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New infinite-dimensional symmetry groups for the stationary axisymmetric Einstein–Maxwell equations with multiple Abelian gauge fields*

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The so-called extended hyperbolic complex (EHC) function method is used to study further the stationary axisymmetric Einstein–Maxwell theory with p Abelian gauge fields (EM-p theory, for short). Two EHC structural Riemann–Hilbert (RH) transformations are constructed and are then shown to give an infinite-dimensional symmetry group of the EM-p theory. This symmetry group is verified to have the structure of semidirect product of Kac–Moody group $SU(\widehat{p+1,1})$ and Virasoro group. Moreover, the infinitesimal forms of these two RH transformations are calculated and found to give exactly the same infinitesimal transformations as in previous author's paper by a different scheme. This demonstrates that the results obtained in the present paper provide some exponentiations of all the infinitesimal symmetry transformations obtained before.

Keywords: general relativity, extended hyperbolic complex function method, symmetry group
PACC: 0420, 0290, 0220

1. Introduction

The stationary axisymmetric (SAS) Einstein–Maxwell (EM) field equations with multiple (say p) Abelian gauge fields (SAS EM-p equations, for short) are a class of very important equations in mathematical physics and attracted much attention in the past years, e.g. Refs.[1–17]. It has been pointed out[1−3,5,11] that the SAS EM (i.e. $p = 1$) theory possesses larger symmetry group than that of the SAS vacuum gravity. However, the complications arising from the existence of Maxwell fields make the study of the field equations more difficult and thus some studying methods previously used need to be extended.

In a recent paper,[17] we presented and used a so-called extended hyperbolic complex (EHC) function method and found that all of the SAS EM-p theories for $p = 0, 1, 2, \cdots$ can be studied in a unified form (as special cases, $p = 0$ corresponds to the vacuum gravity; $p = 1$, the EM theory). Moreover, we introduced a $(p+2)\times(p+2)$ matrix Ernst-like EHC potential and then established two Hauser–Ernst (HE)-type EHC linear systems associated with the EM-p equations for each p. Based on the solutions of these EHC linear systems, some parameterized infinitesimal symmetry transformations of the considered theories were constructed and these infinitesimal symmetry transformations were further verified to constitute an infinite-dimensional Lie algebra: the semidirect product of the Kac–Moody algebra $su(p+1,1)\otimes R(t,t^{-1})$ and the Virasoro algebra (without centre charges), where $R(t,t^{-1})$ is the algebra of Laurent polynomials in t and t^{-1} with real coefficients. These results show that the EM-p theory under consideration possesses very rich symmetry structures. However, only the infinitesimal symmetry transformations were obtained there. In order to generate new solutions (from some old ones) in practice, we need to find finite symmetry transformations and this is the main aim of the present paper.

The present paper is a continuation of our previous paper.[17] In this paper, by virtue of the presented EHC function method, we establish an EHC structural Riemann–Hilbert (RH) transformation for each of the HE-type linear systems mentioned above and further demonstrate that these two EHC RH transformations,

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joined together, give rise to symmetry actions of the semidirect product of full affine Kac–Moody group and Virasoro group on the solution space of the EM-p theory. These essentially realize exponentiations of all the infinite-dimensional semidirect product Lie algebra of the infinitesimal symmetry transformations given in Ref.[17]. Here, in order to obtain new symmetries, the RH problem methods as well as EM-p field equations and related linear systems are generalized to some EHC structural forms, and two distinct EHC RH transformations are simultaneously considered. Moreover, some particular relations that hold in the case of two-dimensional matrix, such as for any 2×2 matrix A: $A^{\top}\epsilon A=(\det A)\epsilon$, $A^{\top}\epsilon+\epsilon A=(\mathrm{tr}A)\epsilon$ $\left(\text{with } \epsilon=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)$, have no general analogues for higher dimensional $n\times n$ ($n\geq 3$) matrices, while these relations are important and useful in some studies of the reduced vacuum gravity.[18,19] However, for the EM-p (especially $p\geq 1$) theories studied in the present paper, the fundamental field equations and the associate potentials are expressed in terms of some $(p+2)\times(p+2)$ matrices, so some studying schemes will be generalized in this paper.

In Section 2, some related concepts and notations of the so-called EHC function method, the matrix Ernst-like EHC potential and two HE-type EHC linear systems for the SAS EM-p theories are recalled. In Section 3, two EHC structural RH transformations for the studied theory and the equivalent integral equations are constructed. Section 4 discusses the relations between the results of this paper and that of Ref.[17]. Finally, Section 5 gives the summary and discussions.

2. EHC function and EHC linear systems associated with EM-p equations

In this section, we briefly recall some related concepts and notations of the EHC function and the EHC matrix Ernst-like potentials, EHC linear systems associated with EM-p theories,[17] which are useful in the later discussions.

2.1. EHC function[17]

Let ε be the extended hyperbolic imaginary unit, i.e. $\varepsilon^2=1$, $\varepsilon\neq\pm 1$. When both a and b are ordinary complex numbers, we call $c(\varepsilon)=a+\varepsilon b$ an extended hyperbolic complex (EHC) number and denote $\mathrm{Re}_{\mathrm{EHC}}(c(\varepsilon)):=a$, $\mathrm{Im}_{\mathrm{EHC}}(c(\varepsilon)):=b$. All the EHC numbers with usual addition and multiplication constitute a commutative ring which we call the EHC ring. Corresponding to the two imaginary units ε and the ordinary imaginary unit i ($\mathrm{i}^2=-1$) in the EHC ring, we have two complex conjugations:

$$c(\varepsilon)^{\star}:=a-\varepsilon b,\qquad \overline{c(\varepsilon)}:=\overline{a}+\varepsilon\overline{b},\tag{1}$$

where $\overline{a}$ and $\overline{b}$ are ordinary complex conjugates of a and b. If a and b are ordinary complex functions of some ordinary complex (or real) variables $z_1,\cdots,z_n$, then $c(z_1,\cdots,z_n;\varepsilon)=a(z_1,\cdots,z_n)+\varepsilon b(z_1,\cdots,z_n)$ is called an EHC function. We say $c(z_1,\cdots,z_n;\varepsilon)$ to be continuous, analytical, etc. if $a(z_1,\cdots,z_n)$ and $b(z_1,\cdots,z_n)$ both, as ordinary complex functions, have the same properties. We also need EHC (function) matrices, and for an EHC matrix A, we define

$$A^{\dagger}:=\overline{A}^{\top},\qquad A^{\ddagger}:=(A^{\dagger})^{\star},\tag{2}$$

here "$\top$" denotes the transposition.

2.2. EHC matrix potentials and HE-type EHC linear systems[17]

When there exist p (nonnegative integer) Maxwell fields A^a_{μ} ($a=1,2,\cdots,p$) in space–time, the essentially complete set of SAS EM-p field equations can be written as[17]

$$\mathrm{d}(\rho M^{-1}\mathrm{d}M)=0,\tag{3}$$

where $M=M(x,y)$ is a $(p+2)\times(p+2)$ ordinary complex matrix function of the remainder coordinates x, y and satisfies the conditions

$$M^{\dagger}=M,\qquad M\gamma M=\rho^2\gamma,\tag{4}$$

the real function $\rho=\rho(x,y)>0$ and

$$\gamma:=\begin{pmatrix}0 & 0 & 1\\ 0 & \mathrm{i}I_{\mathrm{p}} & 0\\ -1 & 0 & 0\end{pmatrix},\quad \gamma^{\dagger}=-\gamma,\quad \gamma^2=-I,\tag{5}$$

I_{p} and I are, respectively, the p-dim and $(p+2)$-dim unit matrices. Moreover, from (3) and (4), we can introduce a matrix twist potential $Q(x,y)$ and a $(p+2)\times(p+2)$ Ernst-like matrix EHC potential (or called EHC H-potential) $H(\varepsilon):=M+\varepsilon Q$ such that $\mathrm{d}Q=-\rho^{-1}M\gamma\,{}^{*}\mathrm{d}M$ and the EM-p field equations can be equivalently written as an EHC matrix equation

$$2(z+\rho^{*})\mathrm{d}H(\varepsilon)=(H(\varepsilon)+H(\varepsilon)^{\ddagger})\varepsilon\gamma\mathrm{d}H(\varepsilon),\tag{6}$$

where another real field $z=z(x,y)$ is introduced such that ${}^{*}\mathrm{d}\rho=\mathrm{d}z$ and $Q-Q^{\dagger}=-2z\gamma$.

