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Dielectric responses of graded composites having generalized gradation profiles*

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Under an external uniform electric field, the dielectric response of graded cylindrical composites having generalized dielectric profile inclusions is investigated. The generalized dielectric profile of graded cylindrical inclusion is expressed in the form, $\varepsilon_i(r) = c(b+r)^k e^{\beta r}$ where r is the radial variable of the cylindrical inclusions and c, b, k and β are parameters. The local potential solution of generalized dielectric profile graded composites is derived by means of the power series method and the effective dielectric response is predicted in the dilute limit. Moreover, from the result of generalized profile, the analytical solutions of local potentials and the effective responses of graded composites having three cases of dielectric profiles, i.e., the exponential profile $\varepsilon_{\rm i}(r) = ce^{\beta r}$, the general power law profile $\varepsilon_{\rm i}(r) = c(b+r)^k$ and the profile $\varepsilon_{\rm i}(r) = cr^k e^{\beta r}$, are sorted out, respectively. In the dilute limit, our exact results are used to test the validity of differential effective dipole approximation (DEDA) for estimating the effective response of graded cylindrical composites, and it is shown that the DEDA is in excellent agreement with the exact result.

Keywords: generalized profiles, graded composite, effective response
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1. Introduction

Physical properties of composites have attracted much attention because of their applications to the material science.[1−9] As one kind of composites, the physical properties of a graded composite are investigated for their specific needs of designing new functionally graded materials, such as electric property, optical property, mechanical and thermal properties.[10−15] For example, the mechanical properties of a graded composite can be designed by changing their microstructure and/or inclusion particle physical properties along the radial direction of the cylindrical or spherical inclusions. Recently, many authors have focused their attention on studying the properties of graded composites because the classical effective property formulas of homogeneous and isotropic composites are not suitable to predict the properties of graded composites.[15−22] In graded composites, the differential physical property profiles of inclusions play different roles in controlling the overall properties of the graded composites, such as the simple power-law, linear, and exponential profiles.[15,22,23] For instance, with different dielectric profiles of the inclusions, the distributions of local electric potential and electric field induced by external electric field result in some different interesting optical properties.[24] In fact, the more parameters in a gradient profile offer more freedom in the control of the effective material properties. More importantly, the analytical results of graded composites give an insight into the property of graded composite. Therefore, it is one of our aims to find a suitable grading profile with more freedom parameters for a specific engineering application. In addition, using our exact results, we have tested the validity of differential effective dipole approximation (DEDA) to estimate the effective response of arbitrary graded profiles in practice applications.[18−20]

As a theoretical model, we will investigate the exact local potential and the effective response of a

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graded cylindrical composite having the generalized dielectric profile $\varepsilon_{\rm i}(r) = c(b+r)^k {\rm e}^{\beta r}$. This generalized dielectric profile of inclusion has the coupling properties of the general power-law, linear and exponential profiles, and the dielectric response of graded composites with generalized profile is very different from that of the simple graded composites. From our generalized model, we have sorted out the analytical potentials and effective responses of graded composites having the exponential profile $\varepsilon_{\rm i}(r) = c{\rm e}^{\beta r}$, the general power-law profile $\varepsilon_{\rm i}(r) = c(b+r)^k$ and the product profile $\varepsilon_i(r) = cr^k {\rm e}^{\beta r}$, respectively.

2. Potential distributions in a cylindrical graded composite with generalized profile

For a linear constitutive relationship $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ between the electric displacement $\boldsymbol{D}$ and the electric field $\boldsymbol{E}$ of both the host and the inclusion regions, we investigate the potential distribution in a composite material having generalized dielectric profile of cylindrical inclusions in an external uniform electric field. Under the quasi-electrostatic condition, the corresponding governing equations are $\nabla \cdot \boldsymbol{D} = 0$ and $\nabla \times \boldsymbol{E} = 0$. The boundary conditions are the continuities of the local electric potentials and the normal components of the electric displacement at the interface between the inclusions and the host medium. At infinity the potential must match that of the external applied electric field.

Let us consider a long cylindrical fibre having a generalized dielectric profile $\varepsilon_{\rm i}(r) = c(b+r)^k {\rm e}^{\beta r}$ embedded in a homogeneous and isotropic host having dielectric constant $\varepsilon_{\rm h}$, where the radius of the long cylindrical fibre is a, and r is the radial variable of a cylindrical inclusion in cylindrical coordinates. The equations of local potentials Φ_α in whole composite regions are written as follows:

$$\nabla \varepsilon_\alpha(r) \cdot \nabla \Phi_\alpha + \varepsilon_\alpha(r) \nabla^2 \Phi_\alpha = 0, \quad \text{in} \quad \Omega_\alpha, \tag{1}$$

where the subscripts $\alpha = {\rm i}, {\rm h}$ are the quantities in inclusion and host regions, respectively. Φ_α and Ω_α are the potentials and the regions of the α type material, respectively.

