Vol 15 No 10, October 2006
1009-1963/2006/15(10)/2193-04

Chinese Physics

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Three-order pseudo-Hamilton canonical equations

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(Received 2 September 2005; revised manuscript received 23 May 2006)

Based on the three-order Lagrangian equations, Hamilton's function of acceleration H^* and generalized acceleration momentum P^*_α are defined, and pseudo-Hamilton canonical equations corresponding to three-order Lagrangian equations are obtained. The equations are similar to Hamilton's canonical equations of analytical mechanics in form.

Keywords: three-order Lagrangian equations, pseudo-Hamilton canonical equations, time rate of change of force

PACC: 0320

1. Introduction

The theories of analytical mechanics are becoming more and more important in the fields of modern physics. The development of the theories of analytical mechanics is very vapid accordingly. In the 1990's, three-order Lagrangian equations were given first by Mei *et al.*[1] Before long, the dynamical theory of the Birkhoff system was established by Mei *et al.*[2−4] Recently, the symmetrical properties and invariance of Hamiltonian system have been investigated separately by Qiao *et al*,[5,11,12] Jia,[6] Zhang,[7,8] Fang *et al*,[9] Xu,[10] Xu *et al*,[13] Gu and Zhang,[14] and Ge and Zhang.[15] In this paper, based on three-order Lagrangian equations given by Mei *et al*,[1] Hamilton's function of acceleration H^* and generalized acceleration momentum P^*_α are first defined. Then, considering the definitions of quantities H^* and P^*_α and the form of three-order Lagrangian equations, the pseudo-Hamilton canonical equations corresponding to three-order Lagrangian equations are obtained. These results can provide a new method to solve some mechanical problems concerning time rates of change of force and acceleration energy, and also make the theory of three-order Lagrangian equations in analytical mechanics more perfect and general.

2. Obtaining pseudo-Hamilton canonical equations

Three-order Lagrangian equations are given by Mei[1] as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \dddot{q}_\alpha}\right)-\frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}=Q^*_\alpha,\quad(\alpha=1,2,\ldots,n)\,,\quad(1)$$

where

$$S=\frac{1}{2}\sum_{i=1}^{N}m_i\ddot{\boldsymbol{r}}_i\cdot\ddot{\boldsymbol{r}}_i,\quad Q^*_\alpha=\sum_{i=1}^{N}\dot{\boldsymbol{F}}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}.$$

When the time rate of change of force is a potential function, i. e. $\dot{\boldsymbol{F}}_i=-\dfrac{\partial U}{\partial \dot{\boldsymbol{r}}_i}$, one has

$$Q^*_\alpha=\sum_{i=1}^{N}\dot{\boldsymbol{F}}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}=\sum_{i=1}^{N}-\frac{\partial U}{\partial \dot{\boldsymbol{r}}_i}\cdot\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_\alpha}=\sum_{i=1}^{N}-\frac{\partial U}{\partial \dot{q}_\alpha},\quad(2)$$

where U is the acceleration potential energy, and the relation[1] $\dfrac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_\alpha}=\dfrac{\partial \boldsymbol{r}_i}{\partial q_\alpha}$ has been used in Eq.(2). If the acceleration potential energy U is not a function of generalized acceleration, i. e. $\dfrac{\partial U}{\partial \ddot{q}_\alpha}=0$, then letting $L^*=S-2U$ and substituting it into Eq.(1), one can have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dddot{q}_\alpha}\right)-\frac{1}{2}\frac{\partial L^*}{\partial \dot{q}_\alpha}=0,\quad\left(\frac{\partial U}{\partial \ddot{q}_\alpha}=0\right),\quad(\alpha=1,2,\ldots,n)\,.\quad(3)$$

