Vol 15 No 1, January 2006
1009-1963/2006/15(01)/0232-03

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

Frame dragging in the field of Kerr family*

He Tang-Mei(何唐梅) and Wang Yong-Jiu(王永久)†

Institute of Physics, Hunan Normal University, Changsha 410081, China

(Received 16 May 2005; revised manuscript received 19 August 2005)

The interesting phenomenon of frame dragging which is associated with the rotation of the source in the field of Kerr family is discussed, and the angular velocity of an uncharged test particle is obtained with a straightforward mathematical method.

Keywords: frame dragging, general relativity, uncharged test particle
PACC: 9760L, 0460K, 0470D

1. Introduction

The phenomenon of rotation always exhibits interesting and often intriguing effects in the context of general relativity. One of the most typical examples of these rotational effects is the frame dragging, which is also called "dragging of inertial frames". Frame dragging, which appears when particles move in the field of a rotating source, is of great importance because it may lead to a better understanding of the essence of general relativity. The effect is also very important for the study of some astrophysical phenomena, since it may give rise to some additional information about the structure and the state of motion of the sources of the gravitational fields of astrophysical objects. For this reason it is interesting to examine the properties of the gravitational field of the Kerr family from this point of view since they are the only exact solutions of Einstein's equations known at present which could represent the exterior field of a rotating neutral or electrically charged or electrically and magnetically charged body in asymptotically flat space. They are corresponding to the Kerr solution,[1] the Kerr–Newman (KN) solution[2] and the Kerr–Newman–Kasuya (KNK) solution,[3] respectively.

In this paper, the angular velocity of the frame dragging in the field of the Kerr family is calculated with a straightforward mathematical method, and then the effect is also discussed.

2. The frame dragging in the field of Kerr family

The most useful form of the solution is given in terms of coordinates t, r, θ and φ which can be interpreted most simply and naturally on a manifold where t and r are Boyer–Lindquist coordinates running from $-\infty$ to $+\infty$; θ and φ are ordinary spherical coordinates in which φ is periodic with a period of 2π, and θ runs from 0 to π. The covariant form of the metric tensor with signature (+, −, −, −) is expressed as[4]

$$\begin{aligned} \mathrm{d}s^2 =& g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 \\ &+ g_{\varphi\varphi}\mathrm{d}\varphi^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi, \end{aligned} \tag{1}$$

where

$$g_{tt} = \rho^{-2}(\Delta - a^2\sin^2\theta),$$

$$g_{rr} = -\rho^2\Delta^{-1},$$

$$g_{\theta\theta} = -\rho^2,$$

$$g_{\varphi\varphi} = -\rho^{-2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\sin^2\theta,$$

$$g_{t\varphi} = \rho^{-2}a\sin^2\theta(2mr - e^2).$$

The parameters m, e, a are, respectively, associated with the mass, charge and angular momentum per unit mass of the source. And the charge is

$$e = \sqrt{Q^2 + P^2},$$

*Project supported by the State Key Program for Basic Research of China (Grant No 2003CB716300) and the Natural Science Foundation of Hunan Education Department of China (Grant No NSF 03A028 and 02C203).
†Correspondence author. E-mail: wyj@hunnu.edu.cn

where Q is the electric charge and P is the magnetic charge. When the metric represents the KNK solution, both Q and P are non vanishing; when it represents the KN solution, P vanishes; and when if represents the Kerr solution, both Q and P are vanishing. And the abbreviations are

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Delta = r^2 - 2mr + a^2 + e^2.$$

To examine the properties of the field, we assume that an uncharged test particle with mass μ and zero angular momentum is released from a non-rotating frame, far from the source of the Kerr family.

Thus, the Lagrangian of the test particle can be expressed as

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \tag{2}$$

where the dot over a symbol denotes ordinary differentiation with respect to an affine parameter λ. And λ is related to the proper time τ by

$$\tau = \mu\lambda.$$

From Eq.(2), we can obtain the momentum of the test particle as

$$P_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\dot{x}^j. \tag{3}$$

Thus, the corresponding momentum in the coordinates of t, r, θ and φ are expressed as

$$P_t = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi},$$

$$P_r = g_{rr}\dot{r},$$

$$P_\theta = g_{\theta\theta}\dot{\theta},$$

$$P_\varphi = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}.$$

Since the space–time of the Kerr family is stationary and axially symmetric, and the g_{ij} is a function of r and θ only, the momenta P_t and P_φ are conserved along the geodesics. Immediately, we obtain two constants of motion: one is corresponding to the conservation of energy, E, and the other is the angular momentum about the symmetry axis, Φ; thus we have

$$P_t \equiv -E = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi},$$

$$P_\varphi \equiv \Phi = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi}.$$

We may then write

$$\dot{t} = \frac{g_{\varphi\varphi}E + g_{t\varphi}\Phi}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}, \tag{4}$$

$$\dot{\varphi} = \frac{g_{t\varphi}E + g_{tt}\Phi}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}. \tag{5}$$

Thus, we can obtain the angular velocity of the frame dragging as

$$\omega \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\dot{\varphi}}{\dot{t}} = -\frac{g_{t\varphi}E + g_{tt}\Phi}{g_{\varphi\varphi}E + g_{t\varphi}\Phi}. \tag{6}$$

The velocity is then made up of two contributions. One is due to the orbital motion of the source in the inertial frame and the other is due to the source's intrinsic angular momentum. And the spatial curvature plays no role. If we let $\Phi = 0$, we obtain the angular velocity $\bar{\omega}$ of a zero angular momentum particle as measured by an observer for whom t is the proper time, and the velocity is given by

$$\bar{\omega} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a(2mr - e^2)}{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}. \tag{7}$$

