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Performance of an irreversible quantum refrigeration cycle*

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A new model of a quantum refrigeration cycle composed of two adiabatic and two isomagnetic field processes is established. The working substance in the cycle consists of many non-interacting spin-1/2 systems. The performance of the cycle is investigated, based on the quantum master equation and semi-group approach. The general expressions of several important performance parameters, such as the coefficient of performance, cooling rate, and power input, are given. It is found that the coefficient of performance of this cycle is in the closest analogy to that of the classical Carnot cycle. Furthermore, at high temperatures the optimal relations of the cooling rate and the maximum cooling rate are analysed in detail. Some performance characteristic curves of the cycle are plotted, such as the cooling rate versus the maximum ratio between high and low "temperatures" of the working substances, the maximum cooling rate versus the ratio between high and low "magnetic fields" and the "temperature" ratio between high and low reservoirs. The obtained results are further generalized and discussed, so that they may be directly applied to describing the performance of the quantum refrigerator using spin-J systems as the working substance. Finally, the optimum characteristics of the quantum Carnot and Ericsson refrigeration cycles are derived by analogy.

Keywords: spin systems, quantum refrigeration cycle, performance parameters
PACC: 0365, 0570L, 0720

1. Introduction

In recent years, the optimal analysis of the performance characteristics of thermodynamic cycles has been extended to the regime of quantum cycles. The performances of quantum Carnot, Ericsson and Stirling cycles have been intensively studied.[1−5] Furthermore, finite-time thermodynamic analyses of some quantum cycles have also been made.[6−10] Quite a number of novel conclusions have been obtained. However, the investigations have rarely dealt with the performance of the quantum Brayton cycle.[11] In fact, the Brayton cycle is one of very importance cycles in engineering thermodynamics. The investigation related to the Brayton cycle has continuously attracted much attention.[12−14] It has some distinctive merits which are noteworthy in theory and practice.

In classical thermodynamic cycles there are the Stirling cycle, Ericsson cycle, Brayton cycle, etc, besides the Carnot cycle. The performance of the Carnot cycle is independent of the property of the working substance, while the performances of other cycles are, in general, related to the property of the working substance.[15,16] The quantum-mechanical cycle differs from the classical cycle in two respects. One is the working substances, which may be the spin systems, harmonic oscillator systems, ideal quantum gases, and micro-particle systems confined in a potential well;[17−19] the other is the relaxation dynamics, which is modelled by the semigroup approach. The advantage of quantum cycles is that the phenomenological heat transfer law can be avoided.

In the present paper, a new model of quantum Brayton refrigeration cycle using the spin-1/2 systems as the working substance is established which is composed of two adiabatic and two isomagnetic field processes. Firstly, The thermodynamic property of a spin-1/2 system is given, based on the quantum master equation and semi-group approach. The performance characteristics of the quantum Brayton refrigeration cycle are analysed. Secondly, the impor-

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tant performance parameters such as the coefficient of performance, cooling rate, and power input are optimized. Especially, at high temperatures the optimal relation of the cooling rate and the maximum cooling rate are derived in detail. Finally, these results obtained here are generalized for the spin-J systems. The optimum performances of the quantum Ericsson or Carnot refrigeration cycle are derived similarly.

2. Quantum refrigeration cycle

Let us consider a quantum spin-1/2 system with a magnetic moment $\boldsymbol{M}$ in a magnetic field $\boldsymbol{B}$. The magnitude of the magnetic field can change with time, but is not allowed to reach zero. The Hamiltonian of the interaction between the magnetic moment $\boldsymbol{M}$ in the quantum system and the magnetic field $\boldsymbol{B}$ is given by[20]

$$\hat{H}(t) = -\hat{\boldsymbol{M}} \cdot \boldsymbol{B} = 2\mu_{\mathrm{B}} \hat{\boldsymbol{S}} \cdot \boldsymbol{B} = 2\mu_{\mathrm{B}} B_z(t)\, \hat{S}_z, \quad (1)$$

where μ_{B} is the Bohr magnetron, $\boldsymbol{S}$ is the spin angular momentum, $\hbar = h/(2\pi)$, and h is the Planck constant. Throughout this paper we adopt $\hbar$=1 and define $\omega(t) = 2\mu_{\mathrm{B}} B_z(t)$ for simplicity. ω is positive since the spin angular momentum and magnetic moment are opposite in direction. One can refer to ω rather than B_z as "the field". Thus, the Hamiltonian of an isolated single spin-1/2 system in the presence of the field $\omega(t)$ may be expressed as

$$\hat{H}(t) = \omega(t)\, \hat{S}_z. \quad (2)$$

The internal energy of the spin-1/2 system is the expectation value of the Hamiltonian, i.e.

