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New exact solutions of nonlinear Klein–Gordon equation

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New exact solutions, expressed in terms of the Jacobi elliptic functions, to the nonlinear Klein–Gordon equation are obtained by using a modified mapping method. The solutions include the conditions for equation's parameters and travelling wave transformation parameters. Some figures for a specific kind of solution are also presented.

Keywords: nonlinear Klein–Gordon equation, Jacobi elliptic functions, modified mapping method, travelling wave solution

PACC: 0340K, 0290

1. Introduction

Many phenomena in physics are often described by nonlinear partial differential equations (NPDEs). Exploring the exact solutions of NPDEs has become one of most important topics in mathematics physics. In recent years, a number of methods have been reported, such as the inverse scattering method,[1] bilinear method,[2] tanh expansion method,[3] homogeneous balance method,[4] separation-of-variable method,[5] sin-cosine method,[6] Jacobi elliptic function expanding method,[7,8] etc.

The Jacobi elliptic function expanding method is a very powerful method. Liu and Fu *et al*[7−9] presented Jacobi elliptic function periodic solutions of NPDEs including the KdV equation, mKdV equation, nonlinear Schrödinger equation. Yan *et al*[10−13] further developed this method and proposed an extended Jacobi elliptic function method so that more types of Jacobi elliptic function solutions could be obtained. Liu and Fan[14] proposed a more general Jacobi elliptic function method by using of a general function transform. The key idea of Zhu *et al*[15] is extending the range of j from $-n$ to n. Chen and Zhang[16] extended the base of the Jacobi elliptic function method from a single Jacobi elliptic function such as $\mathrm{sn}(x,k)$ or $\mathrm{cn}(x,k)$ to pairs of Jacobi elliptic functions such as $\mathrm{sn}(x,k)$ and $\mathrm{cn}(x,k)$. Liu and Li[17] presented a Maple package automated Jacobi elliptic function method. And recently, Peng[18] has obtained the (2+1)-D breaking soliton equation by using of modified mapping method.

The aim of this paper is to apply the so-called modified mapping method to solve the nonlinear Klein–Gordon equation and obtain some new solutions.

2. Jacobi elliptic functions and modified mapping method review

2.1. The Jacobi elliptic functions[19]

$$\mathrm{sn}(x)=\mathrm{sn}(x,m),\quad \mathrm{cn}(x)=\mathrm{cn}(x,m),$$
$$\mathrm{dn}(x)=\mathrm{dn}(x,m)\quad et\quad al, \tag{1}$$

where m $(0<m<1)$ is the modulus of Jacobi elliptic functions, which are double periodic and possess properties of triangular functions, namely

$$\mathrm{sn}^2(x)+\mathrm{cn}^2(x)=1,\quad \mathrm{dn}^2(x)+m^2\mathrm{sn}^2(x)=1. \tag{2a}$$

$$(\mathrm{sn}(x))'=\mathrm{cn}(x)\mathrm{sn}(x),$$
$$(\mathrm{cn}(x))'=-\mathrm{sn}(x)\mathrm{dn}(x),$$
$$(\mathrm{dn}(x))'=-m^2\mathrm{cn}(x)\mathrm{sn}(x). \tag{2b}$$

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When $m \to 0$, the Jacobi elliptic functions degenerate to triangular functions, i.e.

$$\mathrm{sn}(x) \to \sin(x), \quad \mathrm{cn}(x) \to \cos(x), \quad \mathrm{dn}(x) \to 1. \quad (3a)$$

When $m \to 1$, the Jacobi elliptic functions degenerate to hyperbolic functions, i.e.

$$\mathrm{sn}(x) \to \tanh(x), \quad \mathrm{cn}(x) \to \mathrm{sech}(x),$$

$$\mathrm{dn}(x) \to \mathrm{sech}(x). \quad (3b)$$

We obtain the figures of $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$ versus x when m=0.4 (see Fig.1).