Obviously, $\bar{\omega}$ vanishes at infinity and at $r = e^2/(2m)$. That is to say, the test particle, which is released from a non-rotating frame, far from the source, with a zero angular momentum, accumulates a nonzero angular velocity in the same angular direction as that of the source of the metric, as it plunges toward the origin. Then at $r = e^2/(2m)$, the angular velocity of the particle vanishes again. This is just the interesting phenomenon of frame dragging in general relativity. When $e^2 = Q^2 + P^2$, i.e. in the KNK field, $r_{\mathrm{KNK}} = (Q^2 + P^2)/(2m)$; when the source is endowed only with electric charge Q, i.e. in the KN field, $r_{\mathrm{KN}} = Q^2/(2m)$; when it is considered in the Kerr field, $r_{\mathrm{K}} = 0$, i.e. the centre of the source. Thus, it is apparent that both the electric charge and the magnetic charge spread over the zero angular velocity surface far away from the centre of the source.

Following Steadman,[5] we can also obtain the motion region of the test particle in the field of the Kerr family. Here, we consider only the case of $\Delta > 0$, i.e. $a^2 + e^2 > m^2$. In this case, the metric has a ring singularity at $r = 0$, and $\theta = \pi/2$. Such a singularity is naked.

Since

$$g_{rr} < 0, \quad \text{and} \quad g_{\theta\theta} < 0,$$

we have

$$g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 < 0.$$

For null geodesics we have $\mathrm{d}s^2 = 0$.
Thus

$$g_{tt}\mathrm{d}t^2 + g_{\varphi\varphi}\mathrm{d}\varphi^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi > 0.$$

It leads to

$$g_{tt}\dot{t}^2 + g_{\varphi\varphi}\dot{\varphi}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} > 0. \quad (8)$$

Substituting Eqs.(4) and (5) into the inequality (8), we have

$$g_{\varphi\varphi}E^2 + g_{tt}\Phi^2 + 2g_{t\varphi}E\Phi < 0.$$

This is just the condition that the motion region must satisfy.

When $\Delta < 0$, i.e. $a^2 + e^2 < m^2$, the metric is referred to as a rotating black hole. The event horizon $r_{\pm}^{\rm h}$ and the Killing horizon $r_{\pm}^{\rm K}$ of the black hole are, respectively, given by

$$r_{\pm}^{h} = m \pm \sqrt{m^2 - a^2 - e^2},$$
$$r_{\pm}^{K} = m \pm \sqrt{m^2 - a^2\cos^2\theta - e^2}.$$

There is a region between the outer event horizon $r_{+}^{\rm h}$ and the Killing horizon $r_{+}^{\rm K}$, known as ergosphere. The test particle is dragged along with the black hole's rotation once it is inside the Killing horizon. This dragging continues as it approaches the event horizon. That is to say, inside the ergosphere, the test particle must move in the direction of the rotation of the black hole (the φ direction). It can be interpreted as the fact that the test particle outside the Killing horizon must have positive energy; therefore the particle inside the ergosphere with negative energy must either remain on a geodesic inside the Killing horizon, or be accelerated until its energy is positive if it is to escape.

It is more interesting that, $\bar{\omega}$ is negative in the situation where $g_{\varphi\varphi}$ is positive in Eq.(7). That is to say, in this situation, the zero angular momentum test particle may acquire an angular velocity in the opposite direction of rotation from the source of the metric. Such "negative frame dragging" is also called "frame dragging anomaly" in general relativity, which has attracted a lot of interest recently. Its occurrence is dependent on the coefficient of the metric in the φ coordinate. Yet, negative frame dragging occurs in non-rotating reference frames for the metrics of all solutions of the Kerr family that are associated with rotating sources.

The phenomenon of the frame dragging is, in a sense, the intrinsic angular momentum of the source that, associated with a, drags the inertial frame into rotation about it. One direct manifestation of this frame dragging in the gravitational field of the Kerr family is the gyroscopic precession with respect to the stationary observer.[6−9] Yet, the precession is extremely small. A one percent measurement of the frame dragging effect would be a significant and unique test of general relativity. Gravity Probe B-a satellite that would check the frame dragging effect was finally launched from Vanderburg Air Force Base in California last year. If successful, it would assuredly be ranked as one of the classical experiments of physics.

3. Conclusion

In this paper, we have discussed the frame dragging in the field of Kerr family. The frame dragging is a general relativistic feather. It is due to the influence of gravity alone, and has no counterpart in Newtonian physics.[10] It is of great significance to astrophysical applications.

References

[1] Kerr R P 1963 *Phys. Rev. Lett.* **11** 237
[2] Newman E T, Couch E, Chinnapered K, Exton A, Prakash A and Torrence R 1965 *J. Math. Phys.* **6** 918
[3] Kasuya M 1982 *Phys. Rev.* D **25** 995
[4] Wang Y J and Tang Z M 1990 *Theory and Effects of Gravitation* (Changsha: Press of Hunan Science and Technology)
[5] Steadman B R 1995 *Class. Quantum Grav.* **16** 3685
[6] Iyer B R and Vishveshwara C V 1993 *Phys. Rev.* D **48** 5706
[7] Chen J H and Wang Y J 2003 *Class. Quantum Grav.* **20** 3897
[8] Chen J H and Wang Y J 2004 *Chin. Phys.* **13** 583
[9] Wang Y J and Tang Z M 2004 *Chin. Phys.* **13** 579
[10] Wang Y J and Tang Z M 2002 *Sci. Chin. Ser.* A **45** 528