left(\mathrm{sn}\left(\xi\right)\right)^2+\left(2\sqrt{B^2-C^2}B^2-2\sqrt{B^2-C^2}k^2C^2\right)\mathrm{sn}\left(\xi\right)+\left(B^2-C^2\right)\sqrt{B^2-k^2C^2}]}.\end{aligned}$$

15) If $a=\dfrac{1}{4\ \left(B^2+C^2k^2\right)},\ b=\dfrac{1}{2}-k^2$ and $c=\dfrac{B^2+C^2k^2}{4}$, then the solution to Eq.(1) is

$$\phi_{15}=\frac{\sqrt{\dfrac{C^2k^2+B^2-C^2}{B^2+C^2k^2}}+\mathrm{cn}\left(\xi,k\right)}{B\mathrm{sn}\left(\xi,k\right)+C\mathrm{dn}\left(\xi,k\right)},$$

and the new solution to Eq.(1) is

$$\begin{aligned}u_{15}=&a_0-\frac{6c_1\sqrt{c}\left(\sqrt{B^2-C^2+k^2C^2}+\sqrt{B^2+k^2C^2}\mathrm{cn}\left(\xi\right)\right)^2}{\left(B^2+k^2C^2\right)\left(B\mathrm{sn}\left(\xi\right)+C\mathrm{dn}\left(\xi\right)\right)^2}-\frac{ac_1\sqrt{c}\left(B^2+k^2C^2\right)\left(B\mathrm{sn}\left(\xi\right)+C\mathrm{dn}\left(\xi\right)\right)^2}{c\left(\sqrt{B^2-C^2+k^2C^2}+\sqrt{B^2+k^2C^2}\mathrm{cn}\left(\xi\right)\right)^2}\\&+\frac{c_1\left\{\sqrt{B^2+k^2C^2}[C\mathrm{sn}\left(\xi\right)\left(-1+k^2\right)-B\mathrm{dn}\left(\xi\right)]-\sqrt{B^2-C^2+k^2C^2}\mathrm{cn}\left(\xi\right)\left[B\mathrm{dn}\left(\xi\right)+Ck^2\mathrm{sn}\left(\xi\right)\right]\right\}}{\sqrt{B^2+k^2C^2}[\left(k^2C^2-B^2\right)\left(\mathrm{sn}\left(\xi\right)\right)^2-2\,B\mathrm{sn}\left(\xi\right)C\mathrm{dn}\left(\xi\right)-C^2]}\\&-\frac{c_1\sqrt{\mathrm{ac}}\left(B^2+k^2C^2\right)\left[\mathrm{sn}\left(\xi\right)C\left(-1+k^2\right)-B\mathrm{dn}\left(\xi\right)\right]-\sqrt{B^2-C^2+k^2C^2}\mathrm{cn}\left(\xi\right)\left[B\mathrm{dn}\left(\xi\right)+Ck^2\mathrm{sn}\left(\xi\right)\right]}{c\left\{\sqrt{B^2+k^2C^2}[2\,B^2-C^2+2\,k^2C^2-\left(B^2+k^2C^2\right)\mathrm{sn}\left(\xi\right)]+2\ \left(B^2+k^2C^2\right)\sqrt{B^2-C^2+k^2C^2}\mathrm{cn}\left(\xi\right)\right\}}.\end{aligned}$$

16) If $a=\dfrac{k^4}{4(B^2+C^2)},\ b=\dfrac{k^2}{2}-1$ and $c=\dfrac{B^2+C^2}{4}$, then the solution to Eq.(1) is

$$\phi_{16}=\frac{\sqrt{\dfrac{B^2+C^2-k^2C^2}{B^2+C^2}}+\mathrm{dn}\left(\xi,k\right)}{B\mathrm{sn}\left(\xi,k\right)+C\mathrm{cn}\left(\xi,k\right)},$$

