Vol 15 No 1, January 2006
1009-1963/2006/15(01)/0060-06
Chinese Physics
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The effect of quantum noise on the restricted quantum game*

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(Received 15 July 2005; revised manuscript received 26 September 2005)

It has recently been established that quantum strategies have great advantage over classical ones in quantum games. However, quantum states are easily affected by the quantum noise resulting in decoherence. In this paper, we investigate the effect of quantum noise on the restricted quantum game in which one player is restricted in classical strategic space, another in quantum strategic space and only the quantum player is affected by the quantum noise. Our results show that in the maximally entangled state, no Nash equilibria exist in the range of $0 < p \leq 0.422$ (p is the quantum noise parameter), while two special Nash equilibria appear in the range of $0.422 < p < 1$. The advantage that the quantum player diminished only in the limit of maximum quantum noise. Increasing the amount of quantum noise leads to the increase of the classical player's payoff and the reduction of the quantum player's payoff, but is helpful in forming two Nash equilibria.

Keywords: entanglement, quantum game, quantum noise, Nash equilibria
PACC: 0365, 4250

1. Introduction

Game theory has been widely used in economics, biology and social sciences to model decision made in conflict situation. Recent development in quantum computation and quantum information theory[1−4] allows to extend the scope of game theory to quantum world. Actually, various problems in quantum computation and quantum information, such as quantum cryptography,[5] quantum cloning, quantum algorithms,[6] can be regarded as quantum game.[7] Quantum game model has been developed by Eisert *et al*[8] in which the paradox in the classical (two) Prisoner's Dilemma can be solved under the maximal entangled state. Then quantum game is also realized on a NMR quantum computer[9] and its properties in different entanglement degrees are presented in Ref.[10]. The investigations on multi-player quantum games,[7] multi-player and multi-choice quantum games,[11,12] continuous-variable quantum games[13] have been presented recently.

The restricted quantum game in which the players have different strategic space has also been studied.[8,14,15] In Ref.[8], Bob is restricted to applying only classical strategies characterized by $\phi_{\rm B} = 0$($\phi_{\rm B}$ is one of Bob's strategy parameter), while Alice can use quantum strategies. Eisert *et al*[8] find that Alice is advised to play $\hat{M} = \hat{U}(\pi/2, \pi/2)$, giving her reward at least r=3. In Ref.[14], Zhao *et al* showed that the restricted quantum game has no Nash equilibria in the maximally entangled state. However, they all obtain their result only under the condition of no quantum noise. As is known, in any realistic quantum information processing systems, the quantum noise from the environment cannot entirely eliminated. In Ref.[16], Chen *et al* discussed quantum noise in quantum game. Quantum noise was found to have no effect on the Nash equilibria in this model. In Ref.[17], Flitney and Abbott discussed the effect of quantum noise on other type of quantum game. However, in the restricted quantum game, quantum player is more easily affected by the quantum noise. In this paper, we consider the situation in which only the quantum player (Alice) is affected by the quantum noise, while the classical player (Bob) is not affected by the quantum noise. The paper is organized as follows. In Section 2 the model is set up for introducing noise into the re-

*Project supported by the National Natural Science Foundation of China (Grant No 10374025).
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stricted quantum game. In Section 3 the calculations and discussion are given. Conclusions are presented in Section 4.

2. Quantum game with quantum noise

It is most convenient to use the density matrix notation for the state of the system and the operator sum representation for the quantum operators. Phase damping is the ideal model of quantum noise that captures many of most important features of quantum noise occurring in quantum systems. Dephase can result in decoherence, which randomizes the relative phases of the quantum states, and dissipation that modifies the populations of the quantum states. Pure dephasing of a qubit can be expressed as

$$a|0\rangle + b|1\rangle \to a|0\rangle + b\ \mathrm{e}^{\phi}|1\rangle. \tag{1}$$

If we assume that the phase kick ϕ is a random variable with a Gaussian distribution of mean zero and variance 2λ, then the density matrix obtained after averaging over all values of ϕ is[1]

$$\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} \to \begin{pmatrix} |a|^2 & a\bar{b}\,\mathrm{e}^{-\lambda} \\ \bar{a}b\,\mathrm{e}^{-\lambda} & |b|^2 \end{pmatrix}. \tag{2}$$

