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Two classes of fractal structures for the (2+1)-dimensional dispersive long wave equation*

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Using the mapping approach via a Riccati equation, a series of variable separation excitations with three arbitrary functions for the (2+1)-dimensional dispersive long wave (DLW) equation are derived. In addition to the usual localized coherent soliton excitations like dromions, rings, peakons and compactions, etc, some new types of excitations that possess fractal behaviour are obtained by introducing appropriate lower-dimensional localized patterns and Jacobian elliptic functions.

Keywords: mapping approach, DLW equation, explicit solution, fractal
PACC: 0365, 0340, 0545

1. Introduction

Fractals and solitons are two important instances of nonlinearity.[1−4] They are widely applied in many natural sciences, particularly in fluid dynamics, plasma physics, field theory, optics and condensed matter physics, etc.[5−7] Conventionally, the two problem are treated independently since one often considers that solitons are the basic excitations of an integrable model while fractals are the elementary behaviours of a nonintegrable system. In other words, one does not take into account the possibility of the existence of fractals in a soliton system. However, the above consideration may not be complete, especially for some higher-dimensional systems. In the recent study of soliton systems, scientists[8−19] have found that some characteristic lower-dimensional arbitrary functions exist in exact excitations of some two-dimensional integrable models. This means that any lower-dimensional fractal solutions can be used to construct exact solutions of a higher-dimensional integrable model, which also implies that any exotic behaviour may propagate along this characteristics. In this paper, we consider this phenomenon in the following celebrated (2+1)-dimensional dispersive long wave (DLW) system:

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0,$$
$$v_t + u_x + (uv)_x + u_{xxy} = 0. \quad (1)$$

The system was first derived by Boiti *et al*[20] as a compatibility condition for a "weak" Lax pair. In Ref.[21], Paquin and Winternitz showed that the symmetry algebra of Eq.(1) is infinite-dimensional and has a Kac–Moody–Virasoro structure. Some special similarity solutions were also given in Ref.[21] by using symmetry algebra and the classical theoretical analysis. The more general symmetry algebra, W_∞ symmetry algebra, was given in Ref.[22]. In Ref.[23], Lou gave out nine types of two dimensional similarity reduction and thirteen types of ordinary differential equation reduction. In Ref.[24], Lou showed that Eq.(1) has no Painlevé property, though the system is Lax or IST integrable. Abundant propagating localized excitations were also derived by Lou *et al*[18,19] with the help of Painlevé–Bäcklund transformation and a multilinear variable separation approach. However, to the best of our knowledge, its excitations obtained here that possess fractal behaviour by introducing appropriate lower-dimensional localized patterns

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and Jacobian elliptic functions has not been reported in previous literature.

As is known, to search for the solitary wave solutions to a nonlinear physical model, we can make different approaches. One of the most efficient methods of finding soliton excitations for a physical model is the so-called mapping transformation method.[25,26] Along with the mapping deformation idea and based on the general reduction theory,[27−29] we introduce the mapping approach via a Riccati equation. The basic idea of the algorithm is: For a given nonlinear partial differential equation (NPDE) with independent variables $X(x_0 = t, x_1, x_2, \ldots, x_m)$, and dependent variable u,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where P is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. We assume that its solution is in the form of nonstandard truncated Painlevé expansion, namely

$$u = \sum_{i=-n}^{n} \alpha_i(X)\phi^i(q(X)) \tag{3}$$

with

$$\phi' = \sigma + \phi^2, \tag{4}$$

where $\alpha_i(X)(i = -n, \ldots, -1, 0, 1, \ldots, n)$, $q(X)$ are arbitrary functions to be determined, σ is a constant and the prime denotes differentiation with respect to q. To determine u explicitly, one can take the following procedure: First, similar to the usual mapping approach, we can determine n by balancing the highest-order nonlinear term with the highest-order partial derivative term in Eq.(2). Second, substituting Eq.(3) and Eq.(4) into the given NPDE, collecting coefficients of polynomial of ϕ and eliminating each of them, we can derive a set of partial differential equations of $\alpha_i(X)(i = -n, \ldots, -1, 0, 1, \ldots, n)$ and $q(X)$. Third, to obtain $\alpha_i(X)$ and $q(X)$, we solve these partial differential equations. Finally, as Eq.(4) has the general solution

$$\phi = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}q), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}q), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}q), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}q), & \sigma > 0, \\ \dfrac{-1}{\omega}, & \sigma = 0, \end{cases} \tag{5}$$

substituting $\alpha_i(X)$, $q(X)$ and the solutions (5) into Eq.(3), we can derive the exact solutions of the given NPDE.