and the new solution to Eq.(6) is

$$\begin{aligned}u_{16}=&a_0-\frac{6c_1\sqrt{c}\left(\sqrt{B^2+C^2+k^2C^2}+\sqrt{B^2+C^2}\mathrm{dn}\left(\xi\right)\right)^2}{\left(B^2+C^2\right)\left(B\mathrm{sn}\left(\xi\right)+C\mathrm{cn}\left(\xi\right)\right)^2}-\frac{ac_1\sqrt{c}\left(B^2+C^2\right)\left(B\mathrm{sn}\left(\xi\right)+C\mathrm{cn}\left(\xi\right)\right)^2}{c\left(\sqrt{B^2+C^2+k^2C^2}+\sqrt{B^2+C^2}\mathrm{dn}\left(\xi\right)\right)^2}\\&+\frac{c_1\left\{\sqrt{B^2+C^2}[C\mathrm{sn}\left(\xi\right)\left(1-k^2\right)-B\mathrm{cn}\left(\xi\right)]-\sqrt{B^2+C^2-k^2C^2}\mathrm{dn}\left(\xi\right)\left[B\mathrm{cn}\left(\xi\right)+\mathrm{sn}\left(\xi\right)C\right]\right\}}{\sqrt{B^2+C^2}[\left(C^2-B^2\right)\left(\mathrm{sn}\left(\xi\right)\right)^2-2\,B\mathrm{sn}\left(\xi\right)C\mathrm{cn}\left(\xi\right)-C^2]}\\&+\frac{c_1\sqrt{\mathrm{ac}}\left(B+C\right)\left\{\sqrt{B^2+C^2}[C\mathrm{sn}\left(\xi\right)\left(1-k^2\right)-B\mathrm{cn}\left(\xi\right)]-\mathrm{dn}\left(\xi\right)\sqrt{B^2+C^2-k^2C^2}\left(B\mathrm{cn}\left(\xi\right)+\mathrm{sn}\left(\xi\right)C\right)\right\}}{c\left\{\sqrt{B^2+C^2}[k^2C^2-2\,B^2-2\,C^2+k^2\left(\mathrm{sn}\left(\xi\right)\right)^2\left(B^2+C^2\right)]-2\sqrt{B^2+C^2-k^2C^2}\mathrm{dn}\left(\xi\right)\left(B^2+C^2\right)\right\}}.\end{aligned}$$

17) If $a=-\dfrac{-2\,k+k^2+1}{B^2},\ b=2\,k^2+2$ and $c=-B^2k^2-B^2-2\,B^2k$, then the solution to Eq.(1) is

$$\phi_{17}=\frac{k\mathrm{sn}^2\left(\xi,k\right)-1}{B\left(k\mathrm{sn}^2\left(\xi,k\right)+1\right)},$$

and the new solution to Eq.(6) is

$$u_{17} = a_0 - \frac{c_1\sqrt{c}\left(k\mathrm{sn}^2(\xi)-1\right)^2}{B^2\left(k\mathrm{sn}^2(\xi)+1\right)^2} - \frac{ac_1\sqrt{c}B^2\left(k\mathrm{sn}^2(\xi)+1\right)^2}{c\left(k\mathrm{sn}^2(\xi)-1\right)^2} + \frac{4c_1k\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{Bk^2\mathrm{sn}^4(\xi)+2\,Bk\mathrm{sn}^2(\xi)+B} + \frac{4kc_1B\sqrt{ac}\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{c\left(k\mathrm{sn}^2(\xi)-1\right)^2}.$$

18) If $a=-\dfrac{2\,k+k^2+1}{B^2}$, $b=2\,k^2+2$ and $c=-B^2k^2-B^2+2\,B^2k$, then the solution to Eq.(1) is

$$\phi_{18}=\frac{k\mathrm{sn}^2(\xi,k)+1}{B\left(k\mathrm{sn}^2(\xi,k)-1\right)},$$

and the new solution to Eq.(6) is

$$u_{18} = a_0 - \frac{c_1\sqrt{c}\left(k\mathrm{sn}^2(\xi)+1\right)^2}{B^2\left(k\mathrm{sn}^2(\xi)-1\right)^2} - \frac{cB^2\left(k\mathrm{sn}^2(\xi)-1\right)^2}{c\left(k\mathrm{sn}^2(\xi)+1\right)^2} + \frac{4c_1k\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{Bk^2\mathrm{sn}^4(\xi)-2\,Bk\mathrm{sn}^2(\xi)+B} + \frac{4kc_1B\sqrt{ac}\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{c\left(k\mathrm{sn}^2(\xi)+1\right)^2}.$$