$$E = \langle \hat{H} \rangle = \omega(t) \langle \hat{S}_z \rangle = \omega S. \quad (3)$$

Based on the statistical mechanics, the expectation value of the spin angular momentum S_z is expressed by the following relation:[21]

$$S = \langle \hat{S}_z \rangle = -\frac{1}{2} \tanh(\beta\omega/2), \quad (4)$$

where $-1/2 < S < 0$.

In quantum refrigeration cycles, the spin-1/2 system is not only coupled mechanically to a given "magnetic field" $\omega(t)$, but also coupled thermally to a heat reservoir at a temperature T. Based on the semi-group formalism,[22] the equation of motion of an operator in the Heisenberg picture is given by the quantum master equation, i.e.

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \mathrm{i}[\hat{H}, \hat{\boldsymbol{X}}] + \frac{\partial \hat{\boldsymbol{X}}}{\partial t} + L_{\mathrm{D}}(\hat{\boldsymbol{X}}), \quad (5)$$

where

$$L_{\mathrm{D}}(\hat{\boldsymbol{X}}) = \sum_{\alpha} \gamma_{\alpha} (\hat{\boldsymbol{V}}_{\alpha}^{+} [\hat{\boldsymbol{X}}, \hat{\boldsymbol{V}}_{\alpha}] + [\hat{\boldsymbol{V}}_{\alpha}^{+}, \hat{\boldsymbol{X}}] \hat{\boldsymbol{V}}_{\alpha}) \quad (6)$$

is a dissipation term and originates from a thermal coupling of the spin to a heat reservoir, $\hat{\boldsymbol{V}}_{\alpha}$ and $\hat{\boldsymbol{V}}_{\alpha}^{+}$ are operators in the Hilbert space of the system and are Hermitian conjugates, and γ_{α} are phenomenological positive coefficients. Substituting $\hat{\boldsymbol{X}}$ in Eq.(5) by $\hat{H}$ and using Eq.(3), one can obtain the rate of change of the internal energy as follows:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} < \hat{H} > = \left\langle \frac{\partial \hat{H}}{\partial t} \right\rangle + < L_{\mathrm{D}}(\hat{H}) > \\ &= \frac{\mathrm{d}\omega}{\mathrm{d}t} S + \omega \frac{\mathrm{d}S}{\mathrm{d}t}. \end{aligned} \quad (7)$$

Comparing Eq.(7) with the time derivative of the first law of thermodynamics[23]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t} + \frac{\mathrm{d}Q}{\mathrm{d}t}, \quad (8)$$

one can easily find that the instantaneous power is

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \left\langle \frac{\partial \hat{H}}{\partial t} \right\rangle = \frac{\mathrm{d}\omega}{\mathrm{d}t} S, \quad (9)$$

and the instantaneous heat flow is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = < L_{\mathrm{D}}(\hat{H}) > = \omega \frac{\mathrm{d}S}{\mathrm{d}t}. \quad (10)$$

It is thus clear that for a spin-1/2 system, Eq.(7) gives the time derivative of the first law of thermodynamics.

Figure 1 shows a schematic diagram of a quantum refrigeration cycle, which is composed of two adiabatic and two isomagnetic field processes. This cycle has microscopic analog to the magnetic Brayton refrigeration cycle,[24] where the working substance consists of magnetic salts. For the convenience of writing, "temperature" will refer to β rather than T, where $\beta = 1/T$ and T is the absolute temperature in energy units. In two adiabatic processes, no heat exchange is involved. Increasing ω corresponds to the working substance doing work on the surroundings. The reverse process i.e. adiabatically decreasing ω, corresponds to the surroundings doing work on the working substance. In the isomagnetic field processes, the working substance is coupled to the hot reservoir at a constant "temperature" β_{h} and the cold reservoir at a constant "temperature" β_{c}, respectively. Q_{h} and Q_{c} represent the amounts of heat exchange between the working substance and the heat reservoirs during the two isomagnetic field processes, respectively. The "temperatures"

of the working substance are different from those of the heat reservoirs. They are, respectively, given by β_1, β_2, β_3, and β_4, and they are related to each other as follows: $\beta_4 > \beta_1 \geq \beta_c > \beta_h \geq \beta_3 > \beta_2$. ω_1 and ω_2 represent the high and low "magnetic fields", respectively.