Over time, the random phase kicks cause an exponential decay of the off-diagonal elements of the density matrix. We use the operator-sum representation to model decoherence:

$$\rho' \ = \ \sum_{j=0}^{2} \mathcal{E}_j\, \rho\, \mathcal{E}_j^{\dagger}, \tag{3}$$

where $\mathcal{E}_0 = \sqrt{p}|0\rangle\langle 0|$, $\mathcal{E}_1 = \sqrt{p}|1\rangle\langle 1|$ and $\mathcal{E}_2 = \sqrt{1-p}\,\hat{I}$, where p is the quantum noise parameter and ρ is the density matrix of the system.

Let us briefly recall the scheme of the quantum game presented in Ref.[8]. Classically, there are two players and each player can choose two possible strategies: co-operate ($\widehat{C}$) and defect ($\widehat{D}$). The payoff table for the players is shown in Table 1.

Table 1. Payoff matrix for the Prisoner's Dilemma. The first entry in parentheses denotes the payoff of Alice and the second that of Bob.

	Bob: $\hat{C}$	Bob: $\hat{D}$
Alice: $\hat{C}$	$(3,3)$	$(0,5)$
Alice: $\hat{D}$	$(5,0)$	$(1,1)$

A strategy $\hat{S}_{\mathrm{A}}^*$ is a dominant one of Alice's strategy, if it satisfies

$$\$_{\mathrm{A}}(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}^*) \geq \$_{\mathrm{A}}(\hat{S}_{\mathrm{A}}, \hat{S}_{\mathrm{B}}^*). \tag{4}$$

Similarly, we can define a dominant strategy $\hat{S}_{\mathrm{B}}^*$ for Bob. In terms of game theory, a strategy profile $(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}^*)$ can be called a Nash equilibrium if

$$\begin{aligned} \$_{\mathrm{A}}(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}^*) &\geq \$_{\mathrm{A}}(\hat{S}_{\mathrm{A}}, \hat{S}_{\mathrm{B}}^*), \\ \$_{\mathrm{B}}(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}^*) &\geq \$_{\mathrm{B}}(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}). \end{aligned} \tag{5}$$

A Nash equilibrium implies that no player can increase his payoff by unilaterally changing his strategy. A profile $(\hat{S}_{\mathrm{A}}^*, \hat{S}_{\mathrm{B}}^*)$ can be called a Pareto optimal, if it is not possible to increase one player's payoff without lessening the payoff of the other player. From the table, the dominate strategy for both players is to defect. Mutual defection is a Nash equilibrium. Since the aim of the player is to maximize his own payoff, Alice and Bob will both stick to choosing D. But unfortunately, this situation is worse than when they both choose C, while it happens to be a Pareto optimal. The dilemma comes from the fact that the Pareto optimal strategy is the both players to cooperate. In quantum version, each player possesses a qubit $|C\rangle\,(|D\rangle)$. The qubits firstly are prepared in the initial state $|C\rangle\,|C\rangle$ then the entangling gate $\hat{J}$ acts on it. The initial state of the quantum game is $|\psi_i\rangle = \widehat{J}\,|CC\rangle$, where $\hat{J}$ is $\exp(\mathrm{i}\gamma D \otimes D/2)$, and here $\gamma \in [0, \pi/2]$ measures the entanglement of the state $|\psi_i\rangle$. Then the unitary $\hat{U}_{\mathrm{A}}$ and $\hat{U}_{\mathrm{B}}$ are chosen from the 2-parameter set of unitary $2 \otimes 2$ matrices

$$\hat{U}\,(\theta, \phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi}\cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \mathrm{e}^{-\mathrm{i}\phi}\cos\theta/2 \end{pmatrix} \tag{6}$$

