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New non-Noether conserved quantities of mechanical system in phase space

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This paper focuses on studying non-Noether conserved quantities of Lie symmetry and of form invariance for a mechanical system in phase space under the general infinitesimal transformation of groups. We obtain a new non-Noether conserved quantity of Lie symmetry of the system, and Hojman and Mei's results are of special cases of our conclusion. We find a condition under which the form invariance of the system will lead to a Lie symmetry, and, further, obtain a new non-Noether conserved quantity of form invariance of the system. An example is given finally to illustrate these results.

Keywords: non-Noether conserved quantity, Lie symmetry, form invariance, phase space
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1. Introduction

The symmetries of mechanical and physical systems are closely related to the conserved quantities. There have been proposed mainly two methods of seeking the conserved quantities of mechanical and physical systems: Noether symmetry method[1] and Lie symmetry method.[2] The Noether symmetry is an invariance of the Hamilton action under the infinitesimal transformation of groups, and the Lie symmetry is an invariance of the differential equations under the infinitesimal transformation of groups. In 2001, Mei presented a new theory of symmetry:[3] namely, the form invariance of mechanical and physical systems. The form invariance is an invariance under which transformed physical quantities such as Lagrangians, Hamiltonians, non-potential generalized forces and generalized constrained forces, etc still satisfy original dynamical equations.

As is known, the Noether symmetry can always lead to a conserved quantity, but generally the Lie symmetry and the form invariance cannot. Traditionally, the finding of conserved quantity by using the Lie symmetry is to utilize the Noether identity to obtain its Noether conserved quantity, and so is the finding of form invariance. In 1992, Hojman gave a direct method of finding a conserved quantity by using the Lie symmetry, and obtained a new conserved quantity,[4] which is a non-Noether conserved quantity. Recently, much attention has been paid to the investigation of the non-Noether conserved quantities of some mechanical systems by using Hojman's direct method.[5−13] But the above-mentioned studies are restricted to the special infinitesimal transformation of groups, in which the time parameter is not variable. By using this direct method, the non-Noether conserved quantity of Lie symmetry of a mechanical system in configuration space under the general infinitesimal transformation of groups has been studied.[14,15] In the present paper this direct method is used to study the non-Noether conserved quantities of Lie symmetry and of form invariance of a mechanical system in phase space under the general infinitesimal transformation of groups. We derive a new non-Noether conserved quantity of Lie symmetry of the system. Then we give a condition under which the form invariance of the system leads to a Lie symmetry, and, further, obtain a new non-Noether conserved quantity of form invariance of the system. Finally, an example is given to illustrate these results.

2. New non-Noether conserved quantity of Lie symmetry

Let the position of a mechanical system be determined by the n generalized coordinates $q_s (s = 1, \ldots, n)$ and n generalized momenta p_s. The motion

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of the system is described by the Hamilton canonical equations of the form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s, \quad (s = 1, \cdots, n), \quad (1)$$

where H is the Hamilton function of the system, Q_s is the non-potential generalized forces. Equations (1) can be expressed as follows:

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad (s = 1, \cdots, n). \quad (2)$$

Introduce the general infinitesimal transformations of groups

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \quad (3)$$

where ε is an infinitesimal parameter, and ξ_0, ξ_s and η_s are the generators of infinitesimal transformation of groups.

According to the theory of invariance of differential equations under the infinitesimal transformation, the invariance of Eqs.(2) under the infinitesimal transformations (3) leads to the following determining equations:

$$\begin{aligned} &\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - g_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = X^{(0)}(g_s), \\ &\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - h_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = X^{(0)}(h_s), \\ &(s = 1, \cdots, n), \end{aligned} \quad (4)$$

where

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} &= \frac{\partial}{\partial t} + g_s\frac{\partial}{\partial q_s} + h_s\frac{\partial}{\partial p_s}, \\ X^{(0)} &= \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}. \end{aligned} \quad (5)$$

If the infinitesimal generators ξ_0, ξ_s and η_s satisfy Eqs.(4), then the corresponding transformations are called the Lie symmetrical transformations of the system (1).

Hereafter, we give a new non-Noether conserved quantity of Lie symmetry of this system.