2. New explicit solutions for the (2+1)-dimensional DLW equation

Now, we make the above mapping approach to Eq.(1). According to the balancing procedure, Eq.(3) becomes

$$u = f + g\phi(q) + h\phi^{-1}(q),$$

$$v = F + G\phi(q) + H\phi^2(q) + M\phi^{-1}(q) + N\phi^{-2}(q), \tag{6}$$

where f, g, h, F, G, H, M, N and q are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting Eq.(6) and Eq.(4) into Eq.(1), collecting coefficients of polynomial of ϕ and setting each of the coefficients to zero, we have

$$2Hq_x + g^2 q_y = 0,$$

$$\begin{aligned} &2gg_x q_y + 2gq_y q_t + g^2 q_{xy} + 2fgq_x q_y \\ &+ 4H_x q_x + 2Hq_{xx} + 2gg_y q_x + 2Gq_x^2 = 0, \end{aligned}$$

$$\begin{aligned} &8Hq_x^2\sigma + g_t q_y + g_y q_t + gq_{yt} + gf_x q_y + 4g^2 q_x q_y \sigma \\ &+ f_y g q_x + g_x g_y + 2G_x q_x + fg_x q_y + fg_y q_x \\ &+ H_{xx} + gg_{xy} + fgq_{xy} + Gq_{xx} = 0, \end{aligned}$$

$$\begin{aligned} &2gq_y q_t\sigma + 2fgq_x q_y\sigma + 2gg_x q_y\sigma + G_{xx} + fg_{xy} \\ &+ 2gg_y q_x\sigma + g^2 q_{xy}\sigma + g_{yt} + 2Gq_x^2\sigma + f_x g_y + g_x f_y \\ &+ 2Hq_{xx}\sigma + 4H_x q_x\sigma + gf_{xy} = 0, \end{aligned}$$

$$\begin{aligned} &F_{xx} + gh_{xy} + g_{xy}h + g_t q_y\sigma + 2Nq_x^2 - f_y h q_x \\ &- 2M_x q_x - fh_y q_x + fg_x q_y\sigma + h^2 q_x q_y + 2G_x q_x\sigma \\ &- hf_x q_y + g_y q_t\sigma - fh_x q_y + fg_y q_x\sigma + 2Hq_x^2\sigma^2 \\ &+ fgq_{xy}\sigma - Mq_{xx} + f_{yt} + gf_x q_y\sigma + f_y g q_x\sigma \\ &+ g_y h_x - fhq_{xy} + f_x f_y + ff_{xy} - hq_{yt} + h_y g_x \\ &- h_t q_y + g^2 q_x q_y\sigma^2 + gq_{yt}\sigma + Gq_{xx}\sigma - h_y q_t = 0, \end{aligned}$$

$$\begin{aligned} &2Mq_x^2\sigma - 2hh_y q_x - 2hh_x q_y + 2fhq_x q_y\sigma + h_{yt} \\ &- 2Nq_{xx} - h^2 q_{xy} + f_y h_x - 4N_x q_x + 2hq_y q_t\sigma \\ &+ fh_{xy} + M_{xx} + hf_{xy} + f_x h_y = 0, \end{aligned}$$

$$\begin{aligned} &- Mq_{xx}\sigma + 8Nq_x^2\sigma + hh_{xy} - fh_y q_x\sigma \\ &- h_y q_t\sigma - hq_{yt}\sigma + 4h^2 q_x q_y\sigma \\ &- fh_x q_y\sigma - hf_x q_y\sigma + N_{xx} - fhq_{xy}\sigma + h_x h_y \\ &- f_y h q_x\sigma - h_t q_y\sigma - 2M_x q_x\sigma = 0, \end{aligned}$$

$$2fhq_xq_y\sigma - h^2q_{xy} + 2Mq_x^2\sigma - 2hh_yq_x + 2hq_yq_t\sigma - 2hh_xq_y - 4N_xq_x - 2Nq_{xx} = 0,$$

$$2Nq_x + h^2q_y = 0,$$

$$2gq_xq_y + gH = 0,$$

$$g_xH + 2g_yq_x^2 + 2gq_{xx}q_y + 4gq_xq_{xy} + 2gGq_x + 2Hq_t + gH_x + 2fHq_x + 4g_xq_xq_y = 0,$$