Now we introduce a (ordinary complex) parameter t and define

$$A(t;\varepsilon) := I - t(H(\varepsilon) + H(\varepsilon)^{\ddagger})\varepsilon\gamma, \tag{7}$$

$$\Gamma(t;\varepsilon) := t\Lambda(t)^{-1}\mathrm{d}H(\varepsilon), \tag{8}$$

$$\Lambda(t) := 1 - 2t(z+\rho^*),$$

$$\Lambda(t)^{-1} = \lambda(t)^{-2}[1 - 2t(z-\rho^*)], \tag{9}$$

$$\lambda(t) := [(1-2zt)^2 + (2\rho t)^2]^{1/2}. \tag{10}$$

then equation (6) can be rewritten as

$$t\mathrm{d}H(\varepsilon) = A(t;\varepsilon)\Gamma(t;\varepsilon), \tag{11}$$

which is the integrability condition of the following EHC linear differential system:

$$\mathrm{d}F(t;\varepsilon) = \Gamma(t;\varepsilon)\varepsilon\gamma F(t;\varepsilon), \tag{12}$$

where $F(t;\varepsilon) = F(x,y,t;\varepsilon)$ is an EHC $(p+2)\times(p+2)$ matrix function of x, y and t and is called an EHC F-potential.

The HE-type EHC linear system (12) does not determine $F(t;\varepsilon)$ uniquely, we can require that $F(t;\varepsilon)$ be holomorphic in a neighbourhood of $t=0$ and suppress subsidiary conditions consistently as

$$F(0;\varepsilon) = I, \quad \dot{F}(0;\varepsilon) = H(\varepsilon)\varepsilon\gamma, \tag{13}$$

$$\lambda(t)F(t;\varepsilon)^{+}\varepsilon\gamma F(t;\varepsilon) = \varepsilon\gamma, \tag{14}$$

$$F(t;\varepsilon)^{\times}\varepsilon\gamma A(t;\varepsilon)F(t;\varepsilon) = \varepsilon\gamma, \tag{15}$$

here $\dot{F}(t;\varepsilon) := \partial F(t;\varepsilon)/\partial t$, $F(t;\varepsilon)^{+} := F(\bar{t};\varepsilon)^{\dagger}$, $F(t;\varepsilon)^{\times} := F(\bar{t};\varepsilon)^{\ddagger}$, $\bar{t}$ is the complex conjugate of t.

Besides, we can establish another EHC linear system associated with SAS EM-p field equation (6). To this end, we introduce another ordinary complex parameter τ and define

$$\tilde{A}(\tau;\varepsilon) := \tau - (H(\varepsilon) + H(\varepsilon)^{\ddagger})\varepsilon\gamma, \tag{16}$$

$$\tilde{\Gamma}(\tau;\varepsilon) := \tilde{\Lambda}(\tau)^{-1}\mathrm{d}H(\varepsilon), \tag{17}$$

$$\tilde{\Lambda}(\tau) := \tau - 2(z+\rho^*),$$

$$\tilde{\Lambda}(\tau)^{-1} = \tilde{\lambda}(\tau)^{-2}[\tau - 2(z-\rho^*)], \tag{18}$$

$$\tilde{\lambda}(\tau) := [(\tau-2z)^2 + (2\rho)^2]^{1/2}. \tag{19}$$

Then the field equation (6) can be rewritten as

$$\mathrm{d}H(\varepsilon) = \tilde{A}(\tau;\varepsilon)\tilde{\Gamma}(\tau;\varepsilon), \tag{20}$$

and this is the compatibility condition of the following linear system:

$$\mathrm{d}\tilde{F}(\tau;\varepsilon) = \tilde{\Gamma}(\tau;\varepsilon)\varepsilon\gamma\tilde{F}(\tau;\varepsilon), \tag{21}$$

here $\tilde{F}(\tau;\varepsilon) = \tilde{F}(x,y,\tau;\varepsilon)$ is an EHC $(p+2)\times(p+2)$ matrix function of x, y and τ. In addition, we may require consistently that $\tilde{F}(\tau;\varepsilon)$ be analytic in a neighbourhood of $\tau=0$ and satisfy

$$\tilde{\lambda}(\tau)\tilde{F}(\tau;\varepsilon)^{+}\varepsilon\gamma\tilde{F}(\tau;\varepsilon) = \varepsilon\gamma, \tag{22}$$

$$\tilde{F}(\tau;\varepsilon)^{\times}\varepsilon\gamma\tilde{A}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon) = \varepsilon\gamma. \tag{23}$$

We would like to point out that, although equations (12) and (21) are formally interrelated by $\tau \leftrightarrow t^{-1}$, that is not the case for the two linear systems (12)–(15) and (21)–(23) because the analytic properties of $F(t;\varepsilon)$ and $\tilde{F}(\tau;\varepsilon)$, the conditions (14), (15) and (22), (23) are not interrelated simply through $\tau \leftrightarrow t^{-1}$. The two linear systems give rise to different symmetries of the EM-p theory (see Ref.[17] and the following results of the present paper).

3. EHC structural Riemann–Hilbert transformations

At first, we generalize the usual RH problem into an EHC formulation. Let L denote a smooth contour surrounding the origin in the ordinary complex plane and be symmetric with respect to the real axis, L_+ and L_- be the inside and outside (including ∞) of L, respectively. For an ordinary complex variable s, if a given EHC matrix function $G(s;\varepsilon)$ is holomorphic and invertible on L, then there exists a pair of EHC matrix functions $X_{\pm}(s;\varepsilon)$ which are (respectively for $\pm$) holomorphic in $L_{\pm}$, continuous and invertible on $L\cup L_{\pm}$ such that

$$X_-(s;\varepsilon) = X_+(s;\varepsilon)G(s;\varepsilon), \qquad s\in L. \tag{24}$$

We call (24) an EHC structural RH problem. For a fixed kernel $G(s;\varepsilon)$, the fundamental solution $X_{\pm}(s;\varepsilon)$ of the RH problem is unique up to a non-singular constant matrix factor. A suitable boundary condition can cancel this indefiniteness.

3.1. EHC structural RH transformation for Eq.(12)

Motivated by Ref.[19], we introduce a scalar function $v(t)$ which is independent of x, y and holomorphic on $L\cup L_-$ except infinity where it tends to linear divergence such that $v(t)$ is a linear function of t or has singularities in L_+. In addition, $v(t)$ is real when t is real. Furthermore, we introduce two new real functions $\rho' = \rho'(x,y)$, $z' = z'(x,y)$ such that (for fixed x, y) $\lambda(v(t)) = [(1-2zv(t))^2 + (2\rho v(t))^2]^{1/2}$

and $\lambda'(t) := [(1-2z't)^2 + (2\rho't)^2]^{1/2}$ have the same zeros in t, and $\dot{v}(t) \neq 0$ at these zeros. Thus we have

$$\frac{1}{2(z' \pm \mathrm{i}\rho')} = v^{-1}\left(\frac{1}{2(z \pm \mathrm{i}\rho)}\right), \tag{25a}$$

$$\mathrm{d}\rho' = \mathrm{d}z'. \tag{25b}$$

Equation (25a) can be interpreted as a variable transformation under which equations (7)–(15) and their EHC solutions are changed. We shall use the notations with prime to denote the changed functions, e.g. $F(t;\varepsilon) \mapsto F'(t;\varepsilon)$, etc.

Following the spirit of Refs.[18,19], for a given EHC solution $F(t;\varepsilon)$ of equation (12) with conditions (13)–(15), we select the contour L such that $F(t;\varepsilon)$ is holomorphic on $L \cup L_+$ for fixed x, y, and take the kernel $G(t;\varepsilon)$ of (24) as

$$G(t;\varepsilon) = F'(t;\varepsilon)u(t)F(v(t);\varepsilon)^{-1}, \tag{26}$$

where the $(p+2)\times(p+2)$ matrix function $u(t)$ (independent of x, y) is holomorphic in $L \cup L_-$ and satisfies

$$u(t)^+ \gamma u(t) = \gamma, \qquad \det u(t) = 1, \tag{27}$$

i.e. $u(t) \in SU(p+1,1)$.

