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La Shalle's invariant-set-theory based asymptotic synchronization of duffing system with unknown parameters

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A novel La Shalle's invariant set theory (LSIST) based adaptive asymptotic synchronization (LSISAAS) method is proposed to asymptotically synchronize Duffing system with unknown parameters which also are considered as system states. The LSISASS strategy depends on the only information, i.e. one state of the master system. According to the LSIST, the LSISASS method can asymptotically synchronize fully the states of the master system and the unknown system parameters as well. Simulation results also validate that the LSISAAS approach can obtain asymptotic synchronization.

Keywords: chaotic synchronization, adaptive state observer, La Shalle's invariant set theory
PACC: 0545

1. Introduction

In the last decade, theoretic and practical application researches on chaotic synchronizations have emerged as an attractive and hot field[1] and many chaotic synchronization strategies were proposed since the pioneering work of Pecora and Carroll in Ref.[2], which synchronized two identical master–slave Lorenz systems with different initial conditions using a coupling signal. Obviously, in the master-slave configuration, the slave system can be considered as a state-observer of the master system using only the information of a coupling signal from master system. Consequently, many results related to nonlinear observer theory can directly be used in the chaotic synchronization;[3−5] and it is not surprising that the state-observer based synchronization method was widely used in the last decade. Since uncertainties always exist in some or even all systems, some strategies were advanced such as neural networks,[6] fuzzy system[7] and variable structure control,[8,9] etc, to cope with this problem. However, those approaches always depend on the knowledge of full states of master system. As to the design tool of the state-observer based synchronization controller, the direct Lyapunov method is commonly used in almost all published works. Very recently, the well known La Shalle invariant set theory (LSIST) was first applied to chaotic synchronization in Ref.[10]. Consequently, a new tool is introduced and more convenience and freedom can be obtained in the design and analysis of synchronization controller.

In the present paper, a novel LSIST-based adaptive asymptotic synchronization (LSISAAS) method is proposed to asymptotically synchronize the Duffing system with unknown system parameters which are also considered as system states. The LSISASS strategy depends on the only information, i.e. one state of the master system. According to LSIST, the LSISASS method can asymptotically synchronize full states of the master system and the unknown system parameters as well. Compared with Ref.[10], the proposed LSISAAS introduces a reduced-order observer and extends the LSISAAS method.

2. Problem statement and design of LSISAAS

The master Duffing system is described as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a\,x_1 + bx_1^3 + cx_2 + d + f\cos(wt), \\ y &= x_1, \end{aligned} \tag{1}$$

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where y denotes master system output; a, b, c, d are unknown positive constants and f, w are known positive ones.

The slave system is designed as

$$\begin{aligned}
&\dot{\hat{x}}_1 = \hat{x}_2 - K_1(\hat{x}_1 - y),\\
&\dot{\hat{x}}_2 = \hat{a}y + \hat{b}y^3 + \hat{c}\hat{x}_2 + \hat{d} + f\cos(wt) - K_2(\hat{x}_2 - \dot{y}),\\
&\dot{\hat{a}} = -y(\hat{x}_2 - \dot{y}),\\
&\dot{\hat{b}} = -y^3(\hat{x}_2 - \dot{y}),\\
&\dot{\hat{c}} = -\hat{x}_2(\hat{x}_2 - \dot{y})\\
&\dot{\hat{d}} = -(\hat{x}_2 - \dot{y}).
\end{aligned} \tag{2}$$

Remark 1 It is a well-known fact that differentiator is sensitive to noise. Since slave system (2) depends on $\dot{y}$, it cannot be directly applied to synchronize the master system (1). For system (2), we choose a reversible transformation described as

$$\begin{aligned}
&\tilde{x}_1 = \hat{x}_1,\\
&\tilde{x}_2 = \hat{x}_2 - K_2\, y,\\
&\tilde{a} = \hat{a} - y^2/2,\\
&\tilde{b} = \hat{b} - y^4/4,\\
&\tilde{c} = \hat{c} - (\tilde{x}_2 + K_2 y)y + K_2 y^2/2,\\
&\tilde{d} = \hat{d} - y.
\end{aligned} \tag{3}$$