$$gq_x + Fgq_x + gq_{xxy} + g_yq_{xx} + hHq_x + 8gq_x^2q_y\sigma + gG_x + 3gHq_x\sigma + fGq_x + Gq_t + g_xG + f_xH + g_{xx}q_y + H_t + 2g_xq_{xy} + 2g_{xy}q_x + fH_x = 0,$$

$$G_t + Fg_x + h_xH + fG_x + g_{xxy} + 2gGq_x\sigma + gF_x + 4gq_xq_{xy}\sigma + 2g_yq_x^2\sigma + 2Hq_t\sigma + 2gq_yq_{xx}\sigma + f_xG + 4g_xq_xq_y\sigma + g_x + hH_x + 2fHq_x\sigma = 0,$$

$$- hq_x + 2gq_yq_x^2\sigma^2 - fMq_x - gNq_x - hFq_x - 2hq_x^2q_y\sigma + hHq_x\sigma + g_{xx}q_y\sigma + gFq_x\sigma + fGq_x\sigma - 2h_xq_{xy} - h_{xx}q_y - h_yq_{xx} + 2g_{xy}q_x\sigma + f_{xxy} - hq_{xxy} - 2h_{xy}q_x + gq_{xxy}\sigma + f_x - Mq_t + f_xF + fF_x + F_t + g_xM + h_xG + 2g_xq_{xy}\sigma + g_yq_{xx}\sigma + gM_x + hG_x + Gq_t\sigma + gq_x\sigma = 0,$$

$$- 2hMq_x + 4hq_xq_{xy}\sigma + h_{xxy} + 2h_yq_x^2\sigma + M_t - 2fNq_x + f_xM + h_xF + fM_x + g_xN + h_x - 2Nq_t + 4h_xq_xq_y\sigma + gN_x + hF_x + 2hq_{xx}q_y\sigma = 0,$$

$$- 2h_{xy}q_x\sigma - h_{xx}q_y\sigma + h_xM - hq_{xxy}\sigma - h_yq_{xx}\sigma - Mq_t\sigma - hq_x\sigma + N_t + fN_x - gNq_x\sigma - 2h_xq_{xy}\sigma - 8hq_x^2q_y\sigma^2 + hM_x - fMq_x\sigma - 3hNq_x + f_xN - hFq_x\sigma = 0,$$

$$4h_xq_xq_y\sigma^2 + 4hq_xq_{xy}\sigma^2 + 2hq_{xx}q_y\sigma^2 - 2hMq_x\sigma + hN_x + h_xN - 2fNq_x\sigma - 2Nq_t\sigma + 2h_yq_x^2\sigma^2 = 0,$$

$$2hq_xq_y\sigma^2 + hN = 0.$$

It is very difficult to obtain the general solution of above algebraic equations. Fortunately, in the case of $q = \chi(x) + \varphi(y) + \tau(t)$, where $\chi \equiv \chi(x), \varphi \equiv \varphi(y), \tau \equiv \tau(t)$ are three arbitrary variable separation functions of x, y and t, respectively, we can derive two couples of meaningful solutions, i.e.

(A)

$$u_1 = \frac{\chi_{xx} - \tau_t}{\chi_x} + 2\chi_x\phi(\chi + \varphi + \tau),$$

$$v_1 = -1 - 2\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\phi^2(\chi + \varphi + \tau); \quad (7)$$

(B)

$$u_2 = \frac{\chi_{xx} + \tau_t}{\chi_x} + 2\chi_x\phi(\chi + \varphi + \tau) - \frac{2\chi_x\sigma}{\phi(\chi + \psi + \tau)},$$

$$v_2 = -1 - 4\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\phi^2(\chi + \varphi + \tau) - \frac{2\chi_x\varphi_y\sigma^2}{\phi^2(\chi + \varphi + \tau)}. \quad (8)$$

Based on the solutions of Eq.(4), one can obtain the explicit solutions of Eq.(1) corresponding to (A) and (B) respectively:

Case A (1) For $\sigma < 0$, we can derive the following solitary wave solutions of Eq.(1):

$$u_{11} = \frac{\chi_{xx} - \tau_t}{\chi_x} - 2\chi_x\sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi+\tau)), \quad (9)$$

$$v_{11} = -1 - 2\chi_x\varphi_y\sigma + 2\chi_x\varphi_y\sigma\tanh^2(\sqrt{-\sigma}(\chi+\varphi+\tau)); \quad (10)$$

$$u_{12} = \frac{\chi_{xx} - \tau_t}{\chi_x} - 2\chi_x\sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi+\tau)), \quad (11)$$

$$v_{12} = -1 - 2\chi_x\varphi_y\sigma + 2\chi_x\varphi_y\sigma\coth^2(\sqrt{-\sigma}(\chi+\varphi+\tau)) \quad (12)$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