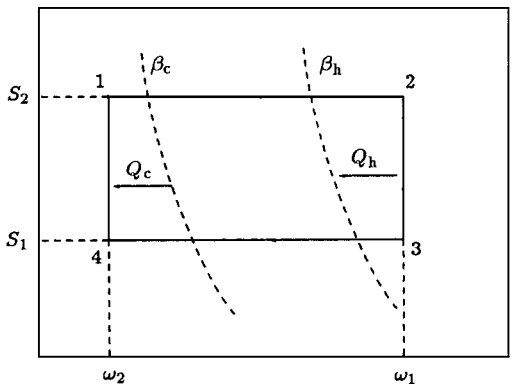


Fig.1. The $S-\omega$ diagram of a spin-1/2 Brayton refrigeration cycle, where ω is in units of Joule.

3. Performance characteristics

Using Eqs.(4) and (10), we can calculate the amounts of heat exchange in the two isomagnetic field processes as follows:

$$Q_h = \int_{S_2}^{S_1} \omega \, dS = \omega_1 \left[-\frac{1}{2}\tanh(\beta_3\omega_1/2) + \frac{1}{2}\tanh(\beta_2\omega_1/2) \right], \quad (11)$$

and

$$Q_c = \int_{S_1}^{S_2} \omega \, dS = \omega_2 \left[-\frac{1}{2}\tanh(\beta_1\omega_2/2) + \frac{1}{2}\tanh(\beta_4\omega_2/2) \right], \quad (12)$$

where S_1 and S_2 are the mean values of the spin angular momentum in the two adiabatic processes respectively and $S_1 < S_2$. During the adiabatic process S remains constant, so there are the following equations,

$$S_1 = -\frac{1}{2}\tanh(\beta_3\omega_1/2) = -\frac{1}{2}\tanh(\beta_4\omega_2/2), \quad (13)$$

and

$$S_2 = -\frac{1}{2}\tanh(\beta_1\omega_2/2) = -\frac{1}{2}\tanh(\beta_2\omega_1/2). \quad (14)$$

Using Eqs.(11) and (12), we obtain the work input per cycle and the coefficient of performance as

$$W = |Q_h + Q_c| = \omega_1 \left[\frac{1}{2}\tanh(\beta_3\omega_1/2) - \frac{1}{2}\tanh(\beta_2\omega_1/2) \right] + \omega_2 \left[\frac{1}{2}\tanh(\beta_1\omega_2/2) - \frac{1}{2}\tanh(\beta_4\omega_2/2) \right], \quad (15)$$

and

$$\varepsilon = \frac{Q_c}{W} = \frac{\omega_2 \left[\frac{1}{2}\tanh(\beta_4\omega_2/2) - \frac{1}{2}\tanh(\beta_1\omega_2/2) \right]}{\omega_1 \left[\frac{1}{2}\tanh(\beta_3\omega_1/2) - \frac{1}{2}\tanh(\beta_2\omega_1/2) \right] + \omega_2 \left[\frac{1}{2}\tanh(\beta_1\omega_2/2) - \frac{1}{2}\tanh(\beta_4\omega_2/2) \right]} = \frac{\omega_2}{\omega_1 - \omega_2}. \quad (16)$$

It is found that the coefficient of performance only depends on high and low "magnetic fields". It is well known that the coefficient of performance of the Carnot refrigeration cycle is $\varepsilon_c = T_c/(T_h - T_c)$. Comparing with these two results, we can find the analogy between the classical thermodynamic result and the result of Carnot cycle as long as the temperature T is replaced by the "magnetic fields" ω.