By virtue of the above EHC structural RH problem, we have the following theorems:

Theorem 1 If, for fixed x and y, $X_\pm(t;\varepsilon) = X_\pm(x,y,t;\varepsilon)$ is a fundamental solution of the EHC RH problem (24), (26) with boundary condition

$$X_+(0;\varepsilon) = I, \tag{28}$$

then the EHC matrix function given by

$$\begin{aligned} \underline{F}(t;\varepsilon) &= X_+(t;\varepsilon)F'(t;\varepsilon) && \text{in } L \cup L_+ \\ &= X_-(t;\varepsilon)F(v(t);\varepsilon)u^{-1}(t) && \text{in } L \cup L_- \end{aligned} \tag{29}$$

is holomorphic on $L \cup L_+$ and satisfies

$$\mathrm{d}\underline{F}(t;\varepsilon) = \underline{\Gamma}'(t;\varepsilon)\varepsilon\gamma\underline{F}(t;\varepsilon), \tag{30a}$$

$$\underline{A}(t;\varepsilon)\mathrm{d}\underline{F}(t;\varepsilon) = t\mathrm{d}\underline{H}(\varepsilon)\varepsilon\gamma\underline{F}(t;\varepsilon), \tag{30b}$$

$$\underline{F}(0;\varepsilon) = I, \tag{31a}$$

$$\dot{\underline{F}}(0;\varepsilon) = \underline{H}(\varepsilon)\varepsilon\gamma, \tag{31b}$$

$$\lambda'(t)\underline{F}(t;\varepsilon)^+\varepsilon\gamma\underline{F}(t;\varepsilon) = \varepsilon\gamma, \tag{31c}$$

$$\underline{F}(t;\varepsilon)^\times\varepsilon\gamma\underline{A}(t;\varepsilon)\underline{F}(t;\varepsilon) = \varepsilon\gamma \tag{31d}$$

with

$$\underline{H}(\varepsilon) := H'(\varepsilon) - \dot{X}_+(0;\varepsilon)\varepsilon\gamma, \tag{32}$$

$$\underline{\Gamma}'(t;\varepsilon) := t\Lambda'(t)^{-1}\mathrm{d}\underline{H}(\varepsilon), \tag{33a}$$

$$\underline{A}(t;\varepsilon) := I - t(\underline{H}(\varepsilon) + \underline{H}(\varepsilon)^\ddagger)(\varepsilon\gamma). \tag{33b}$$

Theorem 2 The new EHC function $\underline{H}(\varepsilon)$ given by (32) is also an H-potential of the considered EM-p theory. Explicitly, $\underline{H}(\varepsilon)$ satisfies

$$2(z' + \rho'^*)\mathrm{d}\underline{H}(\varepsilon) = (\underline{H}(\varepsilon) + \underline{H}(\varepsilon)^\ddagger)\varepsilon\gamma\mathrm{d}\underline{H}(\varepsilon), \tag{34a}$$

$$\frac{1}{2}(\underline{H}(\varepsilon) - \underline{H}(\varepsilon)^\dagger) = -\varepsilon\gamma z', \tag{34b}$$

$$\underline{M} := \mathrm{Re}_{\mathrm{EHC}}(\underline{H}(\varepsilon)), \qquad \underline{M}^\dagger = \underline{M}, \tag{34c}$$

$$\underline{M}\gamma\underline{M} = \rho'^2\gamma. \tag{34d}$$

Before proving the Theorems 1 and 2, we first prove the following

Lemma 1 Based on the fundamental solution $X_\pm(t;\varepsilon)$ of the EHC RH problem (24), (26) with (28), we can consistently define the following EHC functions of t as

$$\begin{aligned} W_1(t;\varepsilon) &= X_+^+(t;\varepsilon)^{-1}\varepsilon\gamma X_+(t;\varepsilon)^{-1}, && \text{on } L \cup L_+ \\ &= (\lambda(v(t))/\lambda'(t))X_-^+(t;\varepsilon)^{-1}\varepsilon\gamma X_-(t;\varepsilon)^{-1}; && \text{on } L \cup L_- \end{aligned} \tag{35a}$$

$$\begin{aligned} W_2(t;\varepsilon) &= X_+^\times(t;\varepsilon)^{-1}\varepsilon\gamma A'(t;\varepsilon)X_+(t;\varepsilon)^{-1}, && \text{on } L \cup L_+ \\ &= X_-^\times(t;\varepsilon)^{-1}\varepsilon\gamma A(v(t);\varepsilon)X_-(t;\varepsilon)^{-1}; && \text{on } L \cup L_- \end{aligned} \tag{36a}$$

$$\begin{aligned} W_3(t;\varepsilon) &= dX_+(t;\varepsilon)X_+(t;\varepsilon)^{-1} \\ &\quad + X_+(t;\varepsilon)\Gamma'(t;\varepsilon)\varepsilon\gamma X_+(t;\varepsilon)^{-1}, && \text{on } L \cup L_+ \\ &= dX_-(t;\varepsilon)X_-(t;\varepsilon)^{-1} \\ &\quad + X_-(t;\varepsilon)\Gamma(v(t);\varepsilon)\varepsilon\gamma X_-(t;\varepsilon)^{-1}; && \text{on } L \cup L_- \end{aligned} \tag{37a}$$

$$\begin{aligned} W_4(t;\varepsilon) &= \varepsilon\gamma\underline{A}(t;\varepsilon)\mathrm{d}X_+(t;\varepsilon)X_+(t;\varepsilon)^{-1} \\ &\quad + tX_+^{\times}(t;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H'(\varepsilon)\varepsilon\gamma X_+(t;\varepsilon)^{-1}, \quad \text{on } L\cup L_+ \\ &= \varepsilon\gamma\underline{A}(t;\varepsilon)\mathrm{d}X_-(t;\varepsilon)X_-(t;\varepsilon)^{-1} \\ &\quad + v(t)X_-^{\times}(t;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H(\varepsilon)\varepsilon\gamma X_-(t;\varepsilon)^{-1} \quad \text{on } L\cup L_- \end{aligned} \tag{38a}$$

and have

$$W_1(t;\varepsilon) = \varepsilon\gamma; \tag{35b}$$

$$W_2(t;\varepsilon) = \varepsilon\gamma\underline{A}(t;\varepsilon); \tag{36b}$$

$$W_3(t;\varepsilon) = \underline{\Gamma}'(t;\varepsilon)\varepsilon\gamma; \tag{37b}$$

$$W_4(t;\varepsilon) = t\varepsilon\gamma\mathrm{d}\underline{H}(\varepsilon)\varepsilon\gamma. \tag{38b}$$

$$\underline{A}^+(t;\varepsilon)\varepsilon\gamma\underline{A}(t;\varepsilon) = \lambda'^2(t;\varepsilon)\varepsilon\gamma. \tag{39}$$

proof: From (14), (27), (26) and (24) we obtain

$$\begin{aligned} &X_+^+(s;\varepsilon)^{-1}\varepsilon\gamma X_+(s;\varepsilon)^{-1} \\ =&(\lambda(v(s))/\lambda'(s))X_-^+(s;\varepsilon)^{-1}\varepsilon\gamma X_-(s;\varepsilon)^{-1}, \quad s\in L. \end{aligned}$$

Noticing the properties of the functions $X_\pm(t;\varepsilon)$, $v(t)$, $\lambda(t)$ and since $[\lambda(v(t))/\lambda'(t)]$ is non-singular in $L\cup L_-$, the above equation implies that $W_1(t;\varepsilon)$ in (35a) is consistently defined and gives an EHC entire function of t. Note that

$$(\lambda(v(t))/\lambda'(t))X_-^+(t;\varepsilon)^{-1}\varepsilon\gamma X_-(t;\varepsilon)^{-1}$$

is regular at $t=\infty$, so $W_1(t;\varepsilon)$ is equal to a constant matrix. From the boundary condition (28), this constant matrix is $\varepsilon\gamma$. Thus we get equation (35b).

To prove (36b), we use (15), (27), (26) and (24) to get

$$\begin{aligned} &X_+^{\times}(s;\varepsilon)^{-1}\varepsilon\gamma A'(s;\varepsilon)X_+(s;\varepsilon)^{-1} \\ =&X_-^{\times}(s;\varepsilon)^{-1}\varepsilon\gamma A(v(s);\varepsilon)X_-(s;\varepsilon)^{-1}, \quad s\in L. \end{aligned}$$

This implies that $W_2(t;\varepsilon)$ is consistently defined and gives EHC entire function of t. From the expression of $A(v(t);\varepsilon)$ and the property of $v(t)$ at $t=\infty$, we conclude that $W_2(t;\varepsilon)$ is linear in t. By using (28) we obtain the coefficients in this linear function such that $W_2(t;\varepsilon)$ turns to be $\varepsilon\gamma\underline{A}(t;\varepsilon)$ by definition (33c). That proves Eq.(36b).

Equation (37b) is proven as follows. From (24), (26) and (12) we have

$$\begin{aligned} =&\mathrm{d}X_+(s;\varepsilon)X_+(s;\varepsilon)^{-1} \\ &+ X_+(s;\varepsilon)\Gamma'(s;\varepsilon)\varepsilon\gamma X_+(s;\varepsilon)^{-1} \\ =&\mathrm{d}X_-(s;\varepsilon)X_-(s;\varepsilon)^{-1} \\ &+ X_-(s;\varepsilon)\Gamma(v(s);\varepsilon)\varepsilon\gamma X_-(s;\varepsilon)^{-1}, s\in L. \end{aligned} \tag{40}$$

Thus the function $W_3(t;\varepsilon)$ defined in (37a) is a meromorphic function of t and has simple singularities at the zeros of $\lambda'(t)$ (or, equivalently, $\lambda(v(t))$). According to the theory of meromorphic function, $W_3(t;\varepsilon)$ can be expressed as $P(\varepsilon)+t\Lambda'(t)^{-1}Q(\varepsilon)$. By using (28) and (32) we obtain

$$\begin{aligned} &P(\varepsilon) = 0, \\ &Q(\varepsilon) = \mathrm{d}\dot{X}_+(0;\varepsilon) + \mathrm{d}H'(\varepsilon)\varepsilon\gamma = \mathrm{d}\underline{H}(\varepsilon)\varepsilon\gamma, \end{aligned}$$

thus $W_3(t;\varepsilon) = t\Lambda'(t)^{-1}\mathrm{d}\underline{H}(\varepsilon)\varepsilon\gamma$ and that gives (37b) by (33).