(2) For $\sigma > 0$, we can obtain the following periodic wave solutions of Eq.(1):

$$u_{13} = \frac{\chi_{xx} - \tau_t}{\chi_x} + 2\chi_x\sqrt{\sigma}\tan(\sqrt{\sigma}(\chi+\varphi+\tau)), \quad (13)$$

$$v_{13} = -1 - 2\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\sigma\tan^2(\sqrt{\sigma}(\chi+\varphi+\tau)); \quad (14)$$

$$u_{14} = \frac{\chi_{xx} - \tau_t}{\chi_x} - 2\chi_x\sqrt{\sigma}\cot(\sqrt{\sigma}(\chi+\varphi+\tau)), \quad (15)$$

$$v_{14} = -1 - 2\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\sigma\cot^2(\sqrt{\sigma}(\chi+\varphi+\tau)) \quad (16)$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

(3) For $\sigma = 0$, we can derive the following variable separation solution of Eq.(1):

$$u_{15} = \frac{\chi_{xx} - \tau_t}{\chi_x} - \frac{2\chi_x}{\chi + \varphi + \tau}, \quad (17)$$

$$v_{15} = -1 - \frac{2\chi_x\varphi_y}{(\chi + \varphi + \tau)^2}; \quad (18)$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

Case B (1) For $\sigma < 0$, we can derive the following solitary wave solutions of Eq.(1):

$$\begin{aligned} u_{21} =& -\frac{\chi_{xx}+\tau_t}{\chi_x} \\ & - 2\chi_x\sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi+\tau) \\ & - \frac{2\chi_x\sqrt{-\sigma}}{\tanh(\sqrt{-\sigma}(\chi+\varphi+\tau))}, \end{aligned} \tag{19}$$

$$\begin{aligned} v_{21} =& -1-4\chi_x\varphi_y\sigma \\ & + 2\chi_x\varphi_y\sigma\tanh^2(\sqrt{-\sigma}(\chi+\varphi+\tau)) \\ & + \frac{2\chi_x\varphi_y\sigma}{\tanh^2(\sqrt{-\sigma}(\chi+\varphi+\tau))}; \end{aligned} \tag{20}$$

$$\begin{aligned} u_{22} =& -\frac{\chi_{xx}+\tau_t}{\chi_x} \\ & - 2\chi_x\sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi+\tau) \\ & - \frac{2\chi_x\sqrt{-\sigma}}{\coth(\sqrt{-\sigma}(\chi+\varphi+\tau))}, \end{aligned} \tag{21}$$

$$\begin{aligned} v_{22} =& -1-4\chi_x\varphi_y\sigma \\ & + 2\chi_x\varphi_y\sigma\coth^2(\sqrt{-\sigma}(\chi+\varphi+\tau)) \\ & + \frac{2\chi_x\varphi_y\sigma}{\coth^2(\sqrt{-\sigma}(\chi+\varphi+\tau))} \end{aligned} \tag{22}$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

(2) For $\sigma > 0$, we can obtain the following periodic wave solutions of Eq.(1):

$$\begin{aligned} u_{23} =& -\frac{\chi_{xx}+\tau_t}{\chi_x} - 2\chi_x\sqrt{\sigma}\tan(\sqrt{\sigma}(\chi+\varphi+\tau)) \\ & + \frac{2\chi_x\sqrt{\sigma}}{\tan(\sqrt{\sigma}(\chi+\varphi+\tau))}, \end{aligned} \tag{23}$$

$$\begin{aligned} v_{23} =& -1-4\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\sigma\tan^2(\sqrt{\sigma}(\chi+\varphi+\tau)) \\ & - \frac{2\chi_x\varphi_y\sigma}{\tan^2(\sqrt{\sigma}(\chi+\varphi+\tau))}; \end{aligned} \tag{24}$$

$$\begin{aligned} u_{24} =& -\frac{\chi_{xx}+\tau_t}{\chi_x} + 2\chi_x\sqrt{\sigma}\cot(\sqrt{\sigma}(\chi+\varphi+\tau)) \\ & - \frac{2\chi_x\sqrt{\sigma}}{\cot(\sqrt{\sigma}(\chi+\varphi+\tau))}, \end{aligned} \tag{25}$$

$$\begin{aligned} v_{24} =& -1-4\chi_x\varphi_y\sigma - 2\chi_x\varphi_y\sigma\cot^2(\sqrt{\sigma}(\chi+\varphi+\tau)) \\ & - \frac{2\chi_x\varphi_y\sigma}{\cot^2(\sqrt{\sigma}(\chi+\varphi+\tau))}; \end{aligned} \tag{26}$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