4. Time evolution of the spin angular momentum and cycle period

In order to calculate the durations of the heat-exchange processes, one must solve the equation of motion that determines the time evolution of the spin angular momentum. For a spin system, $\hat{\boldsymbol{V}}_\alpha$ are chosen to be the spin creation and annihilation operators: $\hat{S}_+ = \hat{S}_x + \mathrm{i}\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - \mathrm{i}\hat{S}_y$, and $\hat{H} = \omega\hat{S}_z$. Substituting $\hat{S}_+$, $\hat{S}_-$, $\hat{H}$, and $\hat{\boldsymbol{X}} = \hat{S}_z$ into Eq.(5), one can prove[23] that

$$\frac{dS}{dt} = -a\exp(q\beta\omega)\left\{2\left[1+\exp(\beta\omega)\right]S + \left[\exp(\beta\omega) - 1\right]\right\}, \quad (17)$$

where $a > 0$ and $-1 < q < 0$. Solving Eq.(17), one can obtain the expression of time evolution as follows:

$$t=-\frac{1}{a}\int_{S_{\rm i}}^{S_{\rm f}}\frac{{\rm d}S}{\exp\left(q\beta\omega\right)\left\{2\left[1+\exp\left(\beta\omega\right)\right]S+\exp\left(\beta\omega\right)-1\right\}},\tag{18}$$

where $S_{\rm i}$ and $S_{\rm f}$ are the initial and final values of S along a given path $S\left(\beta,\omega\right)$ respectively. Equation (18) is a general expression of time evolution for a spin-1/2 system coupling with the heat reservoir and the external magnetic field.

In the two isomagnetic field processes, the "temperature" of the working substance changes from β_2 to β_3 or from β_4 to β_1, so they need a nonnegligible time. Substituting $S\left(\beta\right)=-\frac{1}{2}\tanh\left(\beta\omega_1/2\right)$, $\beta=\beta_{\rm h}$, $S_{\rm i}=S_{\rm i}(\beta_2,\omega_1)$, and $S_{\rm f}=S_{\rm f}(\beta_3,\omega_1)$ into Eq.(18), one can obtain the duration of one isomagnetic field ω_1 process as follows:

$$t_1=\frac{\omega_1}{2a}\int_{\beta_2}^{\beta_3}\left\{\exp\left(q\beta_{\rm h}\omega_1\right)\left[\exp\left(\beta_{\rm h}\omega_1\right)-\exp\left(\beta\omega_1\right)\right]\times\left[1+\exp\left(-\beta\omega_1\right)\right]\right\}^{-1}{\rm d}\beta,\tag{19}$$

where $\beta_{\rm h}>\beta$ because heat is transferred from the working substance to the hot reservoir in the isomagnetic field (ω_1) process. Similarly, substituting $S\left(\beta\right)=-\frac{1}{2}\tanh\left(\beta\omega_2/2\right)$, $\beta=\beta_{\rm c}$, $S_{\rm i}=S_{\rm i}(\beta_4,\omega_2)$, and $S_{\rm f}=S_{\rm f}(\beta_1,\omega_2)$ into Eq.(18), one can obtain the duration of another isomagnetic field (ω_2) process as follows:

$$t_2=\frac{\omega_2}{2a}\int_{\beta_4}^{\beta_1}\left\{\exp\left(q\beta_{\rm c}\omega_2\right)\left[\exp\left(\beta_{\rm c}\omega_2\right)-\exp\left(\beta\omega_2\right)\right]\times\left[1+\exp\left(-\beta\omega_2\right)\right]\right\}^{-1}{\rm d}\beta,\tag{20}$$

where $\beta_{\rm c}<\beta$ because heat is transferred from the cold reservoir to the working substance in the isomagnetic field (ω_2) process.