for all $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$ indeed, the strategies C and D can be associated with $U(0,0)$ and $U(\pi,0)$ respectively. Finally the disentangling gate $\widehat{J}^{+}$ is carried out and the system is measured in the computational basis, giving rise to one of the four outcomes $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, $|DD\rangle$. Alice's and Bob's payoffs are $\$_{\mathrm{A}} = 3P_{\mathrm{CC}} + 5P_{\mathrm{DC}} + P_{\mathrm{DD}}$ and $\$_{\mathrm{B}} = 3P_{\mathrm{CC}} + 5P_{\mathrm{CD}} + P_{\mathrm{DD}}$ respectively, where $P_{\sigma\sigma'} = \langle\sigma\sigma'|\,\rho\,|\sigma\sigma'\rangle$ is the probability of finding the state $|\sigma\sigma'\rangle$ (with $\sigma, \sigma' \in \{C, D\}$).

In Ref.[8], Alice is advised to play $\hat{M} = \hat{U}(\pi/2, \pi/2)$, giving her at least reward $r = 3$ payoff, since $\$_{\mathrm{A}}[\hat{M}, \hat{U}(\theta, 0)] \geq 3$ for any $\theta \in [0, \pi]$ leaving Bob with $\$_{\mathrm{B}}[\hat{M}, \hat{U}(\theta, 0)] \leq 1/2$. However, the quantum

player is more easily affected by the quantum noise. In order to consider this case, the scheme is set up as in Fig.1.

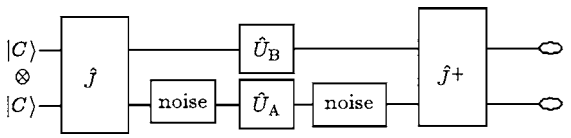


Fig.1. Quantum noise introduced before and after the adoption of the appropriate strategies by the quantum players, based on Eisert *et al*'s model.

3. Calculations and discussion

In this section, we present the major calculations, which can be described as follows: the initial state is specified by the density matrix $\rho_0 = |\psi_0\rangle\langle\psi_0|$,

$$\begin{aligned}
\rho_1 &= \hat{J}\rho_0\hat{J}^\dagger, && \text{(entanglement)}\\
\rho_2 &= D(\rho_1, p_1), && \text{(partial decoherence)}\\
\rho_3 &= (\hat{U}_\mathrm{A}\otimes\hat{U}_\mathrm{B})\,\rho_2\,(\hat{U}_\mathrm{A}\otimes\hat{U}_\mathrm{B})^\dagger, && \text{(players' moves)}\\
\rho_4 &= D(\rho_3, p_2), && \text{(partial decoherence)}\\
\rho_5 &= \hat{J}^\dagger\rho_4\hat{J}; && \text{(disentanglement)}
\end{aligned} \tag{7}$$

and the payoff to Alice is given by

$$\$_\mathrm{A} = 3P_\mathrm{CC} + 5P_\mathrm{DC} + P_\mathrm{DD}. \tag{8}$$

We consider the situation of maximally entangled case, $\gamma = \pi/2$. After choosing Eq.(3) to represent the quantum noise, we can obtain the Alice's payoff:

$$\begin{aligned}
<\$_\mathrm{A}> =& 2\Big[\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\\
&+ 2(1-p_1)\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_2}{2}\sin(\phi_1+\phi_2)\Big]\\
&+ (1-p_2)\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}(-2)\sin(\phi_1+\phi_2)\\
&+ (1-p_1)(1-p_2)\Big[\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}\cos(2\phi_1+2\phi_2) - \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\Big]\\
&+ \frac{5}{2}\Big[\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}\\
&- 2(1-p_1)\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_2}{2}\sin(\phi_1+\phi_2)\Big]\\
&+ \frac{5}{2}\Big[(1-p_2)\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_2}{2}\sin(\phi_1-\phi_2)(-2)\\
&- (1-p_1)(1-p_2)(\cos(2\phi_1)\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} - \cos(2\phi_2)\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2})\Big].
\end{aligned} \tag{9}$$

Setting $p_1 = p_2 = 0$ gives the quantum games of the Eisert *et al*'s model[8] studied in previous papers. From the payoffs of Eq.(9), when Bob's choice is C or D, the best Alice can do is to select $\phi_1 = \dfrac{\pi}{2}$, the result is also consistent with the results of Flitney and Abbott,[15] while Alice's choice of θ_1 depends on Bob's choice.