Under the transformation (3), system (2) then can be rewritten as

$$\begin{aligned}
\dot{\tilde{x}}_1 =& \tilde{x}_2 + K_2 y - K_1(\tilde{x}_1 - y),\\
\dot{\tilde{x}}_2 =& d\cos(wt) + (\tilde{a} + y^2/2)y + (\tilde{b} + y^4/4)y^3\\
&+ (\tilde{c} + \tilde{x}_2 y - K_2 y^2/2 - K)(\tilde{x}_2 + K_2 y) + \tilde{d} + y,\\
\dot{\tilde{a}} =& -y(\tilde{x}_2 + K_2 y),\\
\dot{\tilde{b}} =& -y^3(\tilde{x}_2 + K_2 y),\\
\dot{\tilde{c}} =& -(\tilde{x}_2 + K_2 y)^2 - (\tilde{a} + y^2/2)y^2 - (\tilde{b} + y^4/4)y^4\\
&- (\tilde{c} + \tilde{x}_2 y - K_2 y^2/2 - K_2)(\tilde{x}_2 + K_2 y)y\\
&- d\cos(wt)y - (\tilde{d} + y)y.
\end{aligned} \tag{4}$$

It is clear that system (4) is a reduced-order observer of system (2) and depends only on y. Obviously, system (4) is realizable. Therefore, system (2) also is realizable under the transformation (3).

Let

$$e_1 = \hat{x}_1 - x_1, e_2 = \hat{x}_2 - x_2, e_3 = \hat{a} - a, e_4 = \hat{b} - b,$$
$$e_5 = \hat{c} - c, e_6 = \hat{d} - d, E = [e_1, e_2, e_3, e_4, e_5, e_6]^{\mathrm{T}}. \tag{5}$$

Subtracting (2) from (3) and using definition (4) yield

$$\begin{cases}
\dot{e}_1 = -K_1 e_1 + e_2,\\
\dot{e}_2 = -(K_2 - c)e_2 + ye_3 + y^3 e_4 + \hat{x}_2 e_5 + e_6,\\
\dot{e}_3 = -ye_2,\\
\dot{e}_4 = -y^3 e_2,\\
\dot{e}_5 = -\hat{x}_2 e_2,\\
\dot{e}_6 = -e_2.
\end{cases} \tag{6}$$

To make the synchronization error system (6) asymptotically stabilize at the original point, control law design is given by the following Theorem 1.

Theorem 1 (**LSISAAS Design Theorem**) Choosing $K_1 > 0$ and $K_2 > c$ can guarantee that the slave system (2) asymptotically synchronizes the master system (1), i.e.

$$\lim_{t\to+\infty} E = 0, \tag{7}$$

where E is defined in (5).

Proof Choose the Lyapunov function as

$$V = (e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2)/2. \tag{8}$$

Differentiating V with respect to t and using (6) gives

$$\begin{aligned}
\dot{V} =& -(K_2 - c)e_2^2 + y e_2 e_3 + y^3 e_2 e_4 + \hat{x}_2 e_2 e_5 + e_2 e_6\\
&- ye_2 e_3 - y^3 e_2 e_4 - \hat{x}_2 e_2 e_5 - e_2 e_6\\
=& -(K_2 - c)e_2^2.
\end{aligned} \tag{9}$$

Since $K_2 > c$ is designed, $\dot{V} \le 0$ can be satisfied. In addition, $\dot{V} = 0$ is true if and only if

$$e_2 = 0 \tag{10}$$

is satisfied. Using (10) in the first equation in (6), namely, $\dot{e}_1 = -K_1 e_1 + e_2$, yields

$$\lim_{t\to+\infty} e_1(t) = 0. \tag{11}$$

Substituting (10) and (11) into (6) and using the well known LSIST, we see that the largest invariant set M is described as

$$\begin{aligned}
M :=& \{(e_1, e_2, e_3, e_4, e_5, e_6) |\\
& e_1 = e_2 = 0, && \text{(12a)}\\
& ye_3 + y^3 e_4 + \hat{x}_2 e_5 + e_6 = 0, && \text{(12b)}\\
& \dot{e}_3 = \dot{e}_4 = \dot{e}_5 = \dot{e}_6 = 0. && \text{(12c)}\\
& \}.
\end{aligned}$$