In the two adiabatic processes, since S is a constant of the motion, irrespective of the time dependence of ω, the time spent along the adiabatic process is negligible. Consequently, the cycle period is given by

$$t=t_1+t_2.\tag{21}$$

5. Optimization of performance parameters

The coefficient of performance, cooling rate, and power input are three of the important performance parameters; they are often taken into account in the optimal design and theoretical analysis of refrigerators. Using Eqs.(12), (15), and (21), one can find that cooling rate and power input may be, respectively, expressed as

$$R=\frac{Q_{\rm c}}{t}=\omega_2\left[-\frac{1}{2}\tanh\left(\beta_1\omega_2/2\right)+\frac{1}{2}\tanh\left(\beta_4\omega_2/2\right)\right]/(t_1+t_2),\tag{22}$$

and

$$P=\frac{W}{t}=\left\{\omega_1\left[\frac{1}{2}\tanh\left(\beta_3\omega_1/2\right)-\frac{1}{2}\tanh\left(\beta_2\omega_1/2\right)\right]+\omega_2\left[\frac{1}{2}\tanh\left(\beta_1\omega_2/2\right)-\frac{1}{2}\tanh\left(\beta_4\omega_2/2\right)\right]\right\}/(t_1+t_2).\tag{23}$$

Using Eqs.(22) and (23), one can, in principle, optimize these important performance parameters of the quantum refrigeration cycle.

At very low temperatures, $\beta\omega\to\infty$, so that $\tanh(\beta_1\omega_2/2)\to1$ and $\tanh(\beta_4\omega_2/2)\to1$. In such a case, the amount of refrigeration per cycle $Q_{\rm c}=0$ and the refrigerator has lost its function.

At high temperatures, i.e. $\beta\omega<<1$, the results obtained above may be simplified. For example, Eqs.(11)–(16), (19), (20), (22) and (23) may be, respectively, simplified as

$$Q_{\rm h}=\omega_1^2\left(\beta_2-\beta_3\right)/4,\tag{24}$$

$$Q_{\rm c}=\omega_2^2\left(\beta_4-\beta_1\right)/4,\tag{25}$$

$$\beta_1/\beta_2 = \omega_1/\omega_2 = r, \tag{26}$$

$$\beta_4/\beta_3 = \omega_1/\omega_2 = r, \tag{27}$$

$$W = \left[\omega_1^2(\beta_3 - \beta_2) + \omega_2^2(\beta_1 - \beta_4)\right]/4, \tag{28}$$

$$\varepsilon = \frac{\beta_2}{\beta_1 - \beta_2} = \frac{\beta_3}{\beta_4 - \beta_3} = \frac{1}{r-1}, \tag{29}$$

$$t_1 = \frac{1}{4a}\int_{\beta_2}^{\beta_3}\frac{\mathrm{d}\beta}{\beta_\mathrm{h} - \beta} = \frac{1}{4a}\ln\frac{\beta_\mathrm{h} - \beta_2}{\beta_\mathrm{h} - \beta_3}, \tag{30}$$

$$t_2 = \frac{1}{4a}\int_{\beta_4}^{\beta_1}\frac{\mathrm{d}\beta}{\beta_\mathrm{c} - \beta} = \frac{1}{4a}\ln\frac{\beta_\mathrm{c} - \beta_4}{\beta_\mathrm{c} - \beta_1}, \tag{31}$$

$$R = \frac{a\omega_2^2(\beta_4 - \beta_1)}{\ln\dfrac{\beta_\mathrm{h} - \beta_2}{\beta_\mathrm{h} - \beta_3} + \ln\dfrac{\beta_\mathrm{c} - \beta_4}{\beta_\mathrm{c} - \beta_1}}, \tag{32}$$

and

$$P = \frac{a\left[\omega_1^2(\beta_3 - \beta_2) + \omega_2^2(\beta_1 - \beta_4)\right]}{\ln\dfrac{\beta_\mathrm{h} - \beta_2}{\beta_\mathrm{h} - \beta_3} + \ln\dfrac{\beta_\mathrm{c} - \beta_4}{\beta_\mathrm{c} - \beta_1}}, \tag{33}$$

where $r = \omega_1/\omega_2$ is the ratio between high and low "magnetic fields". It is found that the coefficient of performance only depends on r or the temperature ratio between the original and final states in one adiabatic process. The results obtained here may be compared with those of macroscopic Brayton refrigeration cycle where the working substance consists of magnetic salts.[24] It is well known that the coefficient of performance of an ideal gas Brayton refrigeration cycle is $1/(r_\mathrm{P}^{1-1/\gamma} - 1)$, where $r_P = P_2/P_1$ is the pressure ratio and γ is the specific heat ratio.[25] It is thus clear that the performance of a Brayton refrigeration cycle depends closely on the properties of the working substance, which is different from that of the Carnot refrigeration cycle.