First, assuming Bob's strategy D, Alice's payoff is

$$\begin{aligned}
&\$_\mathrm{A}(\hat{U}_\mathrm{A}(\theta_1,\phi_1),\hat{D})\\
=&2 + \frac{1}{2}\cos^2\frac{\theta_1}{2} - (1-p_1)(1-p_2)\sin^2\frac{\theta_1}{2}\\
&- (1-p_1)(1-p_2)\frac{5}{2}\cos^2\frac{\theta_1}{2}\cos 2\phi_1.
\end{aligned} \tag{10}$$

Obviously, $\$_\mathrm{A}(\hat{U}_\mathrm{A}(\theta_1,\phi_1),\hat{D})$ will reach a maximum at $\theta_1 = 0, \phi_1 = \pi/2$(that is $\hat{Q}$) with the quantum noise parameter p changing from 0 to 1, which can be seen from Fig.2. We then have

$$\begin{aligned}
&\$_\mathrm{A}(\hat{U}_\mathrm{A}(\theta_1,\phi_1),\hat{D}) \le \$_\mathrm{A}(\hat{U}_\mathrm{A}(o,\pi/2),\hat{D})\\
=&2 + 1/2 + 5/2(1-p_1)(1-p_2).
\end{aligned} \tag{11}$$

Figure 2 shows Alice's payoff as a function of the strategies $\hat{U}_\mathrm{A}$ and $\hat{U}_\mathrm{B}$ with increasing values of the quantum noise parameter p. The figures show that as p increases (more noise), the quantum player's (Alice) payoff becomes small, eventually reduces to the fixed value of 5/2 as p approaches unity.

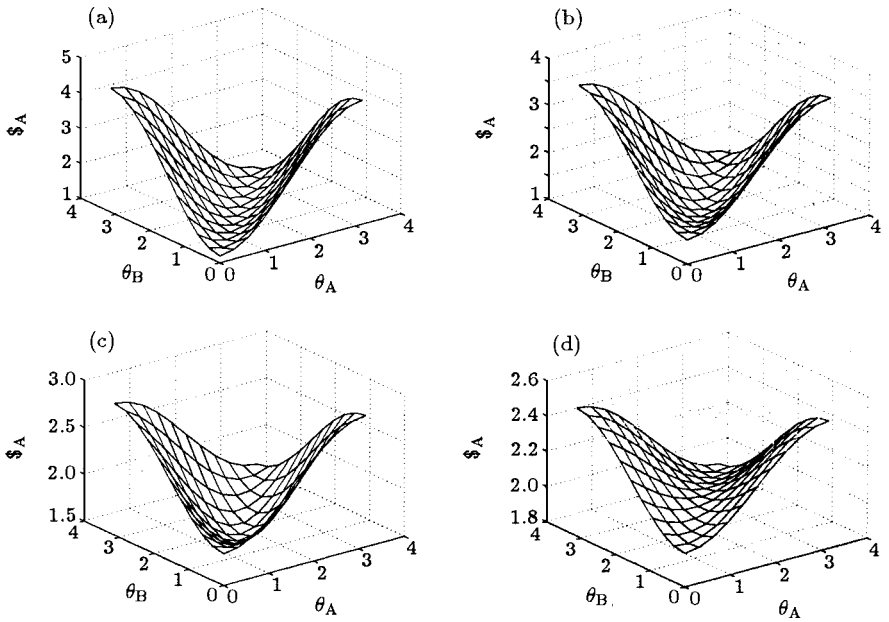


Fig.2. Alice's payoff as a function of the strategies $\theta_A(\theta_A \in [0,\pi])$ and $\theta_B(\theta_B \in [0,\pi])$ with increasing values of the quantum noise parameter p of a dephasing channel. (a) p =0.1, (b) p=0.3, (c) p=0.6 and (d) p=0.9.