When an external uniform electric field $E_a = E_0 \hat{x}$ is applied to the graded composites along $\hat{x}$-direction, the differential equations of potentials in the whole region reduce to a two-dimensional problem in cylindrical coordinates because the electric field of $\hat{z}$-direction is a constant,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_\alpha(r) r \frac{\partial \Phi_\alpha(r,\varphi)}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial \varphi}\left[\varepsilon_\alpha(r)\frac{1}{r}\frac{\partial \Phi_\alpha(r,\varphi)}{\partial \varphi}\right] = 0, \quad \text{in} \quad \Omega_\alpha. \tag{2}$$

Clearly, with equation (2) and boundary condition at infinity, the potential solution of host region is $\Phi_{\rm h}(r,\varphi) = -(r + Br^{-1})E_0 \cos(\varphi)$. The local potential of inclusion region is expressed by means of the method of separable variables $\Phi_{\rm i}(r,\varphi) = \sum_{n=0}^{\infty} R_n(r)\cos(n\varphi)$. Substituting $\Phi_{\rm i}$ into Eq.(2), we have the equation of radial part $R_n(r)$ of inclusion potential,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_{\rm i}(r) r \frac{\partial R_n(r)}{\partial r}\right] - \frac{n^2}{r^2}[\varepsilon_{\rm i}(r) R_n(r)] = 0. \tag{3}$$

For a cylindrical inclusion with dielectric profile $\varepsilon_{\rm i}(r) = c(b+r)^k {\rm e}^{\beta r}$, Eq.(3) is rewritten in the form

$$\frac{\partial^2 R_n(r)}{\partial r^2} + [1/r + k/(b+r) + \beta]\frac{\partial R_n(r)}{\partial r} - n^2 r^{-2} R_n(r) = 0. \tag{4}$$

Here, we shall use the Frobenius method to solve Eq.(4) by letting $R_n(r) = r^p \sum_{i=0}^{\infty} a_i r^i$.[25] Substituting the power series of $R_n(r)$ into Eq.(4), we have

$$\sum_{i=0}^{\infty}(bp^2 - bn^2)a_i r^{p+i} + \sum_{i=0}^{\infty}[(p^2 + pd - p - n^2)a_i + b(i+1)(2p+1)a_{i+1}]r^{p+i+1} + \sum_{i=0}^{\infty}[\beta p a_i + (i+1)(2p+d)a_{i+1} + b(i+2)(i+1)a_{i+2}]r^{p+i+2} + \sum_{i=0}^{\infty}(i+1)[(i+2)a_{i+2} + \beta a_{i+1}]r^{p+i+3} = 0, \tag{5}$$

where $d = \beta b + k + 1$. The characteristic exponent p is determined when we let the coefficient of the lowest power term r^{p+i} be zero,

$$b(p^2 - n^2)a_0 = 0. \tag{6}$$

In Eq.(6), we assume that the constants a_0 and b are not zero (the case of $b = 0$ will be discussed in next part of this section). Thus, for the case of $b \neq 0$, we have $p = \pm n$. In Eq.(5), if we set the coefficient of the power term r^{p+i+3} to be zero, the iteration relationship between the constants a_i, a_{i+1} and a_{i+2}, is obtained:

$$\begin{aligned} a_{i+2} =& \{[n^2 - i(i+1) - d(p+i+1) \\ & - p(p+2i+1)]a_{i+1} \\ & - \beta(p+i)a_i\}/b[(p+i+2)^2 - n^2], \\ & (i = 0, 1, 2, \ldots) \end{aligned} \tag{7}$$

where $a_1 = p(1-d)a_0/b[(p+1)^2 - n^2]$, and a_0 is a constant. Generally, we give, $a_0 = 1$. Furthermore, it can be demonstrated that the power series solution of radial part $R_n(r)$ is convergent in the region $r < |b|$. Because the potential of inclusion is a constant at the polar point $r = 0$, the solution of radial part is $R_n(r) = r^n \sum_{i=0}^{\infty} a_i r^i$ (i.e. $p = n$). The solutions of potentials in inclusion and host regions are determined by using the boundary conditions

$$\Phi_{\rm i}(r, \varphi) = A_1 r \left(\sum_{i=0}^{\infty} a_i r^i \right) \cos(\varphi) E_0, \quad r \leq a < |b| \tag{8}$$

$$\Phi_{\rm h}(r, \varphi) = -(r + Br^{-1}) \cos(\varphi) E_0, \quad r > a \tag{9}$$

where

$$A_1 = -2\varepsilon_{\rm h} / \left[c(b+a)^k {\rm e}^{a\beta} \sum_{i=0}^{\infty} (i+1) a_i a^i + \varepsilon_{\rm h} \sum_{i=0}^{\infty} a_i a^i \right],$$

$$B = -a^2 \left(1 + A_1 \sum_{i=0}^{\infty} a_i a^i \right).$$

Therefore, for the generalized dielectric profile, the solutions of potentials in both inclusion and host regions are given by Eqs.(8) and (9) if the radius of a cylindrical fibre is less than the absolute value of parameter b.