If the following assumptions are further made: $\beta_3 - \beta_2 << \beta_\mathrm{h} - \beta_2$, $\beta_4 - \beta_1 << \beta_4 - \beta_\mathrm{c}$, $\beta_\mathrm{h} \geq 2\beta_2 - \beta_3$, $\beta_\mathrm{c} \leq 2\beta_4 - \beta_1$, the durations of the two isomagnetic field processes may be obtained and their sum is expressed as

$$t = t_1 + t_2 = \frac{\beta_3 - \beta_2}{4a(\beta_\mathrm{h} - \beta_2)} + \frac{\beta_4 - \beta_1}{4a(\beta_4 - \beta_\mathrm{c})}. \tag{34}$$

Substituting Eqs.(26), (27) and (34) into Eqs.(32) and (33) gives

$$R = a\omega_2^2\left[1/r(\beta_\mathrm{h} - \beta_2) + 1/(y\beta_2 - \beta_\mathrm{c})\right]^{-1}, \tag{35}$$

and

$$P = a\omega_2^2(r-1)\left[1/r(\beta_\mathrm{h} - \beta_2) + 1/(y\beta_2 - \beta_\mathrm{c})\right]^{-1}, \tag{36}$$

where $y = \beta_4/\beta_2$ is the maximum ratio between high and low "temperatures" of the working substances in the Brayton cycle. Using Eq.(35) and the extremal condition $\partial R/\partial\beta_2 = 0$, we can obtain an optimal relation

$$\beta_2 = \frac{\sqrt{ry}\beta_\mathrm{h} + \beta_\mathrm{c}}{\sqrt{ry} + y}. \tag{37}$$

Substituting Eq.(37) into Eqs.(35) and (36), we find that the fundamental optimal relations between performance parameters of the cycle and y are, respectively, given by

$$R = \frac{a\omega_2^2\, r\ (y\beta_\mathrm{h} - \beta_\mathrm{c})}{\left(\sqrt{y} + \sqrt{r}\right)^2}, \tag{38}$$

and

$$P = \frac{a\omega_2^2\, r(r-1)\,(y\beta_\mathrm{h} - \beta_\mathrm{c})}{\left(\sqrt{y} + \sqrt{r}\right)^2}. \tag{39}$$

Using Eq.(38), we can plot the $R^* - y$ characteristic curves, as shown in Figs.2–5, where $R^* = R/\left(a\omega_2^2\beta_\mathrm{c}\right)$ at a given ω_2 and β_c, or $R^* = R/\left(a\omega_1^2\beta_\mathrm{h}\right)$ at a given ω_1 and β_h, is the dimensionless cooling rate, $\tau = \beta_\mathrm{c}/\beta_\mathrm{h}$ is the "temperature" ratio between high and low reservoirs. It is seen from Figs.2–3 that the cooling rate monotonically increases as the maximum ratio between high and low "temperatures" of the working

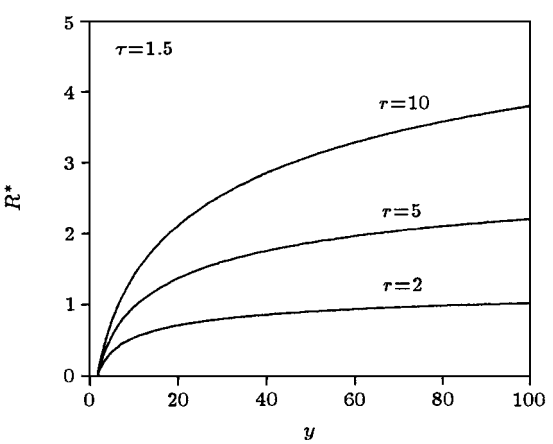


Fig.2. The dimensionless cooling rate $R^* = R/\left(a\omega_2^2\beta_\mathrm{c}\right)$ versus the maximum ratio between high and low "temperatures" of the working substances y at a given "temperature" ratio between high and low reservoirs.