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On the other hand, if $\boldsymbol{r}_i = \boldsymbol{r}_i(q_\alpha, t)$, then one obtains

$$\dot{\boldsymbol{r}}_i = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \sum_\alpha \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \boldsymbol{r}_i}{\partial t}, \tag{4}$$

$$\ddot{\boldsymbol{r}}_i = \sum_\alpha \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\ddot{q}_\alpha + \sum_\beta\sum_\alpha \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\beta \partial q_\alpha}\dot{q}_\alpha\dot{q}_\beta + 2\sum_\beta \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\beta \partial t}\dot{q}_\beta + \frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}, \tag{5}$$

$$\begin{aligned} S =& \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ \sum_{\alpha,\beta} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_\beta}\ddot{q}_\alpha\ddot{q}_\beta + \sum_{\alpha,\beta,\gamma,\tau} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial q_\gamma \partial q_\tau}\dot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma\dot{q}_\tau \right. \\ & + 4\sum_{\alpha,\beta} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial t}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial q_\beta \partial t}\dot{q}_\alpha\dot{q}_\beta + \left(\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\right)^2 + 2\sum_{\alpha,\beta,\gamma} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial q_\beta \partial q_\gamma}\ddot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma \\ & + 4\sum_{\alpha,\beta} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\beta}\ddot{q}_\alpha\dot{q}_\beta + 2\sum_{\alpha} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\ddot{q}_\alpha + 4\sum_{\alpha,\beta,\gamma} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\gamma}\dot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma \\ & \left. + 2\sum_{\alpha,\beta} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\dot{q}_\alpha\dot{q}_\beta + 4\sum_{\alpha} \frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\dot{q}_\alpha \right\} \\ =& S_2 + S_1 + S_0, \end{aligned} \tag{6}$$

where

$$S_2 = \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ \sum_{\alpha,\beta} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_\beta}\ddot{q}_\alpha\ddot{q}_\beta \right\},$$

$$S_1 = \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ 2\sum_{\alpha,\beta,\gamma} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial q_\beta \partial q_\gamma}\ddot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma + 4\sum_{\alpha,\beta} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\beta}\ddot{q}_\alpha\dot{q}_\beta + 2\sum_{\alpha} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\ddot{q}_\alpha \right\},$$

$$\begin{aligned} S_0 =& \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ \sum_{\alpha,\beta,\gamma,\tau} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial q_\gamma \partial q_\tau}\dot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma\dot{q}_\tau + 4\sum_{\alpha,\beta} \frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\beta}\dot{q}_\alpha\dot{q}_\beta + \left(\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\right)^2 \right. \\ & \left. + 4\sum_{\alpha,\beta,\gamma} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\gamma}\dot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma + 2\sum_{\alpha,\beta} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_\alpha \partial q_\beta}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\dot{q}_\alpha\dot{q}_\beta + 4\sum_{\alpha} \frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_\alpha}\cdot\frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}\dot{q}_\alpha \right\}. \end{aligned}$$

Introducing a new generalized acceleration momentum

$$P^*_\alpha = \frac{\partial S}{\partial \ddot{q}_\alpha} = \frac{\partial L^*}{\partial \ddot{q}_\alpha}, \tag{7}$$

one can have the result from expression (7)

$$\ddot{q}_\alpha = \ddot{q}_\alpha\left(P^*_\alpha, \dot{q}_\alpha, q_\alpha, t\right). \tag{8}$$

If a new Hamilton function of acceleration H^* is defined as follows:

$$H^* = H^*\left(P^*_\alpha, \dot{q}_\alpha, q_\alpha, t\right) = -L^* + \sum_\alpha P^*_\alpha \ddot{q}_\alpha, \tag{9}$$

then one has

$$\mathrm{d}H^* = -\mathrm{d}L^* + \sum_\alpha \left(P^*_\alpha \mathrm{d}\ddot{q}_\alpha + \ddot{q}_\alpha \mathrm{d}P^*_\alpha\right), \tag{10}$$

where

$$\begin{aligned} \mathrm{d}L^* =& \sum_\alpha \left(\frac{\partial L^*}{\partial \ddot{q}_\alpha}\mathrm{d}\ddot{q}_\alpha + \frac{\partial L^*}{\partial \dot{q}_\alpha}\mathrm{d}\dot{q}_\alpha + \frac{\partial L^*}{\partial q_\alpha}\mathrm{d}q_\alpha\right) \\ & + \frac{\partial L^*}{\partial t}\mathrm{d}t. \end{aligned} \tag{11}$$