Now, we demonstrate that Eqs.(8) and (9) can be reduced to the classical formula of homogeneous media having $\varepsilon_{\rm i} = c$ if we take the limit $k \to 0, \beta \to 0$ to potential solutions of the generalized dielectric profile composites. In this case, we have $\lim_{k\to 0, \beta \to 0} a_i = 0$ (where $i \neq 0$), $\lim_{k\to 0, \beta \to 0} A_1 = -2\varepsilon_{\rm h}/(c + \varepsilon_{\rm h})$, $\lim_{k\to 0, \beta \to 0} B = a^2(\varepsilon_{\rm h} - c)/(c + \varepsilon_{\rm h})$. Using these limits, equations (8) and (9) become the known Maxwell formulae of cylindrical homogeneous composites with $\varepsilon_{\rm i} = c$.[26]

Next, from the potential solutions (8) and (9) of the generalized profile graded composites, we shall discuss the analytical potentials of graded composites for three cases of dielectric profiles.

Case (A) exponential profile $\varepsilon_{\rm i}(r) = c{\rm e}^{\beta r}$; **Case (B)** the general power law profile $\varepsilon_{\rm i}(r) = c(b + r)^k$; **Case (C)** the product profile $\varepsilon_{\rm i}(r) = cr^k {\rm e}^{\beta r}$.

Case (A) exponential profile $\varepsilon_{\rm i}(r) = c{\rm e}^{\beta r}$. We get the following formulas if we take the limit $k \to 0$ to Eqs.(8) and (9): $a_{i+1} = -\beta a_i/(i+3)$ (or $a_{i+1} = (-\beta)^{i+1} a_0/(3)_{i+1}$, where $(a)_i = a(a+1)(a+2)\cdots(a+i-1)$, $(a)_0 = 1$), $\sum_{i=0}^{\infty} a_i a^i = F(1, 3, -\beta a)$, $\sum_{i=0}^{\infty} (i+1) a_i a^i = F(2, 3, -\beta a)$ and $\sum_{i=0}^{\infty} a_i r^i = F(1, 3, -\beta r)$, where $F(\alpha, \gamma, z)$ is Kummer function.[27] Substituting these limits into equations (8) and (9), we have potentials of graded composites with exponential profile,

$$\Phi_{\rm i}(r, \varphi) = A_1 r F(1, 3, -\beta r) \cos(\varphi) E_0, \quad r \leq a \tag{10}$$

$$\Phi_{\rm h}(r, \varphi) = -(r + Br^{-1}) \cos(\varphi) E_0, \quad r > a \tag{11}$$

where

$$A_1 = -2\varepsilon_{\rm h}/(c{\rm e}^{a\beta} v_1 + \varepsilon_{\rm h} v_2),$$

$$B = a^2(\varepsilon_{\rm h} v_2 - c{\rm e}^{a\beta} v_1)/(c{\rm e}^{a\beta} v_1 + \varepsilon_{\rm h} v_2),$$

$$v_1 = F(2, 3, -a\beta),$$

$$v_2 = F(1, 3, -a\beta).$$

Here, we should note that the potentials (10) and (11) are analytical solutions of graded composites having the exponential profile because the Kummer function is analytic over the whole complex plane, which is not limited by the parameter b. This conclusion has also been given in Ref.[22, 23].

Case (B) the general power law profile $\varepsilon_{\rm i}(r) = c(b + r)^k$. Taking the limit $\beta \to 0$ to Eqs.(8) and (9), we obtain the following limits: $a_i = \left(-\frac{1}{b} \right)^i (\alpha_1)_i (\beta_1)_i / (\gamma_1)_i i!$ (where $\alpha_1 = (2 + k - \sqrt{k^2 + 4})/2$, $\beta_1 = (2 + k + \sqrt{k^2 + 4})/2$, $\gamma_1 = 3$),

$$\sum_{i=0}^{\infty} a_i a^i = F\left(\alpha_1, \beta_1, \gamma_1, -\frac{a}{b} \right),$$

$$\sum_{i=0}^{\infty} a_i r^i = F(\alpha_1, \beta_1, \gamma_1, -\frac{r}{b})$$

and

$$\begin{aligned}&\sum_{i=0}^{\infty}(i+1)a_i a^i\\ =&F\left(\alpha_1, \beta_1, \gamma_1, -\frac{a}{b}\right)\\ &-\frac{a}{b}\frac{\alpha_1\beta_1}{\gamma_1}F\left(\alpha_1+1, \beta_1+1, \gamma_1+1, -\frac{a}{b}\right),\end{aligned}$$