substances increases. At $y = \tau$, we have $R = 0$ and $P = 0$. The optimal region of the maximum ratio between high and low "temperatures" of the working substances should be

$$\beta_\mathrm{c}/\beta_\mathrm{h} < y < r^2\beta_\mathrm{h}/\beta_\mathrm{c}, \tag{40}$$

and

$$r > \tau = \beta_\mathrm{c}/\beta_\mathrm{h}. \tag{41}$$

Using Eq.(38), we can obtain

$$\frac{\mathrm{d}R}{\mathrm{d}y}=\frac{a\omega_2^2 r\left(\sqrt{r}\beta_{\mathrm{h}}+\beta_{\mathrm{c}}/\sqrt{y}\right)}{\left(\sqrt{r}+\sqrt{y}\right)^3}>0. \tag{42}$$

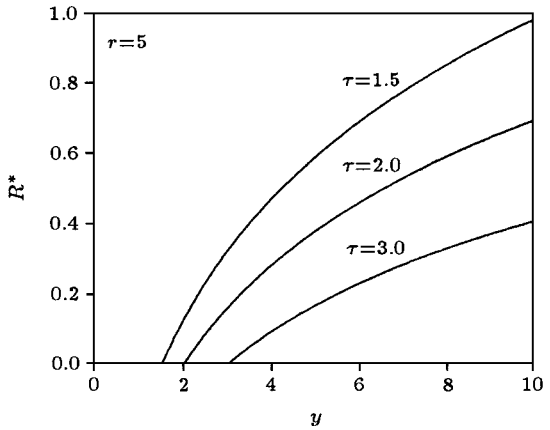


Fig.3. The dimensionless cooling rate $R^*=R/\left(a\omega_2^2\beta_{\mathrm{c}}\right)$ versus the maximum ratio between high and low "temperatures" of the working substances y at a given ratio between high and low "magnetic fields".

It is shown that the cooling rate is a monotonically increasing function of y. Substituting $y_{\max}=r^2\beta_{\mathrm{h}}/\beta_{\mathrm{c}}$ into Eq.(38), we can derive the maximum cooling rate,

$$R_{\max}=\frac{a\omega_2^2\left(r^2\beta_{\mathrm{h}}^2-\beta_{\mathrm{c}}^2\right)}{\left(\sqrt{r\beta_{\mathrm{h}}}+\sqrt{\beta_{\mathrm{c}}}\right)^2}. \tag{43}$$

Using Eq.(38), we can plot the $R_{\max}^*-r$ and $R_{\max}^*-\tau$ characteristic curves, as shown in Figs.4 and 5, where $R_{\max}^*=R_{\max}/\left(a\omega_2^2\beta_L\right)$ is the dimensionless maximum cooling rate. It is shown that at a given τ the maximum cooling rate increases as the ratio between high and low "magnetic fields", r, increases, while at a given r the maximum cooling rate decreases as the "temperature" ratio between high and low reservoirs, τ, increases. Therefore, in order to make the maximum cooling rate as large as possible, we may increase r, or decrease τ.

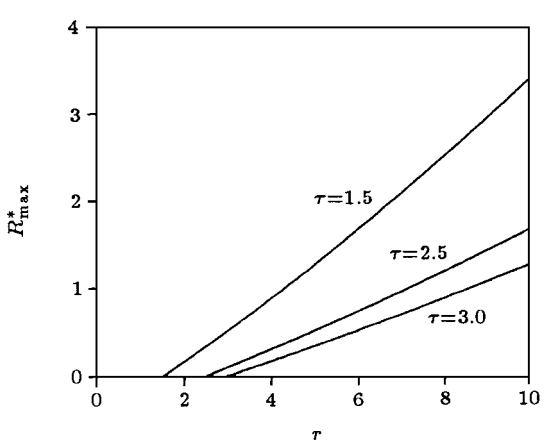


Fig.4. The dimensionless maximum cooling rate $R_{\max}^*=R_{\max}/\left(a\omega_2^2\beta_{\mathrm{c}}\right)$ versus the ratio between high and low "magnetic fields" r.