To prove Eq.(38b), we note that from (40), (36a, 36b) and (11) we have

$$\begin{aligned} &= \varepsilon\gamma\underline{A}(s;\varepsilon)\mathrm{d}X_+(s;\varepsilon)X_+(s;\varepsilon)^{-1} \\ &+ sX_+^{\times}(s;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H'(\varepsilon)\varepsilon\gamma X_+(s;\varepsilon)^{-1} \\ =&\varepsilon\gamma\underline{A}(s;\varepsilon)\mathrm{d}X_-(s;\varepsilon)X_-(s;\varepsilon)^{-1} \\ &+ v(s)X_-^{\times}(s;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H(\varepsilon)\varepsilon\gamma X_-(s;\varepsilon)^{-1}, \quad s\in L. \end{aligned}$$

Thus $W_4(t;\varepsilon)$ in (38a) is an entire function of t and by using (28) we get (38b).

As for (39), we first note that from (14), (15) we have $A^+(t;\varepsilon)\varepsilon\gamma A(t;\varepsilon) = \lambda^2(t)\varepsilon\gamma$. Then from (35a, 35b) and (36a, 36b) it follows that

$$\begin{aligned} &\underline{A}^+(t;\varepsilon)\varepsilon\gamma\underline{A}(t;\varepsilon) \\ &= \begin{cases} X_+^+(t;\varepsilon)^{-1}A'^+(t;\varepsilon)\varepsilon\gamma X_+^\star(t;\varepsilon)^{-1}\varepsilon\gamma X_+^{\times}(t;\varepsilon)^{-1}\varepsilon\gamma A'(t;\varepsilon)X_+(t;\varepsilon)^{-1} & \text{on } L\cup L_+ \\ X_-^+(t;\varepsilon)^{-1}A^+(v(t);\varepsilon)\varepsilon\gamma X_-^\star(t;\varepsilon)^{-1}\varepsilon\gamma X_-^{\times}(t;\varepsilon)^{-1}\varepsilon\gamma A(v(t);\varepsilon)X_-(t;\varepsilon)^{-1} & \text{on } L\cup L_- \end{cases} \\ &= \lambda'^2(t)\varepsilon\gamma. \end{aligned}$$

Proof of Theorem 1 Equation (30a) is derived from (37a, 37b), (29), and (12); Eq.(30b) is deduced from (38a, 38b), (29), (11), (12) and (36a, 36b); Eq.(31a) follows simply from (29), (13) and (28); and (31b) from Eqs.(29), (28), (13) and the definition (32); Eq.(31c) follows from (35a, 35b), (29) and (14); Eq.(31d) is derived from (36a, 36b), (29) and (15).

Proof of Theorem 2 To prove (34a), note that Eqs.(33a) and (30a, 30b) imply $\Lambda'(t)\mathrm{d}\underline{F}(t;\varepsilon) = \underline{A}(t;\varepsilon)\,\mathrm{d}\underline{F}(t;\varepsilon)$. Thus from (9) and (33b) we have

$$2(z' + \rho'^{*})\mathrm{d}\underline{F}(t;\varepsilon) = (\underline{H}(\varepsilon) + \underline{H}(\varepsilon)^{\ddagger})\varepsilon\gamma\,\mathrm{d}\underline{F}(t;\varepsilon).$$

Taking the t-derivative of the above equation and then setting $t = 0$, we obtain (34a) by using (31b). Equation (34b) is derived by taking t-derivative of Eq.(31c) and then setting $t = 0$, and noting (31a, 31b). Equation (34c) is a trivial implication of (34b). To prove (34d), note that from Eqs.(34b, 34c), $\underline{A}(t;\varepsilon) = (1 - 2tz')I - 2t\underline{M}\varepsilon\gamma$. Thus from (39) we have $(1-2tz')^2\varepsilon\gamma - \varepsilon\gamma\underline{M}\varepsilon\gamma\underline{M}\varepsilon\gamma = \lambda'(t)^2\varepsilon\gamma$, this gives (34d).

3.2. EHC structural RH transformation for Eq.(21)

Here we need another scalar function $\tilde{v}(\tau)$, which has the same properties as $v(t)$ in the previous subsection but the variable t is replaced by τ, and according to the properties of $\tilde{v}(\tau)$ we may write

$$\tilde{v}(\tau)|_{\tau\to\infty} = a\tau, \qquad a > 0 \text{ (real number)}. \tag{41}$$

Relating to $\tilde{v}(\tau)$ we introduce two real functions $\rho'' = \rho''(x,y)$, $z'' = z''(x,y)$ such that (for fixed x, y) $\tilde{\lambda}(\tilde{v}(\tau)) = [(\tilde{v}(\tau) - 2z)^2 + (2\rho)^2]^{1/2}$ and $\tilde{\lambda}''(t) := [(\tilde{v}(\tau) - 2z'')^2 + (2\rho'')^2]^{1/2}$ have the same zeros in τ, and $\tilde{v}(\tau) \neq 0$ at these zeros. Thus we have

$$2(z'' \pm \mathrm{i}\rho'') = v^{-1}(2(z \pm \mathrm{i}\rho), \tag{42a}$$

$$\mathrm{d}\rho'' = \mathrm{d}z'', \tag{42b}$$

equation (42a) can be interpreted as another variable transformation, the corresponding transformed functions will be denoted with double prime "$''$", e.g. $\tilde{F}(\tau;\varepsilon) \mapsto \tilde{F}''(\tau;\varepsilon)$, etc.

Consider an EHC structural RH problem relating to (21)–(23) as follows. Similar to L and $L_\pm$, now we use $\tilde{L}$ and $\tilde{L}_\pm$ in τ-plane. For a given EHC solution $\tilde{F}(\tau;\varepsilon)$ of equation (21) with conditions (22), (23), we choose the contour $\tilde{L}$ such that $\tilde{F}(\tau;\varepsilon)$ is holomorphic on $\tilde{L} \cup \tilde{L}_+$ for fixed x, y, and take the kernel of (24) as

$$\tilde{G}(\tau;\varepsilon) = a^{-1/2}\tilde{F}''(\tau;\varepsilon)\tilde{u}(\tau)\tilde{F}(\tilde{v}(\tau);\varepsilon)^{-1}, \tag{43}$$

where the positive real number a is the same as in (41), and $\tilde{u}(\tau)$ has the same properties as $u(t)$ in subsection 3.2 except that t is replaced by τ.

By virtue of EHC RH problem (24) with kernel (43), we can obtain another EHC RH transformation of the EM-p theories under consideration. Firstly, we have the following

Lemma 2 For fixed x and y, let $\tilde{X}_\pm(\tau;\varepsilon) = \tilde{X}_\pm(x,y,\tau;\varepsilon)$ be a fundamental solution of the EHC RH problem (24), (43) with boundary condition

$$\tilde{X}_-(\infty;\varepsilon) = I. \tag{44}$$

Then we can consistently define the following functions of τ as

$$\begin{aligned}\tilde{W}_1(\tau;\varepsilon) &= \tilde{X}_+^+(\tau;\varepsilon)^{-1}\varepsilon\gamma\tilde{X}_+(\tau;\varepsilon)^{-1}, && \text{on } \tilde{L}\cup\tilde{L}_+\\ &= a^{-1}(\tilde{\lambda}(\tilde{v}(\tau))/\tilde{\lambda}''(\tau))\tilde{X}_-^+(\tau;\varepsilon)^{-1}\varepsilon\gamma\tilde{X}_-(\tau;\varepsilon)^{-1}; && \text{on } \tilde{L}\cup\tilde{L}_-\end{aligned} \tag{45a}$$

$$\begin{aligned}\tilde{W}_2(\tau;\varepsilon) &= \tilde{X}_+^\times(\tau;\varepsilon)^{-1}\varepsilon\gamma\tilde{A}''(\tau;\varepsilon)\tilde{X}_+(\tau;\varepsilon)^{-1}, && \text{on } \tilde{L}\cup\tilde{L}_+\\ &= a^{-1}\tilde{X}_-^\times(\tau;\varepsilon)^{-1}\varepsilon\gamma\tilde{A}(\tilde{v}(\tau);\varepsilon)\tilde{X}_-(\tau;\varepsilon)^{-1}; && \text{on } \tilde{L}\cup\tilde{L}_-\end{aligned} \tag{46a}$$