where $F(\alpha_1, \beta_1, \gamma_1, z)$ is the hyper-geometric function.[27] With these limits, we have

$$\Phi_{\rm i}(r,\varphi) = -A_1\frac{r}{b}F\left(\alpha_1, \beta_1, \gamma_1 - \frac{r}{b}\right)\cos(\varphi)E_0, \quad r \le a \tag{12}$$

$$\Phi_{\rm h}(r,\varphi) = -(r + Br^{-1})\cos(\varphi)E_0, \quad r > a \tag{13}$$

where

$$\begin{aligned}A_1 &= -2a\varepsilon_{\rm h}/[cv_1 + \varepsilon_{\rm h}v_2],\\ B &= a^2(A_1F/b - 1),\\ \xi &= -a/b,\\ F &= F(\alpha_1, \beta_1, \gamma_1; \xi),\\ F_1 &= F(\alpha_1+1, \beta_1+1, \gamma_1+1; \xi),\\ v_1 &= (b+a)^k(\xi F + \xi^2\alpha_1\beta_1\gamma_1^{-1}F_1),\\ v_2 &= \xi F.\end{aligned}$$

It is also noted that Eqs.(12) and (13) are the analytical solutions of general power law graded composites since the hyper-geometric function is analytic over all complex plane except its singular points. Thus, the potential solutions are convergent in whole complex plane.

Case (C) the product profile $\varepsilon_{\rm i}(r) = cr^k{\rm e}^{\beta r}$. In this case, it is invalid to take the limit $b \to 0$ to Eqs.(8) and (9) because the potential solution of cylindrical inclusion is convergent in the range of $r < |b|$. However, for the limit $b \to 0$, the power series solution of graded composites having dielectric profile $\varepsilon_{\rm i}(r) = cr^k{\rm e}^{\beta r}$ can be solved from the Eq.(5). The characteristic exponent p is obtained by setting coefficient of the low power term r^{p+1} to be zero, $(p^2 + kp - n^2)a_0 = 0$. So, we have $p = (-k \pm \sqrt{k^2 + 4n^2})/2$. Moreover, if we set the coefficient of the power series term r^{p+i} to be zero, the iteration relationship between the constants a_i and a_{i-1} is obtained, $a_i = -\beta(p+i-1)a_{i-1}/i(2p+k+i)$ or $a_i = (-\beta)^i(p)_i/i!(2p+k+1)_i$. Clearly, the power series $\sum_{i=0}^{\infty} a_i r^i$ is convergent in the whole space. The radial part of potential is $R_n(r) = r^p\sum_{i=0}^{\infty} a_i r^i = r^p F(p, 2p+k+1, -\beta r)$, where $p = (\sqrt{k^2+4n^2} - k)/2$, and $F(p, 2p+k+1, -\beta r)$ is the Kummer function. Using the boundary conditions, we have the potential solutions

$$\Phi_{\rm i}(r,\varphi) = A_1 r^p F(p, 2p+k+1, -\beta r)\cos(\varphi)E_0, \tag{14}$$

$$\Phi_{\rm h}(r,\varphi) = -(r + Br^{-1})\cos(\varphi)E_0, \tag{15}$$

where

$$\begin{aligned}p &= (\sqrt{k^2+4} - k)/2,\\ A_1 &= -2\varepsilon_{\rm h}a^{1-p}/(ca^k{\rm e}^{a\beta}v_1 + \varepsilon_{\rm h}v_2),\\ v_1 &= pF(p+1, 2p+k+1, -a\beta),\\ v_2 &= F(p, 2p+k+1, -a\beta),\\ B &= -a^2(A_1a^{p-1}v_2 + 1).\end{aligned}$$

It is clear that Eqs.(14) and (15) are analytical solutions and valid in the whole complex plane because of the Kummer function, and we have demonstrated this conclusion in Ref.[23]. In addition, it is easy to derive the potentials of graded composites having the simple power law profile $\varepsilon_{\rm i} = cr^k$ from Eqs.(14) and (15) by setting $\beta = 0$.

3. Effective dielectric response

In the dilute limit, we shall estimate the effective dielectric responses of graded long cylindrical composites having the generalized dielectric profile by means of potentials in inclusion region, where the unidirectional parallel cylindrical inclusions are dispersedly immersed in a host. We define the effective response $\varepsilon_{\rm e}$ of graded composites with the volume averages of the electric displacement and the electric field, $\bar{\boldsymbol{D}} = \varepsilon_{\rm e}\bar{\boldsymbol{E}}$. Using Landau's formula, we have[28]

$$\frac{1}{V}\int_{\Omega_i}[\varepsilon_{\rm i}(r) - \varepsilon_{\rm h})\boldsymbol{E}\,{\rm d}v = (\varepsilon_{\rm e} - \varepsilon_{\rm h})\bar{\boldsymbol{E}}, \tag{16}$$