Theorem 1 If the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (3) satisfy the determining equations (4) and there exists a function $\mu = \mu(t, \boldsymbol{q}, \boldsymbol{p})$ satisfying the following equation:

$$\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \quad (6)$$

then the system (1) has the Lie symmetrical non-Noether conserved quantity as follows:

$$I = \frac{1}{\mu}\frac{\partial(\mu\xi_0)}{\partial t} + \frac{1}{\mu}\frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu}\frac{\partial(\mu\eta_s)}{\partial p_s} - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \text{const}. \quad (7)$$

Proof We have

$$\begin{aligned} \frac{\bar{\mathrm{d}}I}{\mathrm{d}t} =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} \\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} \\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right); \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} =& \frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \frac{\partial\xi_0}{\partial q_k}\frac{\partial g_k}{\partial t} - \frac{\partial\xi_0}{\partial p_k}\frac{\partial h_k}{\partial t}, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} =& \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \frac{\partial\xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s}, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} =& \frac{\partial}{\partial p_s}\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - \frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s} - \frac{\partial\eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}. \end{aligned} \quad (9)$$

Substituting Eqs.(9) into Eq.(8) and using Eqs.(4), we obtain

$$\begin{aligned} \frac{\bar{\mathrm{d}}I}{\mathrm{d}t} =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right) \\ &+ \frac{\partial g_s}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\partial h_s}{\partial p_s}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \xi_0\frac{\partial^2 g_s}{\partial q_s\partial t} + \xi_0\frac{\partial^2 h_s}{\partial p_s\partial t} \\ &+ \frac{\partial^2 g_s}{\partial q_s\partial q_k}\xi_k + \frac{\partial^2 g_s}{\partial q_s\partial p_k}\eta_k \\ &+ \frac{\partial^2 h_s}{\partial p_s\partial q_k}\xi_k + \frac{\partial^2 h_s}{\partial p_s\partial p_k}\eta_k. \end{aligned} \quad (10)$$

Performing the partial derivative of Eq.(6) with respect to t, q_k and p_k respectively, and substituting the resulting equation into Eq.(10), we have

$$\begin{aligned} \frac{\bar{\mathrm{d}}I}{\mathrm{d}t} =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial p_s}\eta_s\right) - \xi_0\frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln\mu) \\ &- \xi_k\frac{\partial}{\partial q_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu - \eta_k\frac{\partial}{\partial p_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu \\ =& \frac{\partial}{\partial q_k}\ln\mu\left(\frac{\bar{\mathrm{d}}\xi_k}{\mathrm{d}t} - \xi_0\frac{\partial g_k}{\partial t} - \xi_s\frac{\partial g_k}{\partial q_s} - \eta_s\frac{\partial g_k}{\partial p_s}\right) \\ &+ \frac{\partial}{\partial p_k}\ln\mu\left(\frac{\bar{\mathrm{d}}\eta_k}{\mathrm{d}t} - \xi_0\frac{\partial h_k}{\partial t} - \xi_s\frac{\partial h_k}{\partial q_s} - \eta_s\frac{\partial h_k}{\partial p_s}\right) \\ &+ \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\partial}{\partial t}\ln\mu - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu. \end{aligned} \quad (11)$$

Noting Eqs.(4), we can obtain

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t}=0. \tag{12}$$

3. New non-Noether conserved quantity of form invariance

Under the infinitesimal transformations (3), $H=H(t,\boldsymbol{q},\boldsymbol{p})$ and $Q_s=Q_s(t,\boldsymbol{q},\boldsymbol{p})$ become $H^*=H(t^*,\boldsymbol{q}^*,\boldsymbol{p}^*)$ and $Q_s^*=Q_s^*(t^*,\boldsymbol{q}^*,\boldsymbol{p}^*)$. Expanding H^* and Q_s^*, one has

$$\begin{aligned}H^*=&H(t^*,\boldsymbol{q}^*,\boldsymbol{p}^*)\\ =&H(t,\boldsymbol{q},\boldsymbol{p})+\varepsilon(X^{(0)}(H))+O(\varepsilon^2),\end{aligned} \tag{13}$$

$$\begin{aligned}Q_s^*=&Q(t^*,q^*,p^*)\\ =&Q(t,q,p)+\varepsilon(X^{(0)}(Q))+O(\varepsilon^2).\end{aligned} \tag{14}$$

Definition If the Hamiltonian $H=H(t,\boldsymbol{q},\boldsymbol{p})$ and non-potential generalized forces $Q_s=Q_s(t,\boldsymbol{q},\boldsymbol{p})$ satisfy the Hamilton canonical equations (1) under the infinitesimal transformations (3), i.e.

$$\dot{q}_s=\frac{\partial H^*}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H^*}{\partial q_s}+Q_s^*,\quad (s=1,\cdots,n), \tag{15}$$

then the invariance is called form invariance of the system (1).