(3) For $\sigma = 0$, we can derive the following variable separation solution of Eq.(1):

$$u_{25} = -\frac{\chi_{xx}+\tau_t}{\chi_x} + \frac{2\chi_x}{\chi+\varphi+\tau}, \tag{27}$$

$$v_{25} = -1 - \frac{2\chi_x\varphi_y}{(\chi+\varphi+\tau)^2} \tag{28}$$

with three arbitrary functions $\chi(x), \varphi(y)$ and $\tau(t)$.

3. Two classes of fractal structures for the (2+1)-dimensional DLW equation

Due to the arbitrariness of functions $\chi(x), \varphi(y)$ and $\tau(t)$ included in the above cases, the physical quantity u and v may possess varied structures. For example, when $\chi(x) = kx, \varphi(y) = ly$ and $\tau(t) = ct$, all solutions of the above cases become simple travelling wave excitations or rational singularity excitations. Moreover, based on the derived solutions, we may obtain a variety of stationary localized solutions, which are neither travelling wave excitations nor propagating waves, just as Wu *et al* reported about the non-propagating solitons in 1984.[30] For instance, when the arbitrary functions are chosen appropriately, we may derive many kinds of non-propagating localized solutions like dromion solutions, ring solutions, peakon and compacton solutions and so on.[31−33]

Now an important problem is, whether we can find some new types of soliton/periodic structures that possess fractal excitations for the $(2+1)$-dimensional DLW system, i.e. the fractal excitations of the DLW system. The answer is apparently affirmative since $\chi(x), \varphi(y)$ and $\tau(t)$ are arbitrary functions. Here, we cite some interesting possible fractal patterns. For simplification, we merely discuss some special fractal excitations of the solution v_{11} in case A, namely

$$\begin{aligned} V \equiv v_{11} =& -1-2\chi_x\varphi_y\sigma \\ & + 2\chi_x\varphi_y\sigma\tanh^2(\sqrt{-\sigma}(\chi+\varphi+\tau)). \end{aligned} \tag{29}$$

3.1. Regular fractal dromion and lump

3.1.1. Regular fractal dromion

In a $(2+1)$-dimensional system, one of the most important basic excitations is the so-called dromion solution which is exponentially localized in all directions. Recently, it has been found that many lower-dimensional piecewise smooth functions with fractal structure can be used to construct exact localized solutions for some higher-dimensional soliton systems which also possess fractal structure. This situation

also occurs in the DLW system. If we appropriately select the arbitrary functions χ, φ and τ, some special types of fractal dromions for the physical quantity V can be found. For example, if we take

$$\chi(x) = 1 + \mathrm{e}^{|x|(1+\sin(\ln(x^2)))}, \tag{30}$$

$$\varphi(y) = 1 + \mathrm{e}^{|y|(1+\sin(\ln(y^2)))}, \tag{31}$$

$$\tau = \sin\left(\frac{\pi}{2} + t\right), \tag{32}$$

then a simple fractal dromion is obtained. Figure 1(a) shows a plot of this special type of fractal dromion structure for the physical quantity V given by Eq.(29) with conditions (30)–(32). Figure 1(b) shows the fractal structure of dromion in the region $\{x = [-0.02, 0.02], y = [-0.02, 0.02]\}$. To observe the self-similar structure of the fractal dromion clearly, we enlarge a small region near the centre of Fig.1(b). When zooming in the region of Fig.1(b) to $\{x = [-0.00085, 0.00085],\ y = [-0.00085, 0.00085]\}$ and $\{x = [-0.0000016, 0.0000016],\ y = [-0.0000016, 0.0000016]\}$, completely similar structures to that presented in Fig.1(b) are found (see Fig.1(c) and Fig.1(d)).