where V denotes the volume of the whole composite region. Substituting the potential of a single cylindrical inclusion into the equation (16) under the external uniform electric field $\bar{\boldsymbol{E}}$, we obtain the effective response formula in the dilute limit

$$\varepsilon_{\rm e} = \varepsilon_{\rm h} + f_{\rm i}A_1(\varepsilon_{\rm h}H_1 - ca^{-2}H_2), \tag{17}$$

where $f_{\rm i}$ denotes the volume fraction of inclusions and A_1 is given in Eq.(8),

$$H_1 = \sum_{i=0}^{\infty}\left(\frac{i+1}{i+3}a_{i+1}a^{i+1} + \frac{2}{i+2}a_i a^i\right),$$

$$H_2 = \sum_{i=0}^{\infty}[(i+1)a_{i+1}I_{i+2} + 2a_i I_{i+1}],$$

$$I_i = \int_0^a r^i(b-r)^k \mathrm{e}^{\beta r}\mathrm{d}r, \quad (\text{where } \ i = 1, 2, 3, \cdots).$$

In order to calculate I_i, we introduce a linear transformation $x = r + b$ to I_i so that $I_i = \mathrm{e}^{-b\beta}\int_b^{a+b}(x-b)^i x^k \mathrm{e}^{\beta x}\mathrm{d}x$, where, $i = 1, 2, 3, \cdots$. Thus, we have

$$\begin{aligned} I_i =& \mathrm{e}^{-b\beta}\int_b^{a+b}\left[\sum_{m=0}^{i} C_i^m x^{i-m+k}\mathrm{e}^{\beta x}(-b)^m\right]\mathrm{d}x \\ =& \mathrm{e}^{-b\beta}\sum_{m=0}^{i} C_i^m(-b)^m T_{m,k}, \end{aligned}$$

where $T_{m,k} = \int_b^{a+b} x^{i-m+k}\mathrm{e}^{\beta x}\mathrm{d}x$. Here, we should note that, for the given finite value of parameter k, the value of quantity $T_{m,k}$ is easily calculated in terms of power series expansion of the exponent term, $\mathrm{e}^{\beta x} = \sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!}$.

Now, we shall directly give the effective dielectric response of graded composites for three cases of dielectric profiles by using Eq.(17) or their potentials of Section 2. In the dilute limit, we have the effective dielectric responses of the three cases.

Case (A) exponential profile $\varepsilon_{\mathrm{i}}(r) = c\mathrm{e}^{\beta r}$,

$$\varepsilon_{\mathrm{e}} = \varepsilon_{\mathrm{h}} + 2f_{\mathrm{i}}A_1(\varepsilon_{\mathrm{h}}u_2 - c\mathrm{e}^{a\beta}u_1), \tag{18}$$

where A_1 is given in Eq.(10).

$$\begin{aligned} u_1 =& (1 - \mathrm{e}^{-a\beta} - a\beta\mathrm{e}^{-a\beta})/a^2\beta^2, \\ u_2 =& (\mathrm{e}^{-a\beta} + a\beta - 1)/a^2\beta^2. \end{aligned}$$

In the above derivation process, the Kummer function $F(1,3,-a\beta) = 2(\mathrm{e}^{-a\beta} + a\beta - 1)/a^2\beta^2$ and $F(2,3,-a\beta) = 2(1 - \mathrm{e}^{-a\beta} - a\beta\mathrm{e}^{-a\beta})/a^2\beta^2$ are applied. If the radius of long cylindrical fibre is the unit length, our result is that same as that of Eq.(14) of Ref.[22].

Case (B) the general power law profile $\varepsilon_{\mathrm{i}}(r) = c(b+r)^k$,

$$\varepsilon_{\mathrm{e}} = \varepsilon_{\mathrm{h}} + 2f_{\mathrm{i}}A_1(cu_3 - \varepsilon_{\mathrm{h}}u_4), \tag{19}$$

where the coefficient A_1 is given in Eq.(12). To calculate the coefficients u_3 and u_4, we define an integration

$$\begin{aligned} & I(k,\alpha,\beta,\gamma;\frac{a}{b}) \\ =& \int_0^a (b+r)^k F\left(\alpha,\beta,\gamma,-\frac{r}{b}\right)\mathrm{d}r \\ =& \int_b^{a+b} r^k F\left(\alpha,\beta,\gamma;1-\frac{r}{b}\right)\mathrm{d}r \\ =& \sum_{n=1}^{k+1}(-1)^{n-1}\tilde{v}\left(n,k,\alpha,\beta,\gamma;\frac{a}{b}\right) \end{aligned}$$