Criterion For the mechanical system (1), if the infinitesimal generators ξ_0, ξ_s and η_s satisfy the following determining equations:

$$\begin{aligned}&\frac{\partial}{\partial p_s}(X^{(0)}(H))=0,\\ &\frac{\partial}{\partial q_s}(X^{(0)}(H))-X^{(0)}(Q_s)=0,\end{aligned} \tag{16}$$

then the system is of form invariance under the infinitesimal transformations (3).

Proof Substituting Eqs.(13) and (14) into Eqs.(15), noting Eqs.(1), and ignoring ε^2 and higher infinitesimal terms, we can obtain Eqs.(16).

Compared Eqs.(4) with Eqs.(16), it is apparent that the form invariance is different from the Lie symmetry. The form invariance of the system (1) can lead to a Lie symmetry or not, for which we have the following theorem.

Theorem 2 If the mechanical system (1) is of form invariance under the infinitesimal transformations (3), and satisfies condition

$$\begin{aligned}&\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-g_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial p_s}\\ &+\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial p_s}+\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial p_s}=0,\\ &\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t}-h_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}-\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial q_s}\\ &-\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial q_s}-\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial q_s}=0,\end{aligned} \tag{17}$$

then the form invariance of the system can lead to a Lie symmetry.

Proof Since

$$\begin{aligned}&\frac{\partial}{\partial p_s}(X^{(0)}(H))\\ =&X^{(0)}\left(\frac{\partial H}{\partial p_s}\right)+\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial p_s}+\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial p_s}+\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial p_s},\\ &\frac{\partial}{\partial q_s}(X^{(0)}(H))\\ =&X^{(0)}\left(\frac{\partial H}{\partial q_s}\right)+\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial q_s}+\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial q_s}+\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial q_s}.\end{aligned} \tag{18}$$

Substituting Eqs.(16) into Eqs.(18), and noting Eqs.(1) and (2), one has

$$\begin{aligned}&X^{(0)}(g_s)+\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial p_s}+\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial p_s}+\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial p_s}=0,\\ &X^{(0)}(h_s)-\frac{\partial H}{\partial t}\frac{\partial\xi_0}{\partial q_s}-\frac{\partial H}{\partial q_k}\frac{\partial\xi_k}{\partial q_s}-\frac{\partial H}{\partial p_k}\frac{\partial\eta_k}{\partial q_s}=0.\end{aligned} \tag{19}$$

Substituting Eqs.(19) into Eqs.(17), we can obtain Eqs.(4).

The new non-Noether conserved quantity that the form invariance leads to under the general infinitesimal transformation of groups can be given by the theorem below.

Theorem 3 For the mechanical system (1), if the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (3) satisfy Eqs.(16) and (17), and there exists a function $\mu=\mu(t,\boldsymbol{q},\boldsymbol{p})$ satisfying Eq.(6), then the form invariance of this system can lead to the non-Noether conserved quantity (7).

We can easily obtain Theorem 3 by using Theorem 1 and Theorem 2.

4. Conclusions and discussion

In this paper, we have given new non-Noether conserved quantities of the Lie symmetry and of the form invariance for a mechanical system in phase space under the general infinitesimal transformation

of groups. The result is possessed of more general significance. Particularly, if $\xi_0 = 0$ in Eqs.(3), and $p_s = \dot{q}_s$, $\eta_s = \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s$, then Theorem 1 in this paper reduces to Hojman's theorem.[4] If $\xi_0 = 0$ in Eqs.(3), then Theorem 1 reduces to the result in Ref.[8]. If the mechanical system is a Hamiltonian system, then Eq.(7) becomes

$$I = \frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \text{const.} \tag{20}$$

The conserved quantity (20) is the same as the result in Ref.[16].

5. Example

The Lagrangian of a mechanical system is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + q_1 - q_2, \tag{21}$$

and the non-potential generalized forces are $Q_1 = -\dot{q}_1$ and $Q_2 = -\dot{q}_2$. Now we study the non-Noether conserved quantities of the Lie symmetry and of the form invariance for the system in phase space. One has

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1, \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2. \tag{22}$$

$$H = p_s\dot{q}_s - L = \frac{1}{2}(p_1^2 + p_2^2) - q_1 + q_2. \tag{23}$$

From Eqs.(1) and (2), we can obtain the equations of motion of the mechanical system as

$$\begin{aligned}
&g_1 = \dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1,\\
&g_2 = \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2,\\
&h_1 = \dot{p}_1 = -\frac{\partial H}{\partial q_1} + Q_1 = 1 - p_1,\\
&h_2 = \dot{p}_2 = -\frac{\partial H}{\partial q_2} + Q_2 = -(1 + p_2).
\end{aligned} \tag{24}$$