Note $e_2 = \hat{x}_2 - x_2$, $e_2 = 0$ and $x_2 = \dot{y}$, Eq.(12b) is then rewritten as

$$ye_3 + y^3 e_4 + \dot{y}e_5 + e_6 = 0, \tag{13}$$

where e_3, e_4, e_5 and e_6 are constants.

As proved in the Appendix, the condition $e_5 \neq 0$ cannot be satisfied. Consequently, we only need to consider the case $e_5 = 0$. In such a circumstance, the following equation can be satisfied:

$$ye_3 + y^3 e_4 + e_6 = 0. \tag{14}$$

Since y is a chaotic variable, (14) is satisfied if and only if $e_3 = e_4 = e_6 = 0$. Then, in the case of $e_5 = 0$,

$$e_3 = e_4 = e_5 = e_6 = 0 \tag{15}$$

is guaranteed.

Consequently, the largest invariant set M can actually be described as

$$\begin{aligned} M = &\{(e_1, e_2, e_3, e_4, e_5, e_6)| \\ &e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = 0\}. \end{aligned} \tag{16}$$

The La Shalle invariant set theory implies that the synchronization error system (6) is asymptotically stable at the original point. Thus, Eq.(7) is satisfied. Q.E.D.

3. Simulation

We choose a Duffing system with the parameters $a = 1$, $b = -1$, $c = -0.25$, $d = 0.01$, $f = 0.4$ and $w = 1$ as master system. The initial conditions are taken as $x_1(0) = x_2(0) = 1$. The Duffing system then displays chaotic behaviour (shown in Fig.1). The initial conditions of the slave system are taken as: $\hat{x}_i(0) = 0, \forall i = 1, 2, \ldots, 6$. Based on Theorem 1, the parameters of LSISAAS are taken as

$$K_1 = K_2 = 0.4. \tag{17}$$

Simulation results are plotted in Figs.2–7. It is clear from Figs.2–7 that using LSISAAS can make synchronization error e_1 and e_2 sufficiently small in finite time. In addition, the parameter estimator error e_3, e_4, e_5 and e_6 are also small enough in finite time. That implies that using LSISAAS can estimate not only master system states x_1 and x_2 but also the unknown system parameters, a, b, c and d with high accuracy in finite time.

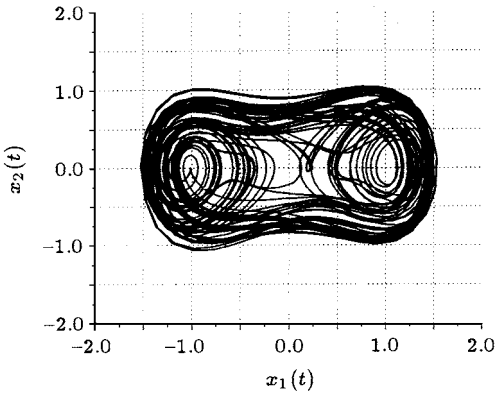


Fig.1. Chaotic behaviour of the chosen Duffing system.

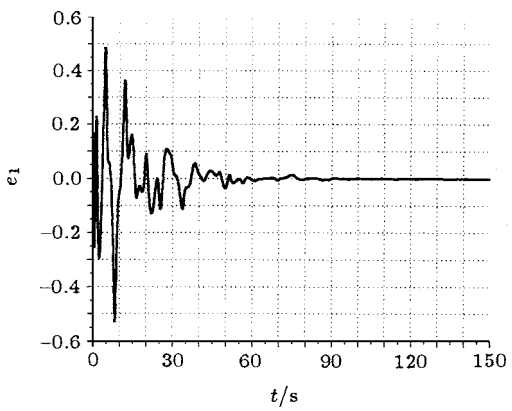


Fig.2. Plot of observe error $e_1(t)$.