where

$$\begin{aligned} \tilde{v}\left(n,k,\alpha,\beta,\gamma;\frac{a}{b}\right) =& v\left(n,k,\alpha,\beta,\gamma;1-\frac{r}{b}\right)\Big|_{r=a+b} \\ & - v\left(n,k,\alpha,\beta,\gamma;1-\frac{r}{b}\right)\Big|_{r=b}, \end{aligned}$$

$$\begin{aligned} & v\left(n,k,\alpha,\beta,\gamma;1-\frac{r}{b}\right) \\ =& (-b)^n(k)_{n-1}r^{k-n+1}F\left(\alpha-n,\beta-n,\gamma-n;1-\frac{r}{b}\right) \\ & \times\prod_{i=1}^{n}\frac{(\gamma-i)}{(\alpha-i)(\beta-i)} \end{aligned}$$

and

$$(k)_{n-1} = k(k-1)(k-2)\cdots(k-n+2), \quad (k)_0 = 1.$$

So, using the formula

$$\begin{aligned} & \mathrm{d}F(\alpha-1,\beta-1,\gamma-1;z)/\mathrm{d}z \\ =& (\alpha-1)(\beta-1)(\gamma-1)^{-1}F(\alpha,\beta,\gamma;z), \end{aligned}$$

we have

$$\begin{aligned} u_3 =& a^{-2}b^{-1}\int_0^a (b+r)^k\left[F\left(\alpha_1,\beta_1,\gamma_1;-\frac{r}{b}\right) + \frac{1}{2}r\frac{\mathrm{d}F\left(\alpha_1,\beta_1,\gamma_1;-\frac{r}{b}\right)}{\mathrm{d}r}\right]r\mathrm{d}r \\ =& a^{-2}b^{-1}\left[I\left(k+1,\alpha_1,\beta_1,\gamma_1;\frac{a}{b}\right) - bI\left(k,\alpha_1,\beta_1,\gamma_1;\frac{a}{b}\right)\right] \\ & - (ab)^{-2}\alpha_1\beta_1(2\gamma_1)^{-1}\left[I\left(k+2,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right)\right. \\ & \left. - 2bI\left(k+1,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right) + b^2 I\left(k,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right)\right], \end{aligned}$$

$$u_4 = a^{-2}b^{-1}\int_0^a \left[F\left(\alpha_1,\beta_1,\gamma_1;-\frac{r}{b}\right)+\frac{1}{2}r\frac{\mathrm{d}F\left(\alpha_1,\beta_1,\gamma_1;-\frac{r}{b}\right)}{\mathrm{d}r}\right]r\mathrm{d}r$$
$$= b^{-1}a^{-2}\left[I\left(1,\alpha_1,\beta_1,\gamma_1;\frac{a}{b}\right)-bI\left(0,\alpha_1,\beta_1,\gamma_1;\frac{a}{b}\right)\right]$$
$$-(ab)^{-2}\alpha_1\beta_1(2\gamma_1)^{-1}\left[I\left(2,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right)\right.$$
$$\left.-2bI\left(1,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right)+b^2I\left(0,\alpha_1+1,\beta_1+1,\gamma_1+1;\frac{a}{b}\right)\right].$$

It is noted that $I\left(0,\alpha_1,\beta_1,\gamma_1;\frac{a}{b}\right) = a$ because $F(\alpha_1,\beta_1,\gamma_1;z)=1$ when $k=0$ (i.e. $\alpha_1=0$).

Case (C) the product profile $\varepsilon_{\mathrm{i}}(r) = cr^k\mathrm{e}^{\beta r}$,

$$\varepsilon_{\mathrm{e}} = \varepsilon_{\mathrm{h}} + f_{\mathrm{i}}A_1(\varepsilon_{\mathrm{h}}u_5 - ca^k\mathrm{e}^{a\beta}u_5 + cu_6), \tag{20}$$

where $u_5 = a^{p-1}F(p,2p+k+1,-a\beta)$,

$$u_6 = a^{-2}\int_0^a F(p,2p+k+1,-\beta r)\times \mathrm{e}^{\beta r}(k+\beta r)r^{p+k}\mathrm{d}r$$
$$= a^{p+k-1}\sum_{m=0}^{\infty}\frac{(p+k+1)_m}{m!(2p+k+1)_m}(a\beta)^m\times\left(\frac{k}{p+k+m+1}+\frac{a\beta}{p+k+n+2}\right).$$

It is clear that, for the unit radius of cylindrical inclusion, the equation (20) is the same as that of Eq.(12) of Ref.[22].