When Alice sets $\theta_1 = 0, \phi_1 = \pi/2$, Bob's payoff is:

$$\begin{aligned}&\$_B(\hat{U}_A(0,\pi/2),\hat{U}_B(\theta_2,0))\\ =&2-(1-p_1)(1-p_2)\\ &+[1/2-3/2(1-p_1)(1-p_2)]\sin^2\frac{\theta_2}{2}. \qquad (12)\end{aligned}$$

After a simple calculation, for $0 < p \leq 0.422$, Bob's best strategy is to take $\hat{U_B}(0,0)$, that is C. But in the range of $0.422 < p < 1$, Bob's best choice is still $\hat{D}(\theta_2 = \pi)$:

$$\begin{aligned}&\$_B(\hat{U}_A(0,\pi/2),\hat{U_B}(\theta_2,0))\\ \leq&\$_B(\hat{U}_A(0,\pi/2),\hat{U_B}(\pi,0))\\ =&5/2-5/2(1-p_1)(1-p_2). \qquad (13)\end{aligned}$$

Figure 3 shows Bob's payoff as a function of the strategies $\hat{U}_A$ and $\hat{U}_B$ with increasing values of the quantum noise parameter p. The figures show that as p increases(more noise), the classical player's(Bob) payoff becomes large, eventually increases to the fixed value of $5/2$ as p approaches unity. So as defined in Eq.(5), from Eqs.(11) and (13), for the pair of the strategies $\hat{Q}\otimes\hat{D}$, Alice and Bob can both obtain their best payoffs. This is the special Nash equilibria for $0.422 < p < 1$.

Then, assuming Bob's strategy C, Alice's payoff is

$$\begin{aligned}&\$_A(\hat{U}_A(\theta_1,\phi_1),\hat{C})\\ =&2+\frac{1}{2}\sin^2\frac{\theta_1}{2}+5/2(1-p_1)(1-p_2)\sin^2\frac{\theta_1}{2}\\ &+(1-p_1)(1-p_2)\cos^2\frac{\theta_1}{2}\cos 2\phi_1. \qquad (14)\end{aligned}$$

So $\$_A(\hat{U}_A(\theta_1,\phi_1),\hat{C})$ will reach maximum at $\theta_1 = \pi, \phi_1 = \pi/2$ with the quantum noise parameter p changing from 0 to 1. Alice's best payoff is

$$\begin{aligned}&\$_A(\hat{U}_A(\theta_1,\phi_1),\hat{C})\\ \leq&\$_A(\hat{U}_A(\pi,\pi/2),\hat{C})\\ =&2+1/2+5/2(1-p_1)(1-p_2), \qquad (15)\end{aligned}$$

just as that shown in Fig.2. When Alice sets $\theta_1 = \pi, \phi_1 = \pi/2$, Bob's payoff is:

$$\begin{aligned}&\$_B(\hat{U}_A(\pi,\pi/2),\hat{U}_B(\theta_2,0)\\ =&2+\frac{1}{2}\cos^2\frac{\theta_2}{2}-(1-p_1)(1-p_2)\\ &-\frac{3}{2}(1-p_1)(1-p_2)\cos^2\frac{\theta_2}{2}. \qquad (16)\end{aligned}$$

After a simple calculation, for the range of $0 < p \leq 0.422$, Bob's best strategy is D, but what is interesting is that for $0.422 < p < 1$, Bob's best strategy is still $\hat{U}_B(0,0)$, that is C, which will give him the best payoff:

$$\begin{aligned}&\$_B(\hat{U}_A(\pi,\pi/2),\hat{U}_B(\theta_2,0))\\ \leq&\$_B(\hat{U}_A(\pi,\pi/2),\hat{U}_B(0,0))\\ =&\frac{5}{2}-\frac{5}{2}(1-p_1)(1-p_2), \qquad (17)\end{aligned}$$

just as shown in Fig.3. According to Eq.(5), from Eqs.(15) and (17), in the pair of the strategies $\hat{U}_{\rm A}(\pi,\pi/2)\otimes\hat{C}$, Alice and Bob will both obtain their best payoffs. No one is willing to change his(her) strategy. This is another special Nash equilibrium for $0.422 < p < 1$.