Differentiating Eq.(7) with respect to time t, and considering Eq.(3), one has

$$\begin{aligned} \dot{P}^*_\alpha &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \ddot{q}_\alpha} = \frac{1}{2}\frac{\partial L^*}{\partial \dot{q}_\alpha}, \\ \ddot{P}^*_\alpha &= \frac{1}{2}\frac{\partial \dot{L}^*}{\partial \dot{q}_\alpha} - \frac{1}{2}\frac{\partial L^*}{\partial q_\alpha}. \end{aligned} \tag{12}$$

Substituting Eqs.(12) and the expression (7) into Eq.(11), one can have

$$\mathrm{d}L^* = \sum_\alpha \left(P^*_\alpha \mathrm{d}\ddot{q}_\alpha + 2\dot{P}^*_\alpha \mathrm{d}\dot{q}_\alpha + \left(\frac{\partial \dot{L}^*}{\partial \dot{q}_\alpha} - 2\ddot{P}^*_\alpha \right) \mathrm{d}q_\alpha \right) + \frac{\partial L^*}{\partial t}\mathrm{d}t. \tag{13}$$

Besides these, from the differentiation of Eq.(9), one can obtain

$$\begin{aligned} \mathrm{d}H^* =& -\mathrm{d}L^* + \sum_\alpha \left(P^*_\alpha \mathrm{d}\ddot{q}_\alpha + \ddot{q}_\alpha \mathrm{d}P^*_\alpha \right) \\ =& -\sum_\alpha \left(P^*_\alpha \mathrm{d}\ddot{q}_\alpha + 2\dot{P}^*_\alpha \mathrm{d}\dot{q}_\alpha + \left(\frac{\partial \dot{L}^*}{\partial \dot{q}_\alpha} - 2\ddot{P}^*_\alpha \right) \mathrm{d}q_\alpha \right) - \frac{\partial L^*}{\partial t}\mathrm{d}t + \sum_\alpha \left(P^*_\alpha \mathrm{d}\ddot{q}_\alpha + \ddot{q}_\alpha \mathrm{d}P^*_\alpha \right) \\ =& \sum_\alpha \left(-2\dot{P}^*_\alpha \mathrm{d}\dot{q}_\alpha - \left(\frac{\partial \dot{L}^*}{\partial \dot{q}_\alpha} - 2\ddot{P}^*_\alpha \right) \mathrm{d}q_\alpha + \ddot{q}_\alpha \mathrm{d}P^*_\alpha \right) - \frac{\partial L^*}{\partial t}\mathrm{d}t, \end{aligned} \tag{14}$$

and

$$\mathrm{d}H^* = \sum_\alpha \left(\frac{\partial H^*}{\partial P^*_\alpha}\mathrm{d}P^*_\alpha + \frac{\partial H^*}{\partial \dot{q}_\alpha}\mathrm{d}\dot{q}_\alpha + \frac{\partial H^*}{\partial q_\alpha}\mathrm{d}q_\alpha \right) + \frac{\partial H^*}{\partial t}\mathrm{d}t. \tag{15}$$

Comparing the terms in Eq.(14) with those in Eq.(15), finally one can have

$$\ddot{q} = \frac{\partial H^*}{\partial P^*_\alpha}, \tag{16}$$

$$\dot{P}^*_\alpha = -\frac{1}{2}\frac{\partial H^*}{\partial \dot{q}_\alpha}, \tag{17}$$

$$\ddot{P}^*_\alpha = \frac{1}{2}\left(\frac{\partial H^*}{\partial q_\alpha} + \frac{\partial \dot{L}^*}{\partial \dot{q}_\alpha} \right), \tag{18}$$

$$\frac{\partial L^*}{\partial t} = -\frac{\partial H^*}{\partial t}. \tag{19}$$

Equations (16) to (19) are the pseudo-Hamilton canonical equations corresponding to three-order Lagrangian equations. They are similar to Hamilton's equations of analytical mechanics in form.