$$\begin{aligned}\tilde{W}_3(\tau;\varepsilon) &= \mathrm{d}\tilde{X}_+(\tau;\varepsilon)\tilde{X}_+(\tau;\varepsilon)^{-1}\\ &\quad + \tilde{X}_+(\tau;\varepsilon)\tilde{I}''(\tau;\varepsilon)\varepsilon\gamma\tilde{X}_+(\tau;\varepsilon)^{-1}, && \text{on } \tilde{L}\cup\tilde{L}_+\\ &= \mathrm{d}\tilde{X}_-(\tau;\varepsilon)\tilde{X}_-(\tau;\varepsilon)^{-1}\\ &\quad + \tilde{X}_-(\tau;\varepsilon)\tilde{I}(\tilde{v}(\tau);\varepsilon)\varepsilon\gamma\tilde{X}_-(\tau;\varepsilon)^{-1}; && \text{on } \tilde{L}\cup\tilde{L}_-\end{aligned} \tag{47a}$$

$$\begin{aligned}\tilde{W}_4(\tau;\varepsilon) &= \varepsilon\gamma\underline{\tilde{A}}(\tau;\varepsilon)\mathrm{d}\tilde{X}_+(\tau;\varepsilon)\tilde{X}_+(\tau;\varepsilon)^{-1}\\ &\quad+\tilde{X}_+^{\times}(\tau;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H''(\varepsilon)\varepsilon\gamma\tilde{X}_+(\tau;\varepsilon)^{-1}, \qquad \text{on } \tilde{L}\cup\tilde{L}_+\\ &= \varepsilon\gamma\underline{\tilde{A}}(\tau;\varepsilon)\mathrm{d}\tilde{X}_-(\tau;\varepsilon)\tilde{X}_-(\tau;\varepsilon)^{-1}\\ &\quad+a^{-1}\tilde{X}_-^{\times}(\tau;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}H(\varepsilon)\varepsilon\gamma\tilde{X}_-(\tau;\varepsilon)^{-1} \quad \text{on } \tilde{L}\cup\tilde{L}_-\end{aligned} \tag{48a}$$

and have

$$\tilde{W}_1(\tau;\varepsilon) = \varepsilon\gamma; \tag{45b}$$

$$\tilde{W}_2(\tau;\varepsilon) = \varepsilon\gamma\underline{\tilde{A}}(\tau;\varepsilon); \tag{46b}$$

$$\tilde{W}_3(\tau;\varepsilon) = \underline{\tilde{\Gamma}}''(\tau;\varepsilon)\varepsilon\gamma; \tag{47b}$$

$$\tilde{W}_4(\tau;\varepsilon) = \varepsilon\gamma\mathrm{d}\underline{\tilde{H}}(\varepsilon)\varepsilon\gamma. \tag{48b}$$

$$\underline{\tilde{A}}^{+}(\tau;\varepsilon)\varepsilon\gamma\underline{\tilde{A}}(\tau;\varepsilon) = \tilde{\lambda}''^{2}(\tau;\varepsilon)\varepsilon\gamma. \tag{49}$$

Where

$$\underline{\tilde{H}}(\varepsilon) := a^{-1}H''(\varepsilon) - \partial_w\tilde{X}_-(\tau;\varepsilon)|_{w=0}\varepsilon\gamma, \quad w := \tau^{-1}, \tag{50}$$

$$\underline{\tilde{\Gamma}}''(\tau;\varepsilon) := \tilde{\Lambda}''(\tau)^{-1}\mathrm{d}\underline{\tilde{H}}(\varepsilon), \tag{51a}$$

$$\underline{\tilde{A}}(\tau;\varepsilon) := \tau I - (\underline{\tilde{H}}(\varepsilon) + \underline{\tilde{H}}(\varepsilon)^{\ddagger})\varepsilon\gamma. \tag{51b}$$

Proof The proof is similar to that of Lemma 1. However, here the constant a is involved and we should use the boundary conditions at $\tau = \infty$ such as (41), (44).

Theorem 3 The EHC matrix function given by

$$\begin{aligned}\underline{\tilde{F}}(\tau;\varepsilon) &= a^{-1/2}\tilde{X}_+(\tau;\varepsilon)\tilde{F}''(\tau;\varepsilon) \qquad \text{in } L\cup L_+\\ &= \tilde{X}_-(\tau;\varepsilon)\tilde{F}(\tilde{v}(\tau);\varepsilon)\tilde{u}(\tau)^{-1} \quad \text{in } L\cup L_-\end{aligned} \tag{52}$$

is holomorphic on $L\cup L_+$ and satisfies

$$\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon) = \underline{\tilde{\Gamma}}''(\tau;\varepsilon)\varepsilon\gamma\underline{\tilde{F}}(t;\varepsilon), \tag{53a}$$

$$\underline{\tilde{A}}(\tau;\varepsilon)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon) = \mathrm{d}\underline{\tilde{H}}(\varepsilon)\varepsilon\gamma\underline{\tilde{F}}(\tau;\varepsilon), \tag{53b}$$

$$\tilde{\lambda}''(\tau)\underline{\tilde{F}}(\tau;\varepsilon)^{+}\varepsilon\gamma\underline{\tilde{F}}(\tau;\varepsilon) = \varepsilon\gamma, \tag{54a}$$

$$\underline{\tilde{F}}(\tau;\varepsilon)^{\times}\varepsilon\gamma\underline{\tilde{A}}(\tau;\varepsilon)\underline{\tilde{F}}(\tau;\varepsilon) = \varepsilon\gamma. \tag{54b}$$

Proof Equations (53a, 53b), (54a, 54b) are, respectively, derived from equations (47a, 47b), (52) and (21), equations (48a, 48b), (52), (20), (21) and (46a, 46b), equations (45a, 45b), (52) and (22), equations (46a, 46b), (52) and (23).

Theorem 4 The new EHC function $\underline{\tilde{H}}(\varepsilon)$ given by (50) is also an H-potential of the considered EM-p theory. Explicitly, $\underline{\tilde{H}}(\varepsilon)$ satisfies

$$2(z''+\rho''^{*})\mathrm{d}\underline{\tilde{H}}(\varepsilon) = (\underline{\tilde{H}}(\varepsilon) + \underline{\tilde{H}}(\varepsilon)^{\ddagger})\varepsilon\gamma\mathrm{d}\underline{\tilde{H}}(\varepsilon), \tag{55a}$$

$$\frac{1}{2}(\underline{\tilde{H}}(\varepsilon) - \underline{\tilde{H}}(\varepsilon)^{\dagger}) = -\varepsilon\gamma z'', \tag{55b}$$

$$\underline{\tilde{M}} := \mathrm{Re}_{\mathrm{EHC}}(\underline{\tilde{H}}(\varepsilon)), \qquad \underline{\tilde{M}}^{\dagger} = \underline{\tilde{M}}, \tag{55c}$$

$$\underline{\tilde{M}}\gamma\underline{\tilde{M}} = \rho''^{2}\gamma. \tag{55d}$$

Proof Firstly we note that equations (53a) and (51a) imply

$$-\tilde{\Lambda}''(\tau)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon)\underline{\tilde{F}}(\tau;\varepsilon)^{-1}\varepsilon\gamma = \mathrm{d}\underline{\tilde{H}}(\varepsilon). \tag{56}$$

To prove (55a), use (53a, 53b) to get

$$\underline{\tilde{A}}(\tau;\varepsilon)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon) = \tilde{\Lambda}''(\tau)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon).$$

Multiplying this equation from the left by $\tilde{\Lambda}''(\tau)$ and from the right by $\underline{\tilde{F}}(\tau;\varepsilon)^{-1}\varepsilon\gamma$, then we obtain (55a) by using (56) and the definitions of $\underline{\tilde{A}}(\tau;\varepsilon)$ and $\tilde{\Lambda}''(\tau)$.

To prove (55b), note that from (54a) and (19) we have

$$\begin{aligned}&\tilde{\lambda}''(\tau)^{-1}\mathrm{d}\tilde{\lambda}''(\tau)\varepsilon\gamma\\ =&-\underline{\tilde{F}}^{+}(\tau;\varepsilon)^{-1}\varepsilon\gamma\mathrm{d}\underline{\tilde{F}}^{+}(\tau;\varepsilon)\varepsilon\gamma\\ &-\varepsilon\gamma\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon)\underline{\tilde{F}}(\tau;\varepsilon)^{-1},\end{aligned}$$

and

$$\tilde{\lambda}''(\tau)^{-1}\mathrm{d}\tilde{\lambda}''(\tau) = -2\tilde{\Lambda}''(\tau)\mathrm{d}z''.$$

These equations give

$$\begin{aligned}2\varepsilon\gamma\mathrm{d}z'' =&\tilde{\Lambda}''(\tau)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon)\underline{\tilde{F}}(\tau;\varepsilon)^{-1}\varepsilon\gamma\\ &-[\tilde{\Lambda}''(\tau)\mathrm{d}\underline{\tilde{F}}(\tau;\varepsilon)\underline{\tilde{F}}(\tau;\varepsilon)^{-1}\varepsilon\gamma]^{+},\end{aligned}$$

this, by using equation (56), is followed by $\mathrm{d}\underline{\tilde{H}}(\varepsilon) - \mathrm{d}\underline{\tilde{H}}(\varepsilon)^{\dagger} = -2\varepsilon\gamma\mathrm{d}z''$ and then gives (55b) by choosing some suitable integral constant.