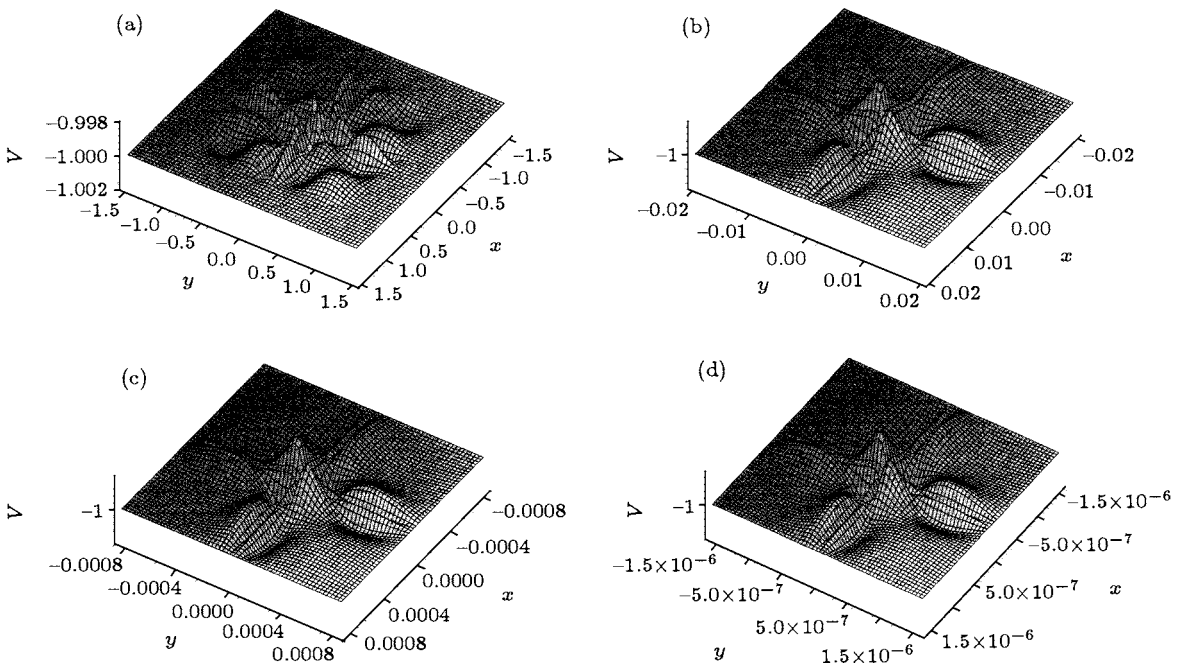


Fig.1. (a) Fractal dromion structure for the physical quantity V with conditions (30)–(32) and $t = 0$. (b)–(d) Fractal structures of dromion in the region $\{x = [-0.02, 0.02], y = [-0.02, 0.02]\}$, $\{x = [-0.00085, 0.00085], y = [-0.00085, 0.00085]\}$ and $\{x = [-0.0000016, 0.0000016],\ y = [-0.0000016, 0.0000016]\}$, respectively.

3.1.2. Regular fractal lump

It is well known that lump solution has been formed by rational functions. This localized coherent soliton structure is another type of significant localized excitation. If we choose the functions χ, φ and τ of the physical quantity V in Eq.(29) appropriately, we can find some types of lump solutions with fractal behaviour. Figure 2(a) shows a fractal lump structure for the physical quantity V given by Eq.(29), where χ, φ and τ in Eq.(29) are chosen as

$$\chi(x) = 1 + \frac{|x|\cos^2(\ln(x^2))}{1 + x^4}, \tag{33}$$

$$\varphi(y) = 1 + \frac{|y|\cos^2(\ln(y^2))}{1 + y^4}, \tag{34}$$

$$\tau = \cos\left(\frac{t}{2}\right). \tag{35}$$

From Fig.2(a), we can see that the solution is localized in all directions. Near the centre, there are infinitely many peaks which are distributed in a fractal manner. Figure 2(b) presents a density plot of the structure of the fractal lump in the region $\{x = [-0.046, 0.046],\ y = [-0.046, 0.046]\}$. More detailed study shows the self-similar structure of this lump. We have reduced the region of Fig.2(b) to $\{x = [-0.00042, 0.00042],\ y = [-0.00042, 0.00042]\}$ and so on, and a totally similar structure to that plotted in Fig.2(b) has been found.