19) If $a=-\dfrac{-2+k^2-2k_1}{B^2}$, $b=-4+2\,k^2$ and $c=-B^2\left(-2+k^2+2k_1\right)$, then the solution to Eq.(1) is

$$\phi_{19}=\frac{k_1+\mathrm{dn}^2(\xi,k)}{B\left(\mathrm{dn}^2(\xi,k)-k_1\right)},$$

and the new solution to Eq.(6) is

$$u_{19} = a_0 - \frac{c_1\sqrt{c}\left(\mathrm{dn}^2(\xi)+k_1\right)^2}{B^2\left(\mathrm{dn}^2(\xi)-k_1\right)^2} - \frac{ac_1\sqrt{c}B^2\left(\mathrm{dn}^2(\xi)-k_1\right)^2}{c\left(\mathrm{dn}^2(\xi)+k_1\right)^2} + \frac{4k^2k_1B\sqrt{ac}\mathrm{dn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{c\left(\mathrm{dn}^2(\xi)+k_1\right)^2} + \frac{4c_1k_1k^2\mathrm{dn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)}{2\,B-Bk^2-2Bk_1+2\,Bk^2(k_1-1)\mathrm{sn}^2(\xi)+Bk^4\mathrm{sn}^4(\xi)}.$$

20) If $a=-B^2\left(-2+k^2+2k_1\right)$, $b=-4+2\,k^2$ and $c=-\dfrac{-2+k^2-2\,k_1}{B^2}$, then the solution to Eq.(1) is

$$\phi_{20}=\frac{B\left(\mathrm{dn}^2(\xi,k)-k_1\right)}{\mathrm{dn}^2(\xi,k)+k_1},$$

and the new solution to Eq.(6) is

$$u_{20} = a_0 - \frac{ac_1\sqrt{c}\left(\mathrm{dn}^2(\xi)+k_1\right)^2}{cB^2\left(\mathrm{dn}^2(\xi)-k_1\right)^2} - \frac{4c_1k_1k^2B\mathrm{dn}(\xi)\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)}{k^2-2k_1-2+2k^2\mathrm{sn}^2(\xi)(k_1+1)-k^4\mathrm{sn}^4(\xi)} - \frac{6cMB^2\left(\mathrm{dn}^2(\xi)-k_1\right)^2}{\left(\mathrm{dn}^2(\xi)+k_1\right)^2} - \frac{4c_1k^2k_1\mathrm{dn}(\xi)\,\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)}{Bk^2-2B+2Bk_1+2Bk^2\mathrm{sn}^2(\xi)(1-k_1)-Bk^4\mathrm{sn}^4(\xi)},$$

where

$$\xi=\sqrt{\frac{\alpha c_1\sqrt{c}}{6c\beta}-\frac{\gamma l^2}{\beta}}x+ly-\frac{\sqrt{\dfrac{\alpha c_1\sqrt{c}}{6c\beta}-\dfrac{\gamma l^2}{\beta}}\left(6\delta c+6\alpha a_0c+b\alpha c_1\sqrt{c}-6c\sqrt{a}\right)}{6c}t,$$

and $k_1 = \sqrt{1-k^2}$, A, B and C are arbitrary constants in solution 1 - solution 20.

It is known that when $k \to 1$, then $\mathrm{sn}\xi \to \tanh\xi$, $\mathrm{cn}\xi \to \mathrm{sech}\xi$, and $\mathrm{dn}\xi \to \mathrm{sech}\xi$, and when $k \to 0$, then $\mathrm{sn}\xi \to \sin\xi$, $\mathrm{cn}\xi \to \cos\xi$ and $\mathrm{dn}\xi \to 1$. So we also can obtain more solutions expressed in terms of hyperbolic functions and trigonometric functions.

Remark 1 The above 20 solutions to Eq.(1) i.e. $\phi_1 - \phi_{20}$ are first used to express the exact solutions to the nonlinear wave equations. They are first reported here to our knowledge. All the above solutions of the Zakharov–Kuznetsov equation are quite new and different from those reported in Ref.[22].

Remark 2 These solutions are obtained by substituting the relevant results in case 4 into Eq.(9), so they are only some solutions of Eq.(6). Other solutions are omitted so as not to be too verbose.

Remark 3 If the modulus $k \to 0$, we can obtain triangular-periodic solutions of Eq.(6). If the modulus $k \to 1$, we can obtain soliton solutions. The soliton solutions and triangular-periodic solutions of Eq.(6) are omitted.

4. Conclusions

In this paper, the further improved Jacobi elliptic function method is used to find various new exact doubly-periodic solutions in terms of rational formal Jacobi elliptic function for the Zakharov–Kuznetsov equation. When the modulus $k \to 1$, these obtained solutions reduce to solitary wave solutions, when the modulus $k \to 0$, these obtained solutions reduce to these triangular solutions. These solutions are new to our knowledge. The algorithm can be also applied to many other nonlinear differential equations in mathematical physics.

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