Equation (55c) is a trivial implication of (55b). The proof of (55d) is similar to that of (34d).

The EHC RH transformations given above can be equivalently written as integral equations. Firstly, note that $X_-(t;\varepsilon)$ in (29) is analytic in $L\cup L_-$ (including $t=\infty$), we have

$$\frac{1}{2\pi\mathrm{i}}\int_L\frac{X_-(s;\varepsilon)}{s(s-t)}\mathrm{d}s = 0, \qquad t\in L_+. \tag{57}$$

Substituting equation (29) into (57) it follows that

$$\frac{1}{2\pi\mathrm{i}}\int_L\frac{\underline{F}(s;\varepsilon)u(s)F(v(s);\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s = 0, \quad t\in L_+, \tag{58}$$

subject to the condition $\underline{F}(0;\varepsilon)=I$. As for EHC RH transformation (52), by condition (44) we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\tilde{L}}\frac{\tilde{X}_-(s;\varepsilon)}{(s-\tau)}\mathrm{d}s=I,\qquad \tau\in\tilde{L}_+. \tag{59}$$

Now from (52) we obtain

$$\frac{1}{2\pi\mathrm{i}}\int_{\tilde{L}}\frac{\underline{\tilde{F}}(s;\varepsilon)\tilde{u}(s)\tilde{F}(\tilde{v}(s);\varepsilon)^{-1}}{(s-\tau)}\mathrm{d}s=I,\qquad \tau\in\tilde{L}_+. \tag{60}$$

In order to show the group structure of the above EHC RH transformations explicitly, we introduce $\xi(t):=v^{-1}(t)$ on the contour L and define the action of $(u,\ \xi)$ on any EHC analytic function $\Psi(t;\varepsilon)$ as

$$(u,\xi)\,\Psi(t;\varepsilon):=u(t)\,\Psi(\xi^{-1}(t);\varepsilon)=u(t)\,\Psi(v(t);\varepsilon). \tag{61}$$

Then integral equation (58) can be rewritten as

$$\frac{1}{2\pi\mathrm{i}}\int_{L}\frac{\underline{F}(s;\varepsilon)(u,\xi)F(s;\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s=0,\qquad t\in L_+. \tag{62}$$

If we carry out the EHC RH transformation two times successively and denote

$$\begin{aligned}&(u,\xi):\ F(t;\varepsilon)\longrightarrow\underline{F}(t;\varepsilon),\\ &(u_1,\xi_1):\ \underline{F}(t;\varepsilon)\longrightarrow\underline{\underline{F}}(t;\varepsilon),\end{aligned} \tag{63}$$

it can be verified that we have an EHC RH transformation $(u_2,\xi_2):\ F(t;\varepsilon)\to\underline{\underline{F}}(t;\varepsilon)$ such that

$$(u_2,\xi_2)=(u_1,\xi_1)(u,\xi)=(u_1\gamma_{\xi_1}(u),\xi_1\xi). \tag{64}$$

Here we have used the homomorphism $\gamma:\ \{\xi\}\to\mathrm{Aut}\{u\}$ defined by

$$\gamma:\xi\longrightarrow\gamma_\xi,\quad \gamma_\xi:u(t)\longrightarrow\gamma_\xi(u)(t)=u(\xi^{-1}(t)). \tag{65}$$

Similarly, we have

$$(\tilde{u}_2,\tilde{\xi}_2)=(\tilde{u}_1,\tilde{\xi}_1)(\tilde{u},\tilde{\xi})=(\tilde{u}_1\tilde{\gamma}_{\tilde{\xi}_1}(\tilde{u}),\tilde{\xi}_1\tilde{\xi}). \tag{66}$$

According to the loop group theory,[20] the composition laws (64) and (66) show that the EHC RH transformations (29) (or (58)) and (52) (or (60)) provide some representation of the semidirect product of the Kac–Moody and Virasoro groups. When $v(t)=t$, $\tilde{v}(\tau)=\tau$ (thus $a=1$), (29) (or (58)) and (52) (or (60)) provide a representation of the Kac–Moody group. When $u(t)=I$, $\tilde{u}(\tau)=I$, we get a representation of the Virasoro group.

4. Infinitesimal RH transformations and their algebraic structure

In the paper Ref.[17], by a different method, we gave an infinite set of infinitesimal symmetry transformations of the EM-p theories and showed that they have an infinite-dimensional algebraic structure. In this section, we discuss the relationship between the ECH RH transformations given by Section 3 of this paper and the results of Ref.[17].

Setting $v(t)=t$ and considering the infinitesimal transformation

$$u(t)=I+\delta u(t),\qquad \underline{F}(t;\varepsilon)=F(t;\varepsilon)+\delta F(t;\varepsilon),$$

then (58) becomes

$$\begin{aligned}&\delta F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ &=-\frac{t}{2\pi\mathrm{i}}\int_L\frac{F(s;\varepsilon)\delta u(s)F(s;\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s,\qquad t\in L_+.\end{aligned} \tag{67}$$

Noticing the analytic property of $u(t)$, without loss of generality, we can choose $\delta^{(k)}_\alpha u(t)=T_\alpha t^{-k}$, $(k\geq 0)$, where $T_\alpha=T_a\alpha^a$, T_a are generators of $su(p+1,1)$ (the Lie algebra of $SU(p+1,1)$), α^a are infinitesimal real constants. Substituting them into (67), we have

$$\begin{aligned}&\delta^{(k)}_\alpha F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ &=-\frac{t}{2\pi\mathrm{i}}\int_L\frac{s^{-k}F(s;\varepsilon)T_\alpha F(s;\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s,\qquad t\in L_+.\end{aligned}$$

Then the parameterized infinitesimal transformation $\delta_\alpha(t')F(t;\varepsilon)=\sum_{k=0}^{\infty}t'^k\delta^{(k)}_\alpha F(t;\varepsilon)$ $(t'\in L_+)$ is given by

$$\begin{aligned}&\delta_\alpha(t')F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ &=-\frac{t}{2\pi\mathrm{i}}\int_L\sum_{k=0}^{\infty}\frac{t'^kF(s;\varepsilon)T_\alpha F(s;\varepsilon)^{-1}}{s^{k+1}(s-t)}\mathrm{d}s\\ &=-\frac{t}{2\pi\mathrm{i}}\int_L\frac{F(s;\varepsilon)T_\alpha F(s;\varepsilon)^{-1}}{(s-t')(s-t)}\mathrm{d}s\\ &=\frac{t}{t-t'}[F(t';\varepsilon)T_\alpha F(t';\varepsilon)^{-1}-F(t;\varepsilon)T_\alpha F(t;\varepsilon)^{-1}].\end{aligned} \tag{68}$$

Similarly, taking $\tilde{v}(\tau)=\tau$ and considering the infinitesimal transformation

$$\begin{aligned}&\tilde{u}(\tau)=I+\tilde{\delta}\tilde{u}(\tau),\\ &\underline{\tilde{F}}(\tau;\varepsilon)=\tilde{F}(\tau;\varepsilon)+\tilde{\delta}\tilde{F}(\tau;\varepsilon),\end{aligned}$$

then (60) becomes

$$\begin{aligned}&\tilde{\delta}\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ &=-\frac{1}{2\pi\mathrm{i}}\int_{\tilde{L}}\frac{\tilde{F}(s;\varepsilon)\tilde{\delta}\tilde{u}(s)\tilde{F}(s;\varepsilon)^{-1}}{(s-\tau)}\mathrm{d}s,\quad \tau\in\tilde{L}_+.\end{aligned} \tag{69}$$

Without loss of generality, we can choose $\tilde{\delta}_\alpha^{(k)}\tilde{u}(\tau) = T_\alpha\tau^{-k}$, $(k \geq 1)$, then from (69) we obtain the parameterized infinitesimal transformation $\tilde{\delta}_\alpha(\tau')\tilde{F}(\tau;\varepsilon) = \sum_{k=1}^{\infty}\tau'^k\tilde{\delta}_\alpha^{(k)}\tilde{F}(\tau;\varepsilon)$ $(\tau' \in \tilde{L}_+)$ as

$$\begin{aligned}&\tilde{\delta}_\alpha(\tau')\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ =&\frac{\tau'}{\tau-\tau'}[\tilde{F}(\tau';\varepsilon)T_\alpha\tilde{F}(\tau';\varepsilon)^{-1}\\ &-\tilde{F}(\tau;\varepsilon)T_\alpha\tilde{F}(\tau;\varepsilon)^{-1}].\end{aligned}\tag{70}$$