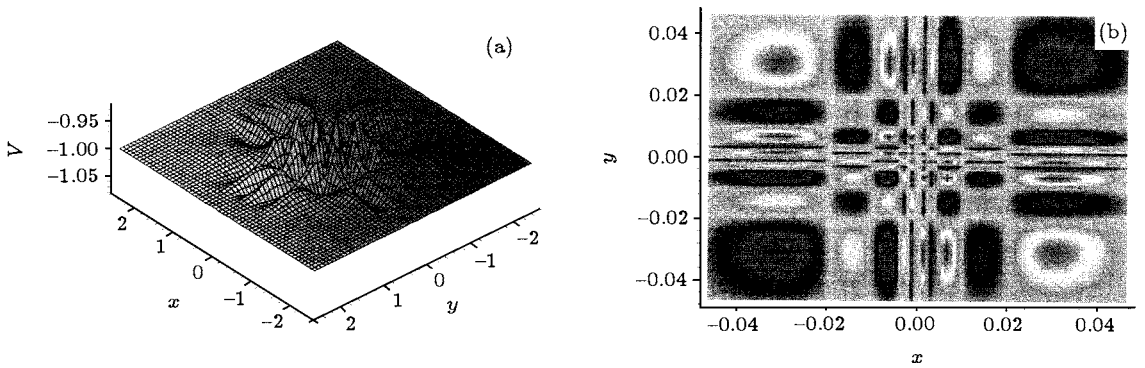


Fig.2. (a) Fractal lump structure for the physical quantity V with conditions (33) – (35) and $t = 0$. (b) Density of the fractal structure of dromion in the region $\{x = [-0.046, 0.046],\ y = [-0.046, 0.046]\}$.

3.2. Jacobian periodic wave structures and their fractal behaviour

In reality, there abounds an intricate variety of periodic waves. How to describe and construct them appropriately is an interesting topic. Recently, a kind of Jacobian elliptic function approach, which is usually regarded as straightforward and effective, was presented and successfully applied to construct certain periodic waves for many nonlinear models,[34−37] because some suitable elliptic functions are useful and helpful to describe the complicated periodic wave phenomenon in physics and other natural sciences.

Now, we try to construct some Jacobian periodic wave structures and their fractal behaviour for the (2 + 1)-dimensional DLW system based on the above variable separation result. Since the functions $\chi(x), \varphi(y)$ and $\tau(t)$ of the physical quantity V in Eq.(29) are arbitrary functions of the indicated arguments, we may choose $\chi(x), \varphi(y)$ and $\tau(t)$ to be some Jacobian elliptic periodic functions. For instance, when taking

$$\chi(x) = \sum_{j=1}^{n_1} P_j(x),$$

$$\varphi(y) = \sum_{j=1}^{n_2} Q_j(y), \tau(t) = \sum_{j=1}^{n_3} R_j(t), \qquad (36)$$

where $P_j(x), Q_j(y)$ and $R_j(t)$ are Jacobian elliptic periodic functions such as sn, cn, dn, etc, and n_1, n_2, n_3 are positive integers, we can construct many kinds of Jacobian periodic wave structures. Two special circumstances are depicted in Fig.3 and Fig.4 respectively. The related functions $\chi(x), \varphi(y)$ and $\tau(t)$ are directly presented in the figure captions.

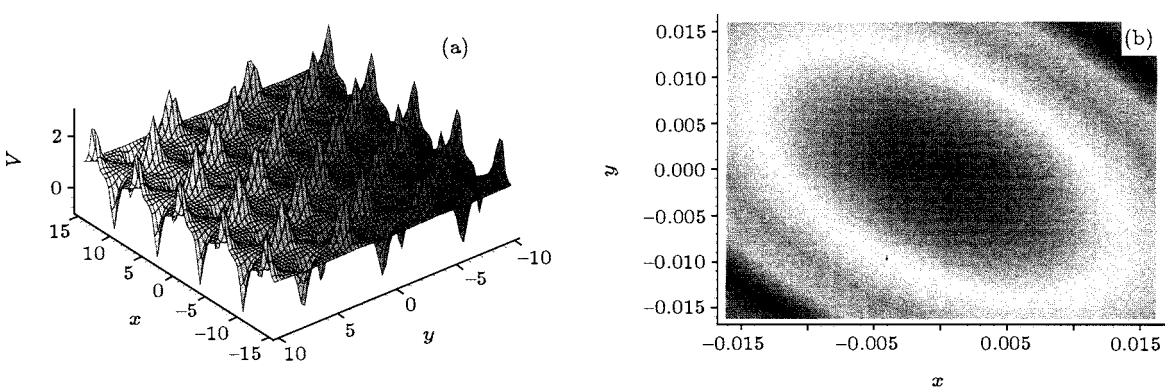


Fig.3. (a) Jacobian periodic wave structure for the physical quantity V give by Eq.(29) with $\chi(x) = \mathrm{sn}(x), \varphi(y) = \mathrm{sn}(y)$ and $\tau(t) = \mathrm{sn}(t)$ on the condition of $t = 0$ and the modulus $m = 0.5$. (b) Density of the fractal structure of Fig.3(a) in the region $\{x = [-0.016, 0.016],\ y = [-0.016, 0.016]\}$.