The determining equations (4) of Lie symmetry give

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - p_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \eta_1,\\
&\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - p_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \eta_2,\\
&\frac{\bar{\mathrm{d}}\eta_1}{\mathrm{d}t} - (1 - p_1)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = -\eta_1,\\
&\frac{\bar{\mathrm{d}}\eta_2}{\mathrm{d}t} + (1 + p_2)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = -\eta_2.
\end{aligned} \tag{25}$$

The determining equations (16) of form invariance give

$$\begin{aligned}
&\frac{\partial \xi_1}{\partial p_1} - \frac{\partial \xi_2}{\partial p_1} - p_1\frac{\partial \eta_1}{\partial p_1} - p_2\frac{\partial \eta_2}{\partial p_1} - \eta_1 = 0,\\
&\frac{\partial \xi_1}{\partial p_2} - \frac{\partial \xi_2}{\partial p_2} - p_1\frac{\partial \eta_1}{\partial p_2} - p_2\frac{\partial \eta_2}{\partial p_2} - \eta_2 = 0,\\
&\frac{\partial \xi_1}{\partial q_1} - \frac{\partial \xi_2}{\partial q_1} - p_1\frac{\partial \eta_1}{\partial q_1} - p_2\frac{\partial \eta_2}{\partial q_1} - \eta_1 = 0,\\
&\frac{\partial \xi_1}{\partial q_2} - \frac{\partial \xi_2}{\partial q_2} - p_1\frac{\partial \eta_1}{\partial q_2} - p_2\frac{\partial \eta_2}{\partial q_2} - \eta_2 = 0.
\end{aligned} \tag{26}$$

From Eqs.(17), one has

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - p_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \frac{\partial \xi_1}{\partial p_1} + \frac{\partial \xi_2}{\partial p_1} + p_1\frac{\partial \eta_1}{\partial p_1} + p_2\frac{\partial \eta_2}{\partial p_1} = 0,\\
&\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - p_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \frac{\partial \xi_1}{\partial p_2} + \frac{\partial \xi_2}{\partial p_2} + p_1\frac{\partial \eta_1}{\partial p_2} + p_2\frac{\partial \eta_2}{\partial p_2} = 0,\\
&\frac{\bar{\mathrm{d}}\eta_1}{\mathrm{d}t} - (1 - p_1)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\partial \xi_1}{\partial q_1} - \frac{\partial \xi_2}{\partial q_1}\\
&- p_1\frac{\partial \eta_1}{\partial q_1} - p_2\frac{\partial \eta_2}{\partial q_1} = 0,\\
&\frac{\bar{\mathrm{d}}\eta_2}{\mathrm{d}t} + (1 + p_2)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\partial \xi_1}{\partial q_2} - \frac{\partial \xi_2}{\partial q_2}\\
&- p_1\frac{\partial \eta_1}{\partial q_2} - p_2\frac{\partial \eta_2}{\partial q_2} = 0.
\end{aligned} \tag{27}$$

Equation (6) yields

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu - 2 = 0, \tag{28}$$

with the solution being

$$u = e^{2t}. \tag{29}$$

Taking

$$\begin{aligned}
&\xi_0 = 1, \quad \xi_1 = -e^t(p_1 - 1)^2,\\
&\xi_2 = -e^t(p_2 + 1)^2,\\
&\eta_1 = e^t(p_1 - 1)^2,\\
&\eta_2 = e^t(p_2 + 1)^2,
\end{aligned} \tag{30}$$

Eqs.(25) are satisfied, but Eqs.(26) are not. Therefore, for generators (30), the system has the Lie symmetry, but not form invariance. From Theorem 1, we obtain the Lie symmetrical non-Noether conserved quantity of the system as follows:

$$I = 2 + 2e^t(p_1 + p_2) = \text{const.} \tag{31}$$

Taking

$$\begin{aligned}
&\xi_0 = t, \quad \xi_1 = q_1, \quad \xi_2 = -p_1,\\
&\eta_1 = 1, \quad \eta_2 = 0,
\end{aligned} \tag{32}$$

then Eqs.(26) are satisfied, but Eqs.(27) are not. Therefore, for generators (32), the system has the form invariance, but from Theorem 3, the form invariance cannot lead to a non-Noether conserved quantity.

Taking

$$\xi_0 = 1, \quad \xi_1 = \xi_2 = (q_1 + p_1 - t)^2,$$
$$\eta_1 = 0, \quad \eta_2 = 0, \tag{33}$$

then Eqs.(26) and (27) are satisfied. From Theorem 3, the form invariance of the system leads to a non-Noether conserved quantity as follows:

$$I = 2 + 2(q_1 + p_1 - t) = \text{const.} \tag{34}$$

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