4. Differential effective dipole approximation for estimating the effective response

In this section, we shall estimate the effective response of graded cylindrical composites by means of the DEDA method[17] and compare this method with the first-principle approach of Section 3. For three-dimensional problem, Dong *et al*[18] tested the validity of the DEDA for linear profile and power-law profile of spherical inclusions. Furthermore, Dong *et al*[19] and Huang *et al*[20] extended the DEDA method to discuss the electric response of graded composites having anisotropic gradient profiles of spherical inclusions using the differential equation of dipole factor. In this paper, for the two-dimensional problem, it is easy to derive the effective dielectric response formulas of graded cylindrical composites having the isotropic gradient profile of cylindrical inclusion $\varepsilon_{\mathrm{i}}(r)$ with the DEDA method,

$$\frac{1}{V}\int_{\Omega_i}(D-\varepsilon_{\mathrm{h}}E)\mathrm{d}v = 2f_{\mathrm{i}}b_{\mathrm{dip}}(a)\varepsilon_{\mathrm{h}}E_0, \tag{21}$$

where V is the volume of the whole composites, and $b_{\mathrm{dip}}(a)$ is the dipole factor of graded cylindrical composites at the radius length of cylindrical inclusion a. For a radius length of cylindrical inclusion, r, the dipole factor function $b_{\mathrm{dip}}(r)$ satisfies the following differential equation,

$$\frac{\mathrm{d}b_{\mathrm{dip}}(r)}{\mathrm{d}r} = -\frac{[1+b_{\mathrm{dip}}(r)]^2\varepsilon_{\mathrm{h}}^2-[b_{\mathrm{dip}}(r)-1]^2\varepsilon_{\mathrm{i}}(r)^2}{2r\varepsilon_{\mathrm{h}}\varepsilon_{\mathrm{i}}(r)}, \tag{22}$$

where the initial condition $b_{\mathrm{dip}}(0) = [\varepsilon_{\mathrm{i}}(0)-\varepsilon_{\mathrm{h}}]/[\varepsilon_{\mathrm{i}}(0)+\varepsilon_{\mathrm{h}}]$. Considering the Eqs.(16) and (21), in the dilute limit, we have the effective dielectric response formulae of the DEDA method,

$$\varepsilon_{\mathrm{e}}/\varepsilon_{\mathrm{h}} = 1+2f_{\mathrm{i}}b_{\mathrm{dip}}(a). \tag{23}$$

Next, we test the validity of Eq.(23) by comparing it with the exact results of Section 3 for four cases of cylindrical dielectric profiles. Clearly, at least, the equation (22) can be solved for arbitrary inclusion dielectric profile $\varepsilon_{\mathrm{i}}(r)$ numerically. For four cases of inclusion dielectric profiles $\varepsilon_{\mathrm{i}}(r) = c(b+r)^k\mathrm{e}^{\beta r}$, $\varepsilon_{\mathrm{i}}(r) = c\mathrm{e}^{\beta r}$, $\varepsilon_{\mathrm{i}}(r) = c(b+r)^k$ and $\varepsilon_{\mathrm{i}}(r) = cr^k\mathrm{e}^{\beta r}$, the effective responses obtained from the Eq.(23) of the DEDA method are compared with the exact results of Section 3 from Fig.1 to Fig.4, respectively. In these figures, the equation (22) is solved numerically by the fourth-order Runge–Kutta method with the step size 0.01. It is clear that the DEDA method is valid and is in good agreement with the exact formulas. Therefore, in the dilute limit, the DEDA is a very useful method for estimating the effective response of graded composites having complex dielectric function of inclusion if we cannot get the exact solutions of local potentials.

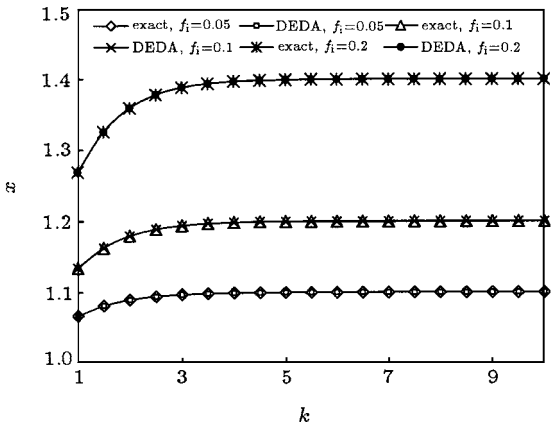


Fig.1. The effective dielectric response contrast ratio $X = \varepsilon_e/\varepsilon_h$ of graded composites with $\varepsilon_i(r) = c(b+r)^k e^{\beta r}$ as function of the parameter k, where, $c = 1$, $b = 3$, $\beta = 0.5$ and the radius of cylindrical inclusion $a = 1$. In this figure, for the volume fraction of inclusion f_i =0.05, 0.1 and 0.2, the effective responses obtained from the exact solution and the DEDA method are compared, respectively.