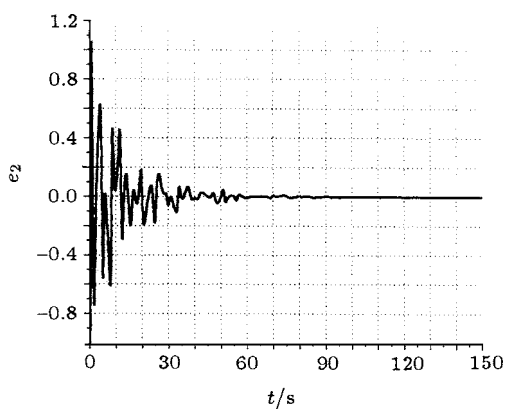


Fig.3. Plot of observe error $e_2(t)$.

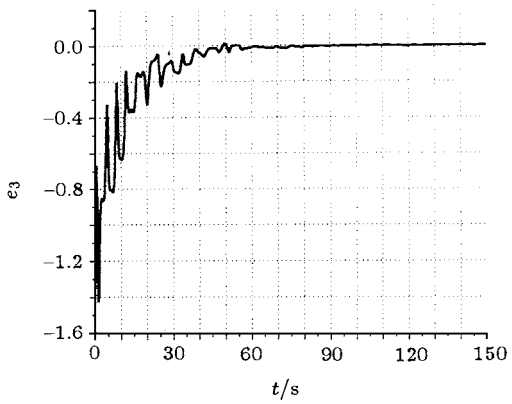


Fig.4. Plot of observe error $e_3(t)$.

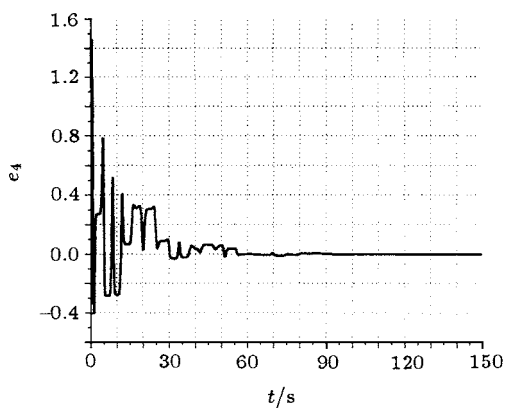


Fig.5. Plot of observe error $e_4(t)$.

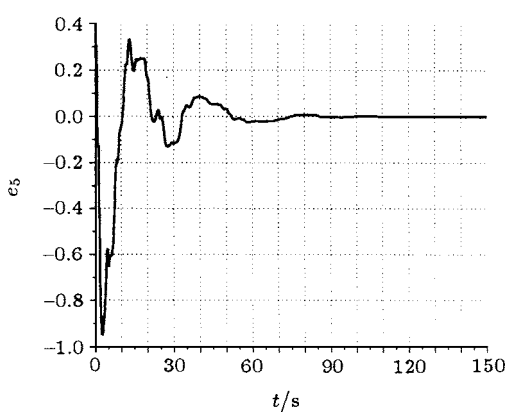


Fig.6. Plot of observe error $e_5(t)$.

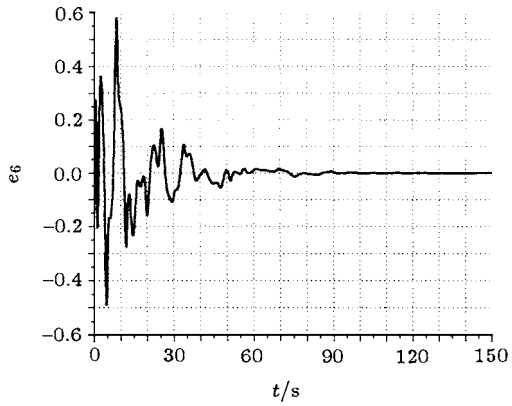


Fig.7. Plot of observe error $e_6(t)$.