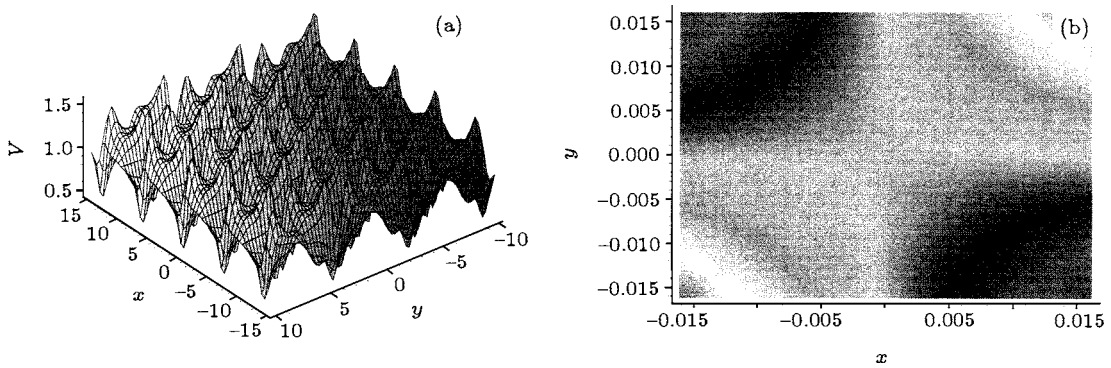


Fig.4. (a) Jacobian periodic wave structure for the physical quantity V give by Eq.(29) with $\chi(x) = \mathrm{cn}(x), \varphi(y) = \mathrm{cn}(y)$ and $\tau(t) = \mathrm{cn}(t) - 1$ on the condition of $t = 0$ and the modulus $m = 0.5$. (b) Density of the fractal structure of Fig.4(a) in the region $\{x = [-0.016, 0.016],\ y = [-0.016, 0.016]\}$.

Certainly, if we choose $\chi(x), \varphi(y)$ and $\tau(t)$ to be different types of Jacobian elliptic functions, we can construct many different kinds of Jacobian double periodic wave structures with fractal behaviour.

4. Summary and discussion

In summary, by means of the mapping approach via the Riccati equation, the (2+1)-dimensional DLW system is successfully solved. Based on one of the derived variable separation solutions with three arbitrary functions, we list some special examples such as regular fractal dromion and lump solutions and Jacobian periodic wave structures with fractal behaviour for the (2+1)-dimensional DLW system. Actually, according to the solution (18)/(28), we can also obtain abundant non-propagating solitons. Comparing the common formula (1.1) in Refs.[18,19] with the solution (18)/(19), one can find easily that they are essentially equivalent when choosing $p(x,t) = \chi(x) + \tau_1(t), q(y,t) = \varphi(y) + \tau_2(t), \tau(t) = \tau_1(t) + \tau_2(t), a_1 = a_2 = 1$ and $a_0 = a_3 = 0$. Therefore, all the localized excitations based on the common formula (1.1) can be derived from the solution (18)/(19) except for partial propagating properties.

As is known, fractals not only belong to the realm of mathematics or computer graphics, but also exist nearly everywhere in nature, such as in fluid turbulence, crystal growth patterns, human veins, fern shapes, galaxy clustering, cloud structures and in numerous other cases. Traditionally, fractals are the counterpart of solitons in nonlinear science since solitons are representative of integrable systems while fractals represent the behaviour of nonintegrable systems. However, in this paper, we have found some fractal structures for localized/periodic solutions of the (2+1)-dimensional dispersive long wave (DLW) equation. Why these localized/periodic excitations possess such kinds of fractal behaviour? If one considers the boundary and/or initial condition of the fractal solutions obtained here, one can find that the initial and/or boundary condition possess the fractal property. This means that the fractal behaviour of these localized/periodic excitations of the integrable models comes from the nonintegrable boundary and/or initial condition. Due to the wide application of the soliton and fractal theory in many physics fields like the fluid dynamics, plasma physics, field theory, optics, etc, it is a significant task to learn more about the localized/periodic fractal excitations of the "integrable" physical model with "nonintegrable" boundary and/or initial condition such as fractal boundary and/or initial condition. This paper is just a beginning. We hope that the approach taken in our present work may be further extended to other nonlinear physical systems.

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