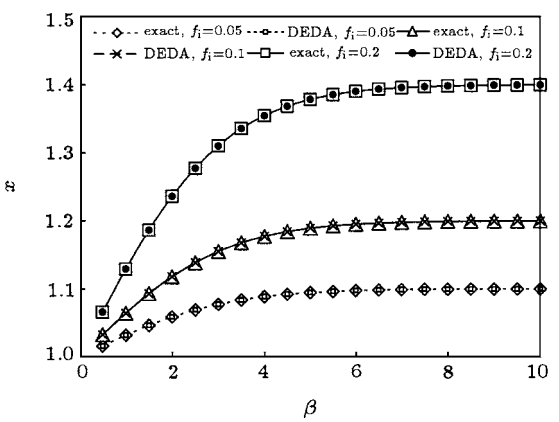


Fig.2. The effective dielectric response contrast ratio $X = \varepsilon_e/\varepsilon_h$ of graded composites with $\varepsilon_i(r) = ce^{\beta r}$ as function of the parameter β, where $c = 1$ and the radius of cylindrical inclusion $a = 1$. In this figure, for the volume fraction of inclusion f_i =0.05, 0.1 and 0.2, the effective responses obtained from the exact solution and the DEDA method are compared, respectively.

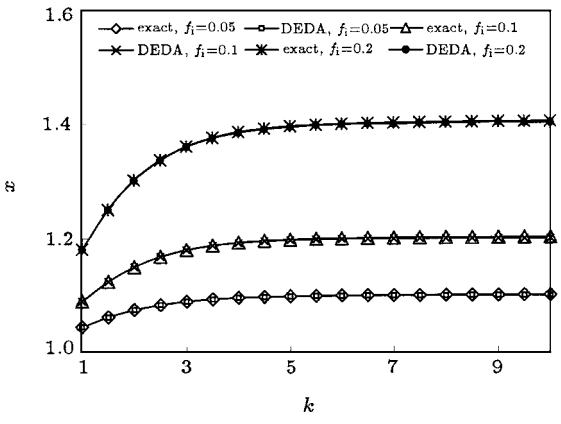


Fig.3. The effective dielectric response contrast ratio $X = \varepsilon_e/\varepsilon_h$ of graded composites with $\varepsilon_i(r) = c(b+r)^k$ as function of the parameter k, where $c = 1$, $b = 2$ and the radius of cylindrical inclusion $a = 1$. In this figure, for the volume fraction of inclusion f_i =0.05, 0.1 and 0.2, the effective responses obtained from the exact solution and the DEDA method are compared, respectively.

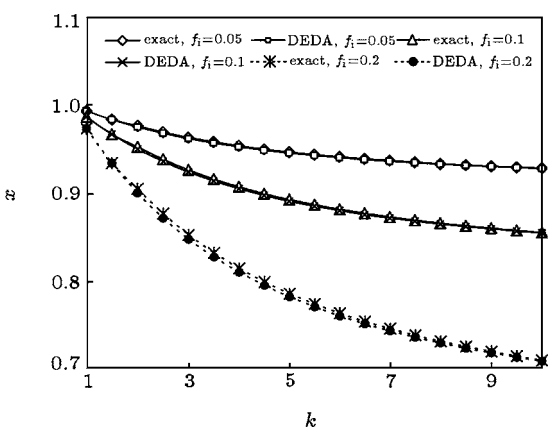


Fig.4. The effective dielectric response contrast ratio $X = \varepsilon_e/\varepsilon_h$ of graded composites with $\varepsilon_i(r) = cr^k e^{\beta r}$ as function of the parameter k, where $c = 1$, $\beta = 0.5$ and the radius of cylindrical inclusion $a = 1$. In this figure, for the volume fraction of inclusion f_i =0.05, 0.1 and 0.2, the effective responses obtained from the exact solution and the DEDA method are compared, respectively.

5. Discussion and conclusion

As a theoretical model, we have studied the local potential solutions and effective dielectric response of graded cylindrical composites having the generalized dielectric profiles in terms of power series method. The local potentials of graded composites with generalized profile are different from that of graded cylindrical composites having the simple dielectric profiles such as general power profile, exponential profile, linear profile and the product of the exponent and the simple power profiles. In functionally graded materials engineering, the effective response of graded composites having generalized profile will be more controllable than other simple profiles due to more controllable parameters in the inclusion property profiles. From our results, we have sorted out the effective responses and the local analytical potentials of graded cylindrical composites having two cases of profiles, i.e. the general power-law and the exponential profiles,

respectively. Therefore, the effective responses and the local potentials of generalized profile composites should be useful to modulate the responses of graded composites from the general power-law profile to the exponential profile and will be very useful in applications in nonlinear optical processes, like the nondegenerate four-wave mixing susceptibility in graded composites. In addition, in the dilute limit, our effective response formulas of graded composites having four differential profiles have demonstrated that the DEDA is actually valid for arbitrary gradient profiles.[17−20] Furthermore, it is easy to investigate the spherical graded composites having the generalized dielectric profiles by the power series method, and, on the basis of our results, the nonlinear responses of weakly nonlinear graded composites can be calculated by combining the perturbation method and the decoupling approximation.[7,8,29]

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