4. Conclusions

In the paper, a novel asymptotic synchronization method, LSISAAS, is proposed to asymptotically synchronize a Duffing system with unknown parameters. The LSISAAS strategy depends on the only information, i.e., one state of the master system. Both theoretic analysis in terms of LSIST and simulation results validate that the LSISAAS approach can asymptotically synchronize fully the states of master system and the unknown system parameter as well.

Appendix

Proof of the assertion "$e_5 \neq 0$ cannot be satisfied".

Proof To prove "$e_5 \neq 0$ cannot be satisfied", we use the method of reductio ad absurdum and start from the Assumption 1.

Assumption 1 $e_5 \neq 0$ is satisfied.

In such circumstances, Eq.(13) is rewritten as

$$\begin{aligned}\dot{y} =& -(e_3/e_5)y - (e_4/e_5)y^3 - e_6/e_5 \\ :=& \delta_1 y + \delta_2 y^3 + \delta_3,\end{aligned} \tag{A1}$$

where $\delta_1 = -(e_3/e_5), \delta_2 = -(e_4/e_5), \delta_3 = -e_6/e_5$. Let's firstly study following equation

$$\dot{y} = \delta_1 y + \delta_2 y^3 = y(\delta_1 + \delta_2 y^2). \tag{A2}$$

Case 1 When $\delta_1\delta_2 = 0$, it is obvious that the solution of Eq.(A2) is not a chaotic one.

Case 2 When $\delta_1\delta_2 \neq 0$, choose a Lyapunov function as $L = y^2/2$, then

$$\dot{L} = \delta_1 y^2 + \delta_2 y^4 = y^2(\delta_1 + \delta_2 y^2). \tag{A3}$$

When $\delta_1 < 0, \delta_2 < 0$ and $\delta_1 > 0, \delta_2 > 0$, it is clear from (A2) and (A3) that system (A2) is stable at original point and unstable, respectively. Consequently, in such circumstances, the solution of Eq.(A2) is not a chaotic one.

If $\delta_1 > 0, \delta_2 < 0$ is assumed, $\dot{L} < 0$ is satisfied when $y^2 > -\delta_1/\delta_2$; and, in Eq.(A2), $\delta_1 + \delta_2 y^2 > 0$ is true when $y^2 < -\delta_1/\delta_2$. Then, $y^2 = -\delta_1/\delta_2$ is guaranteed and the solution of Eq.(A2) also is not a chaotic variable.

As to the case of $\delta_1 < 0, \delta_2 > 0$, $\dot{L} < 0$ is satisfied when $y^2 < -\delta_1/\delta_2$; and, in Eq.(A2), $\delta_1 + \delta_2 y^2 > 0$ is correct when $y^2 > -\delta_1/\delta_2$. Consequently, the solution of Eq.(A2) is not a chaotic one.

Summing up all above cases, we know that the solution of Eq.(A2) is not a chaotic one for any $\delta_1, \delta_2 \in R$. Based on the theory of ordinary differential equations (ODE), the solution of Eq.(A1) includes two parts, namely, the general and the particular. The solution of Eq.(A1) is then expressed as

$$y(t) = y^{\mathrm{P}}(t) + y^{\mathrm{G}}(t), \tag{A4}$$

where the special solution $y^{\mathrm{P}}(t)$ keeps unchanged with respect to t; and the general solution $y^{\mathrm{G}}(t)$ is the solution of $\dot{y} = \delta_1 y + \delta_2 y^3$.

As shown above, the general solution of Eq.(A1), namely $y^{\mathrm{G}}(t)$, is not a chaotic variable. Consequently, item "$y^{\mathrm{P}}(t) + y^{\mathrm{G}}(t)$" is not a chaotic one either. But, note the fact, when the master system is in chaotic state, y is a chaotic variable. Consequently, the Assumption 1 is false and "$e_5 \neq 0$ cannot be satisfied" is proved.

Q.E.D.

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