The RH transformations (58) and (60), in fact, contain more symmetries of the EM-p theories. To show this, we need "cross" infinitesimal variations of $F(t;\varepsilon)$ and $\tilde{F}(\tau;\varepsilon)$, i.e. $\tilde{\delta}F(t;\varepsilon)$ and $\delta\tilde{F}(\tau;\varepsilon)$. They are brought about by $u(t) = I + \tilde{\delta}u(t)$ and $\tilde{u}(\tau) = I + \delta\tilde{u}(\tau)$, respectively. Considering the relation between t and τ in (12) and (21), we can choose $\tilde{\delta}_\alpha^{(k)}u(t) = T_\alpha t^k$, $(k \geq 1)$ and $\delta_\alpha^{(k)}\tilde{u}(\tau) = T_\alpha\tau^k$, $(k \geq 0)$. Correspondingly, equations (67) and (69) give, respectively,

$$\begin{aligned}&\tilde{\delta}_\alpha^{(k)}F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ =&-\frac{t}{2\pi\mathrm{i}}\int_L\frac{s^kF(s;\varepsilon)T_\alpha F(s;\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s;\\ &\delta_\alpha^{(k)}\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ =&-\frac{1}{2\pi\mathrm{i}}\int_{\tilde{L}}\frac{s^k\tilde{F}(s;\varepsilon)T_\alpha\tilde{F}(s;\varepsilon)^{-1}}{(s-\tau)}\mathrm{d}s.\end{aligned}$$

To obtain explicit expressions of the corresponding parameterized "cross" infinitesimal variations of $\tilde{\delta}_\alpha(\tau)F(t;\varepsilon) = \sum_{k=1}^{\infty}\tau^k\tilde{\delta}_\alpha^{(k)}F(t;\varepsilon)$ and $\delta_\alpha(t)\tilde{F}(\tau;\varepsilon) = \sum_{k=0}^{\infty}t^k\delta_\alpha^{(k)}\tilde{F}(\tau;\varepsilon)$, we note that since $F(t;\varepsilon)$, $\tilde{F}(\tau;\varepsilon)$ have different analytic properties, from (12)–(15), (21)–(23) we can set

$$\begin{aligned}&\tilde{F}(\tau;\varepsilon)|_{\tau=\frac{1}{t}} = t^{1/2}F(t;\varepsilon),\\ &F(t;\varepsilon)|_{t=\frac{1}{\tau}} = \tau^{1/2}\tilde{F}(\tau;\varepsilon).\end{aligned}\tag{71}$$

Thus we have

$$\begin{aligned}\tilde{\delta}_\alpha(\tau)F(t;\varepsilon)F(t;\varepsilon)^{-1} =&\frac{t\tau}{1-t\tau}[\tilde{F}(\tau;\varepsilon)T_\alpha\tilde{F}(\tau;\varepsilon)^{-1}\\ &-F(t;\varepsilon)T_\alpha F(t;\varepsilon)^{-1}];\end{aligned}\tag{72}$$

$$\begin{aligned}\delta_\alpha(t)\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1} =&\frac{1}{1-t\tau}[F(t;\varepsilon)T_\alpha F(t;\varepsilon)^{-1}\\ &-\tilde{F}(\tau;\varepsilon)T_\alpha\tilde{F}(\tau;\varepsilon)^{-1}].\end{aligned}\tag{73}$$

Next we consider the cases of $u(t) = I$ and $\tilde{u}(\tau) = I$ of the transformations (58) and (60). We first calculate the infinitesimal transformations brought about by

$$v(t) = t + \Delta(t),\tag{74a}$$

$$\tilde{v}(\tau) = \tau + \tilde{\Delta}(\tau),\tag{74b}$$

where $\Delta(t)$ and $\tilde{\Delta}(\tau)$ are infinitesimal functions of t and τ, respectively. For (74a), we write the variation of $F(t;\varepsilon)$ as

$$\begin{aligned}&\underline{F}(t;\varepsilon) = F(t;\varepsilon) + \Delta F(t;\varepsilon),\\ &F(v(t);\varepsilon)^{-1} = F(t;\varepsilon)^{-1} + \dot{F}(t;\varepsilon)^{-1}\Delta(t),\end{aligned}\tag{75}$$

where $\dot{F}(t;\varepsilon)^{-1} := \partial_v(F(v;\varepsilon)^{-1})|_{v=t}$. Substituting $u(t) = I$ and (75) into (58), we have

$$\begin{aligned}&\Delta F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ =&\frac{t}{2\pi\mathrm{i}}\int_L\frac{\dot{F}(s;\varepsilon)F(s;\varepsilon)^{-1}}{s(s-t)}\Delta(s)\mathrm{d}s,\qquad t\in L_+.\end{aligned}\tag{76}$$

Noticing the properties of $v(t)$, without loss of generality, we can choose $\Delta(t) = \Delta_\sigma^{(k)}(t) = \sigma t^{1-k}$, $(k \geq 0)$, where σ is an infinitesimal real constant. Then (76) gives

$$\begin{aligned}&\Delta_\sigma^{(k)}F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ =&\frac{\sigma t}{2\pi\mathrm{i}}\int_L\frac{s^{1-k}\dot{F}(s;\varepsilon)F(s;\varepsilon)^{-1}}{s(s-t)}\mathrm{d}s,\qquad t\in L_+.\end{aligned}$$

So, the infinitesimal parameterized variation

$$\Delta_\sigma(t')F(t;\varepsilon) = \sum_{k=0}^{\infty}t'^k\Delta_\sigma^{(k)}F(t;\varepsilon)$$

is given by

$$\begin{aligned}\Delta_\sigma(t')F(t;\varepsilon)F(t;\varepsilon)^{-1} =&\frac{\sigma t}{t-t'}[t\dot{F}(t;\varepsilon)F(t;\varepsilon)^{-1}\\ &-t'\dot{F}(t';\varepsilon)F(t';\varepsilon)^{-1}].\end{aligned}\tag{77}$$

Similarly, for (74b), we obtain the infinitesimal parameterized variation

$$\tilde{\Delta}_\epsilon(\tau')\tilde{F}(\tau;\varepsilon) = \sum_{k=1}^{\infty}\tau'^k\tilde{\Delta}_\epsilon^{(k)}\tilde{F}(\tau;\varepsilon)$$

as

$$\begin{aligned}\tilde{\Delta}_\epsilon(\tau')\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1} =&\frac{\epsilon\tau'}{\tau-\tau'}[\tau\dot{\tilde{F}}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ &-\tau'\dot{\tilde{F}}(\tau';\varepsilon)\tilde{F}(\tau';\varepsilon)^{-1}].\end{aligned}\tag{78}$$

As before, we also need "cross" infinitesimal transformations $\tilde{\Delta}F(t;\varepsilon)$ and $\Delta\tilde{F}(\tau;\varepsilon)$ which are brought about by variations

$$v(t) = t + \tilde{\Delta}(t),\tag{79a}$$

$$\tilde{v}(\tau) = \tau + \Delta(\tau).\tag{79b}$$

In these cases, the RH transformations (58) and (60) give

$$\tilde{\Delta}F(t;\varepsilon)F(t;\varepsilon)^{-1}=\frac{t}{2\pi\mathrm{i}}\int_L\frac{\dot{\tilde{F}}(s;\varepsilon)F(s;\varepsilon)^{-1}}{s(s-t)}\tilde{\Delta}(s)\mathrm{d}s; \tag{80}$$

$$\Delta\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}=\frac{1}{2\pi\mathrm{i}}\int_{\tilde{L}}\frac{\dot{\tilde{F}}(s;\varepsilon)\tilde{F}(s;\varepsilon)^{-1}}{(s-\tau)}\Delta(s)\mathrm{d}s. \tag{81}$$

Moreover, from the relations (71), we have

$$\begin{aligned}&[\dot{\tilde{F}}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}]|_{\tau=\frac{1}{t}}\\ =&-t[t\dot{F}(t;\varepsilon)F(t;\varepsilon)^{-1}+\frac{1}{2}];\\ &[\dot{F}(t;\varepsilon)F(t;\varepsilon)^{-1}]|_{t=\frac{1}{\tau}}\\ =&-\tau[\tau\dot{\tilde{F}}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}+\frac{1}{2}].\end{aligned} \tag{82}$$