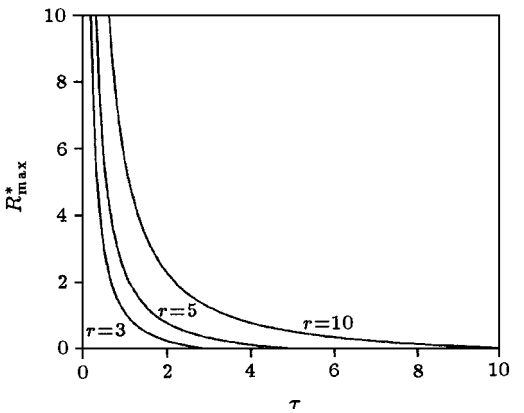


Fig.5. The dimensionless maximum cooling rate $R_{\max}^*=R_{\max}/\left(a\omega_2^2\beta_{\mathrm{c}}\right)$ versus the "temperature" ratio between high and low reservoirs τ.

6. Discussion and generalization

1) When the working substance is composed of a spin-J system (J=1/2, 1, 3/2, 2, ...), the mean value of the spin angular momentum is given by[21,23]

$$S=<\hat{S}_z>=-JB_J(\beta\omega J), \tag{44}$$

where $-J\leq S\leq J$ and

$$B_J(x)=\left(\frac{2J+1}{2J}\right)\coth\left(\frac{2J+1}{2J}x\right)-\frac{1}{2J}\coth\left(\frac{x}{2J}\right)$$

is the Brillouin function. At high temperatures, Eq.(44) may be simplified as

$$S=-\frac{J(J+1)}{3}\beta\omega. \tag{45}$$

Compared with Eq.(4), the heat quantities of the two isomagnetic field processes in the cycle may be obtained by multiplying Eqs.(24) and (25) with a factor of $4J(J+1)/3$. On the other hand, using the method in Section 4, one may prove that the time evolution of the spin angular momentum is determined by

$$\frac{\mathrm{d}S}{\mathrm{d}t}=-2a\left\{2S+\beta\omega\left[J(J+1)-M\right]\right\}, \tag{46}$$

where $M=\dfrac{J(J+1)}{3}$. From Eqs.(45) and (46), we can find that the durations of the two isomagnetic field processes are the same as those expressed by Eqs.(30) and (31). Thus, the coefficient of performance of the quantum cycle consisting of the spin-J systems is the

same as that of the quantum cycle consisting of the spin-1/2 systems, while the cooling rate and power input are $4J(J+1)/3$ times as large as those of the quantum cycle consisting of the spin-1/2 systems, respectively.

2) When $\beta_3 - \beta_2 >> \beta_1 - \beta_2$, the two adiabatic processes in the cycle may be replaced by two isothermal processes, the cycle is close to a quantum Ericsson refrigeration cycle.[9] In this case, $\beta_1 \approx \beta_2$, and $\beta_3 \approx \beta_4$. When $\beta_3 - \beta_2 << \beta_1 - \beta_2$, the two isomagnetic field processes in the cycle may be replaced by the two isothermal processes, the cycle is close to a quantum Carnot refrigeration cycle.[9] In this case, $\beta_2 \approx \beta_3$ and $\beta_1 \approx \beta_4$.

3) The above discussion only refers to a single spin-J system. For the working substance consisting of many non-interacting spin-J systems, the coefficient of performance still holds true, while the internal energy, work input, power input, and heat quantity can be obtained as long as the above results are simply multiplied by the total number of spin systems.

7. Conclusion

We have established a model of a typical quantum Brayton refrigeration cycle consisting of two isoadiabatic and two isomagnetic field processes and using non-interacting spin-1/2 systems as the working substance. Based on the spin theory, motion equation of an operator, and semi-group formalism, we have analysed the optimal performance characteristics of the quantum refrigeration cycles and derived the concrete expressions of several important parameters such as the coefficient of performance, cooling rate, and power input. Especially, the optimal performance of the quantum refrigerator in the high temperature limit is discussed in detail. The maximum cooling rate is calculated.

The obtained results are further generalized, so that they are also suitable for the working substance consisting of non-interacting spin-J systems. Finally, the two adiabatic processes in the cycle may be replaced by the two isothermal processes, and the cycle is close to a quantum Ericsson refrigeration cycle. On the other hand, the replacement of the two isomagnetic field processes by the two adiabatic processes gives directly the cycle model of a quantum Carnot refrigerator, so that the optimal performance of the quantum Carnot or Ericsson refrigeration cycle can be derived by analogy from the present paper.

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