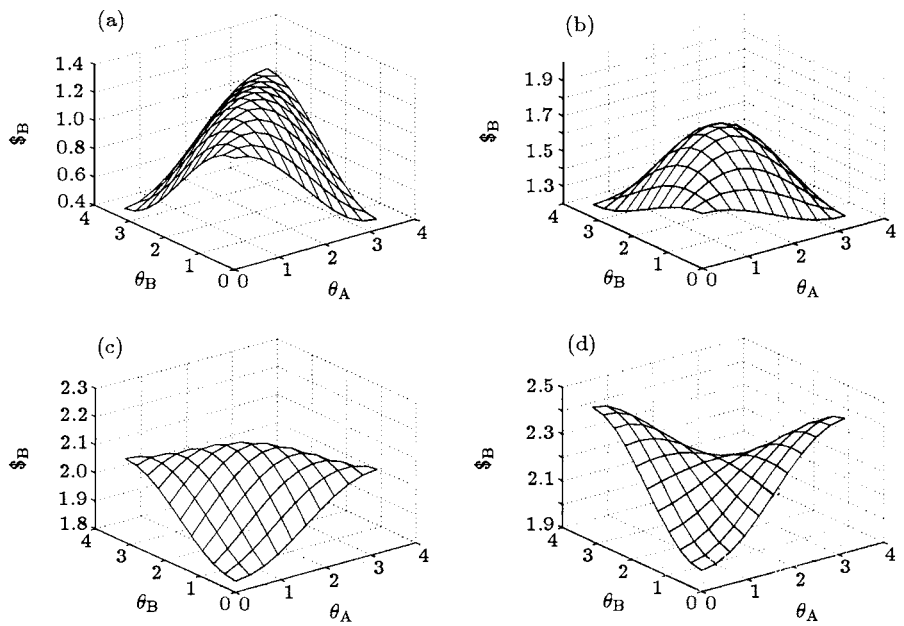


Fig.3. Bob 's payoff as a function of the strategies $\theta_{\rm A}(\theta_{\rm A}\in[0,\pi])$ and $\theta_{\rm B}(\theta_{\rm B}\in[0,\pi])$ with increasing values of the quantum noise parameter p of a dephasing channel (a) p =0.1, (b) p=0.3, (c) p=0.6 and (d) p=0.9.

Comparing Fig.2 with Fig.3, we find that in the whole process, the quantum player (Alice) does not lose her advantage, because she can obtain more payoff than classical player (Bob). However, in the limit of maximum quantum noise (p=1), quantum player's advantage is diminished and the game becomes a fair game, because the two players will have the same payoff. With increasing quantum noise, the payoffs of both players approach the average of the payoffs 5/2. But with the change of the quantum noise parameter p, in the range of $0.422 < p < 1$, there exist two special Nash equilibria $\hat{Q}\otimes\hat{D}$ and $\hat{U}_{\rm A}(\pi,\pi/2)\otimes\hat{C}$. As shown in Figs.2 and 3, at two special points Alice and Bob together have their best payoffs. In contrast to Ref.[14], which showed that the restricted quantum game had no Nash equilibria in the maximally entangled state, our results show that after introducing quantum noise in this quantum game, the quantum game has two special Nash equilibria, so we can see that the quantum noise is helpful in forming two Nash equilibria.

4. Conclusion

In this paper, we have discussed the effects of quantum noise in the phase damping channel on quantum game in which one player(Bob) is restricted in classical space, while another player(Alice) in quantum strategic space. Only the quantum player(Alice) is affected by the quantum noise. We find that with increasing quantum noise parameter, the game exhibits an intriguing structure. For $0 < p \leq 0.422$, there is no Nash equilibrium, however in the range of $0.422 < p < 1$, two special Nash equilibria appear. The advantage of a quantum player is diminished only in the limit of maximum quantum noise. Increasing the amount of quantum noise leads to the increase of the classical player's payoff and the reduction of the quantum player's payoff, and this is helpful in forming two Nash equilibria.

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