Similar to above, we choose $\tilde{\Delta}_\epsilon^{(k)}(t)=\epsilon t^{1+k}$, $k\geq 1$; $\Delta_\sigma^{(k)}(\tau)=\sigma\tau^{1+k}$, $k\geq 0$ and the corresponding parameterized infinitesimal transformations $\tilde{\Delta}_\epsilon(\tau)F(t;\varepsilon)=\sum_{k=1}^{\infty}\tau^k\tilde{\Delta}_\epsilon^{(k)}F(t;\varepsilon)$ and $\Delta_\sigma(t)\tilde{F}(\tau;\varepsilon)=\sum_{k=0}^{\infty}t^k\Delta_\sigma^{(k)}\tilde{F}(\tau;\varepsilon)$ are given, respectively, by

$$\begin{aligned}&\tilde{\Delta}_\epsilon(\tau)F(t;\varepsilon)F(t;\varepsilon)^{-1}\\ =&\frac{\epsilon t\tau}{1-t\tau}[t\dot{F}(t;\varepsilon)F(t;\varepsilon)^{-1}\\ &+\tau\dot{\tilde{F}}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}+\frac{1}{2}];\end{aligned} \tag{83}$$

$$\begin{aligned}&\Delta_\sigma(t)\tilde{F}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ =&\frac{\sigma}{1-t\tau}[\tau\dot{\tilde{F}}(\tau;\varepsilon)\tilde{F}(\tau;\varepsilon)^{-1}\\ &+t\dot{F}(t;\varepsilon)F(t;\varepsilon)^{-1}+\frac{1}{2}].\end{aligned} \tag{84}$$

Finally, we give the transforms of ρ, z under these infinitesimal RH transformations. Write

$$\begin{aligned}&\eta:=z+\mathrm{i}\rho,\qquad \eta':=z'+\mathrm{i}\rho',\qquad \eta'':=z''+\mathrm{i}\rho'',\\ &t_+:=[2(z+\mathrm{i}\rho)]^{-1},\qquad \tau_+:=2(z+\mathrm{i}\rho).\end{aligned} \tag{85}$$

Then from (25a) and (77), we have

$$\begin{aligned}\Delta_\sigma^{(k)}\eta=&(\eta'-\eta)_{(\sigma)}\\ =&\frac{1}{2t_+^2}\Delta_\sigma^{(k)}(t_+)=\sigma\eta(2\eta)^k,\qquad (k\geq 0),\end{aligned}$$

therefore

$$\begin{aligned}\Delta_\sigma(t)\eta=&\sum_{k=0}^{\infty}t^k\Delta_\sigma^{(k)}\eta\\ =&\sigma\eta\sum_{k=0}^{\infty}t^k(2\eta)^k=\sigma\frac{\eta}{1-2t\eta},\end{aligned}$$

this implies that

$$\begin{aligned}\Delta_\sigma(t)z=&\frac{\sigma}{\lambda(t)^2}[z(1-2tz)-2t\rho^2],\\ \Delta_\sigma(t)\rho=&\frac{\sigma}{\lambda(t)^2}\rho.\end{aligned} \tag{86}$$

Similarly, from Eqs.(42a), (79) and (85), we obtain

$$\begin{aligned}\tilde{\Delta}_\epsilon^{(k)}\eta=&(\eta''-\eta)_{(\epsilon)}\\ =&-\frac{1}{2}\tilde{\Delta}_\epsilon^{(k)}(\tau_+)\\ =&-\frac{\epsilon}{2}\tau^{1-k}=-\epsilon\eta(2\eta)^{-k},\quad (k\geq 1),\end{aligned}$$

thus

$$\begin{aligned}\tilde{\Delta}_\epsilon(\tau)\eta=&\sum_{k=1}^{\infty}\tau^k\tilde{\Delta}_\epsilon^{(k)}\eta\\ =&-\epsilon\eta\sum_{k=1}^{\infty}\tau^k(2\eta)^{-k}=\epsilon\tau\frac{\eta}{\tau-2\eta},\end{aligned}$$

this gives

$$\begin{aligned}\tilde{\Delta}_\epsilon(\tau)z=&\frac{\epsilon\tau}{\tilde{\lambda}(\tau)^2}[z(\tau-2z)-2\rho^2],\\ \tilde{\Delta}_\epsilon(\tau)\rho=&\frac{\epsilon\tau^2}{\tilde{\lambda}(\tau)^2}\rho.\end{aligned} \tag{87}$$

Equations (68), (70), (72), (73), (77), (78), (83), (84) give exactly the same infinitesimal transforms of $F(t;\varepsilon)$, $\tilde{F}(\tau;\varepsilon)$ used in Ref.[17], and (86), (87) give the same infinitesimal transformations of ρ, z as obtained in Ref.[17]. By using these infinitesimal transforms we have verified in Ref.[17] that the corresponding infinitesimal transformations of EHC H-potential constitute an infinite-dimensional Lie algebra: the semidirect product of the full Kac–Moody algebra $SU(p+1,1)\otimes R(t,t^{-1})$ and Virasoro algebra (without centre charges). However, in the present paper, these are obtained as the results of infinitesimal forms of the RH transformations (58) and (60). This demonstrates that these two EHC RH transformations, joined together, provide some exponentiations of all the infinitesimal symmetry transformations in Ref.[17]. Moreover, the EHC RH transformations (58) and (60) (or equivalently (29) and (52)) give out finite transformations which can, in principle, be directly used to generate new solutions of EM-p theories from old ones.

5. Summary and discussions

By using the so-called EHC function method, the SAS EM-p theories are studied further. The usual

RH problem is extended into an EHC formulation. Associated with two HE-type EHC linear systems (12)–(15) and (21)–(23), we construct two EHC structural RH transformations of the F-potentials $F(t;\varepsilon)$ and $\tilde{F}(\tau;\varepsilon)$, i.e. (29) and (52) (or equivalently (58) and (60)). These EHC RH transformations generate new F-potentials from old ones and are shown to give an infinite-dimensional symmetry transformation group of the EM-p theory under consideration. This symmetry group is then shown to have the structure of semidirect product of complete Kac–Moody group $SU(\widehat{p+1},1)$ and Virasoro group. (However, the EHC RH transformation (29) (or equivalently (58)) alone gives the "positive half" symmetry subgroup only.) Moreover, we calculate the infinitesimal forms of the above mentioned two RH transformations and find that they give exactly the same infinitesimal transforms as in Ref.[17] by a different scheme, this demonstrates that two EHC RH transformations given in the present paper, joined together, provide some exponentiations of all the infinitesimal symmetry transformations obtained in Ref.[17]. However, the EHC RH transformations in this paper give out finite symmetry transformations which can, in principle, be directly used to generate new solutions of EM-p theories.

This paper treats all of the SAS EM-p theories for $p = 0, 1, 2, \cdots$ in a unified formulation. The case of $p = 0$ gives the result for the SAS vacuum gravity theory but in a "dualized" formalism. The "positive half" infinite-dimensional symmetry subgroup for the SAS vacuum gravity (corresponding to (29) (or (58)) of the case of $p = 0$) in a "non-dualized" formalism was given in the pioneering papers Refs.[18] and [19].

References

[1] Ernst F J 1968 *Phys. Rev.* **168** 1415
Ernst F J 1982 *Lect. Notes Phys.* **189** 128

[2] Kinnersley W 1977 *J. Math. Phys.* **18** 1529
Kinnersley W 1980 *J. Math. Phys.* **21** 2231

[3] Kinnersley W and Chitre D M 1977 *J. Math. Phys.* **18** 1538
Kinnersley W and Chitre D M 1978 *J. Math. Phys.* **19** 1926

[4] Tanabe Y 1978 *J. Math. Phys.* **19** 1808

[5] Hauser I and Ernst F J 1979 *Phys. Rev.* D **20** 1783
Hauser I and Ernst F J 1980 *J. Math. Phys.* **21** 1418

[6] Jones T C 1980 *J. Math. Phys.* **21** 1790

[7] Kaliappan P and Lakshmanan M 1981 *J. Math. Phys.* **22** 2447

[8] Gürses M and Xanthopoulos B C 1982 *Phys. Rev.* D **26** 1912

[9] Eris A, Gürses M and Karasu A 1984 *J. Math. Phys.* **25** 1489

[10] Gürses M 1984 *Phys. Rev.* D **30** 486
Gürses M 1984 *Lect. Notes Phys.* **205** 186

[11] Kramer D and Neugebauer G 1981 *J. Phys.* A **14** L333

[12] Kramer D 1982 *J. Phys.* A **15** 2201

[13] Neugebauer G and Kramer D 1983 *J. Phys.* A **16** 1927

[14] Cosgrove C M 1981 *J. Math. Phys.* **22** 2624

[15] Leaute B and Marcilhacy G 1985 *J. Math. Phys.* **26** 1938

[16] Hashimoto T and Sawae R 1994 *Commun. Math. Phys.* **160** 15

[17] Gao Y J 2004 *Chin. Phys.* **13** 602

[18] Hauser I and Ernst F J 1980 *J. Math. Phys.* **21** 1126

[19] Li W and Hou B Y 1989 *J. Math. Phys.* **30** 1198

[20] Goddard P and Olive D 1986 *J. Mod. Phys.* A **1** 303