3. Discussion

First, because the pseudo-Hamilton canonical equations obtained in this paper come from three-order Lagrangian equations, they are more complicated and the number of the equations involved is more than that of the Hamilton canonical equations in analytical mechanics.

Second, differentiating Eq.(15) with respect to time t and considering the results of Eqs.(16) and (17), one can have

$$\begin{aligned} \frac{\mathrm{d}H^*}{\mathrm{d}t} =& \sum_\alpha \left(\frac{\partial H^*}{\partial P^*_\alpha}\dot{P}^*_\alpha + \frac{\partial H^*}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial H^*}{\partial q_\alpha}\dot{q}_\alpha \right) + \frac{\partial H^*}{\partial t} \\ =& \sum_\alpha \left(-\frac{\partial H^*}{\partial P^*_\alpha}\frac{1}{2}\frac{\partial H^*}{\partial \dot{q}_\alpha} + \frac{\partial H^*}{\partial \dot{q}_\alpha}\frac{\partial H^*}{\partial P^*_\alpha} + \frac{\partial H^*}{\partial q_\alpha}\dot{q}_\alpha \right) \\ &+ \frac{\partial H^*}{\partial t} \\ =& \sum_\alpha \left(\frac{1}{2}\frac{\partial H^*}{\partial \dot{q}_\alpha}\frac{\partial H^*}{\partial P^*_\alpha} + \frac{\partial H^*}{\partial q_\alpha}\dot{q}_\alpha \right) + \frac{\partial H^*}{\partial t}. \end{aligned} \tag{20}$$

Obviously, if $\dfrac{\partial H^*}{\partial \dot{q}_\alpha} = 0$, $\dfrac{\partial H^*}{\partial q_\alpha} = 0$, and $\dfrac{\partial H^*}{\partial t} = 0$, then Hamilton's function of acceleration $H^* = h^*$ is a conservative quantity, where h^* is a constant. Here the conservative conditions of pseudo-Hamilton canonical function H^* and Hamilton's function H are different. The conservative conditions of the former are stricter than the latter.

Third, when the pseudo-Hamilton canonical function H^* satisfies the conservative conditions, if the generalized acceleration energy does not explicitly depend on S_1 and S_0, i.e.

$$S = S_2 = \frac{1}{2}\sum_{i=1}^{N} m_i \left\{ \sum_{\alpha,\beta} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\beta} \ddot{q}_\alpha \ddot{q}_\beta \right\},$$

one can have

$$H^* = -L^* + \sum_\alpha \frac{\partial S}{\partial \ddot{q}_\alpha}\ddot{q}_\alpha$$
$$= -S + 2U + 2S = S + 2U = h^*. \quad (21)$$

If S has the same general form as that in Eq.(6), one can obtain

$$H^* = -L^* + \sum_\alpha \frac{\partial S}{\partial \ddot{q}_\alpha}\ddot{q}_\alpha$$
$$= -S + 2U + 2S_2 + S_1$$
$$= S_2 - S_0 + 2U = h^*. \quad (22)$$

The results of Eqs.(21) and (22) are similar to the energy integral and generalized energy integral of Hamilton's Canonical equations in analytical mechanics.

4. Conclusion

From the above discussion, we know that Eqs.(16) to (19) can be used to obtain the equations of motion of the system, and can also play the same role in solving the mechanical problem as three-order Lagrangian equations. So, they are an alternative to the three-order Lagrangian equation.

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