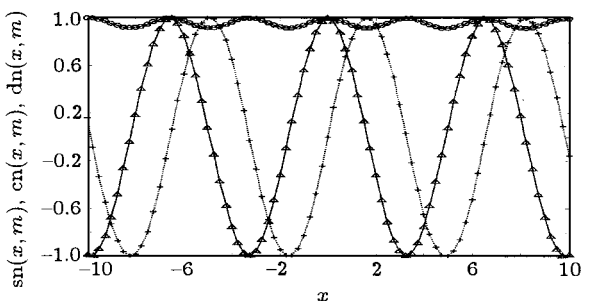


Fig.1. The functions $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$, versus x when $m = 0.4$. The plus ("+") corresponds to $\mathrm{sn}(x,m)$, the triangle ("△") to $\mathrm{cn}(x,m)$ and the circle ("◯") to $\mathrm{dn}(x,m)$.

2.2. Modified mapping method review[18]

For a given NPDE, say, in two variables x and t, we have

$$F(u, u_t, u_x, u_{tx}, \ldots) = 0. \quad (4)$$

We seek for its travelling wave ansatz as follows:

$$u(x,t) = u(\xi), \quad \xi = k_1 x + k_4 t, \quad (5)$$

where $k_j (j = 1, 4)$ are wave parameters to be determined later.

Substituting Eq.(5) into Eq.(4) yields a nonlinear ordinary differential equation

$$K(u, \mathrm{d}u/\mathrm{d}\xi, \mathrm{d}^2u/\mathrm{d}\xi^2, \ldots) = 0. \quad (6)$$

Then we expand $u(\xi)$ into a polynomial of $f(\xi)$ as follows:

$$u(\xi) = \sum_{i=0}^{N} A_i f^i + \sum_{i=1}^{N} B_i f^{-i}, \quad (7)$$

where A_i and B_i are constants to be determined and N can be fixed by balancing the linear terms of highest order derivative with nonlinear terms while f satisfies

$$f'' = pf + qf^3, (f')^2 = r + pf^2 + 1/2qf^4 , \quad (8)$$

where the prime "′" means $\mathrm{d}/\mathrm{d}\xi$; r, p and q are constants for a specific form of Jacobi elliptic function, for example, if $f = \mathrm{dn}(\xi)/(\sqrt{1-m^2}\mathrm{sn}(\xi) + \mathrm{cn}(\xi))$,[20] we have $r = m^2/4$, $p = (m^2-2)/2$, $q = m^2/2$. Substituting Eqs.(7) and (8) into Eq.(6) the coefficients A_i, B_i, k_j, p, q and r may be determined from the equations of their algebraic relations.

3. Examples: nonlinear Klein–Gordon equation

The nonlinear Klein–Gordon equation has been studied extensively [7,9,21,22]. It reads

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0. \quad (9)$$

Substituting Eq.(5) into Eq.(9) leads to

$$Au'' + Bu + Cu^3 = 0, \quad (10)$$

where $A = k_4^2 - c_0^2 k_1^2$, $B = \alpha$, $C = -\beta$,

$$O(u) = N, \quad O(u^3) = 3N, \quad O(u'') = N + 2,$$

and we have

$$N = 1. \quad (11)$$

So the ansatz solution of Eq.(7) can be rewritten as

$$u = A_0 + A_1 f + B_1/f. \quad (12)$$

Substituting Eqs.(12) and (8) into Eq.(10) and collecting each order of f, then setting each coefficient of f^i (i=3, 2, 1, 0, −1, −2, −3) to zeros yield a set of algebraic equations about coefficients A_0, A_1, B_1 and p, q, r, and they are

$$CA_1^3 + AA_1 q = 0, \quad 3CA_0A_1^3 = 0,$$

$$3CA_1^2B_1 + BA_1 + AA_1 p + 3CA_0^2A_1 = 0,$$

$$BA_0 + CA_0^3 + 6CA_0A_1B_1 = 0,$$

$$3CA_0^2B_1 + AB_1 p + BB_1 + 3CA_1B_1^3 = 0,$$

$$3CA_0B_1^2 = 0, \quad CB_1^3 + 2AB_1 r = 0. \quad (13)$$

From Eqs.(13) one can obtain three sets of solutions as follows:

Case 1

$$A_0 = A_1 = 0, \quad B_1 = \pm\sqrt{-2Ar/C} \quad (14a)$$

under **condition 1**:

$$pA + B = 0; \quad (14b)$$

Case 2

$$A_0 = B_1 = 0, \quad A_1 = \pm\sqrt{-Aq/C} \quad (15a)$$

under **condition 2**:

$$pA + B = 0; \tag{15b}$$

Case 3

$$A_0 = 0, \quad A_1 = \pm\sqrt{-Aq/C},$$
$$B_1 = \pm\sqrt{-2Ar/C} \tag{16a}$$

under **condition 3**:

$$3CA_1B_1 + 2B + 2Ap + 3CA_1B_1^2 = 0. \tag{16b}$$

Equations (14), (15) and (16) with Eq.(12) are the solutions of nonlinear Klein–Gordon equation. **Case 2** has been obtained in Refs.[7, 21], while, to the best of the authors' knowledge, **Case 1** and **Case 3** are new solutions. Each case has 32 kinds of solutions if we fix the functions f by following Ref.[20]. So the total of solutions is 96. And there is one thing we must note: our three cases are under the conditions that cover the equation's parameters and travelling wave transformation parameters coming from the solutions of Eqs.(13).

For example, **Case 1**:

$$u = \sqrt{-2Ar/C}(1/f), \tag{11a}$$

$$pA + B = 0. \tag{11b}$$

Substituting A and B into Eq.(11b), we have

$$(k_4^2 - c_0^2k_1^2)p + \alpha = 0. \tag{17a}$$

If we want

$$f = \mathrm{dn}(\xi)/(\sqrt{1-m^2}\mathrm{sn}(\xi) + \mathrm{cn}(\xi))$$

then

$$p = (m^2 - 2)/2. \tag{17b}$$

Equation (17) means that if parameters $\{k_4,\ c_0,\ k_1,\ \alpha\}$ are given, then m is also been fixed. For example, if we let c_0=1, k_1=4, k_4=1, $\alpha = -10.2$, then the m must equal ± 0.8. Han[22] has noticed this point.

Plotting some solutions must be helpful. Therefore, for simplify in each case we only choose two kinds of Jacobi elliptic functions and in order to clearly display the 6 kinds of solutions we only give the one-dimensional figures (see Fig.2, Fig.3 and Fig.4), u versus x (all the figures are obtained at time t=0 and m is always chosen as being positive).

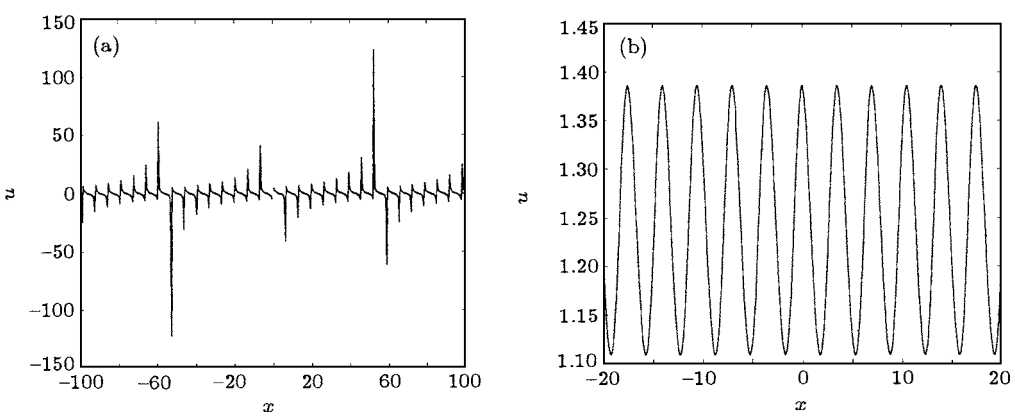


Fig.2. Case 1 solutions of nonlinear Klein–Gordon equation. The parameters are: (a) $\{k_1$=1, k_4=2, c_0=1, α=–1.02, β=1, $f = \mathrm{sn}(\xi)/(1 + \mathrm{cn}(\xi))\}$, so $m = \pm 0.4$; (b) $\{k_1$=1, k_4=0.2, c_0=1, α=1.5744, β=1, $f = \mathrm{nd}(\xi)\}$, so $m = \pm 0.6$.

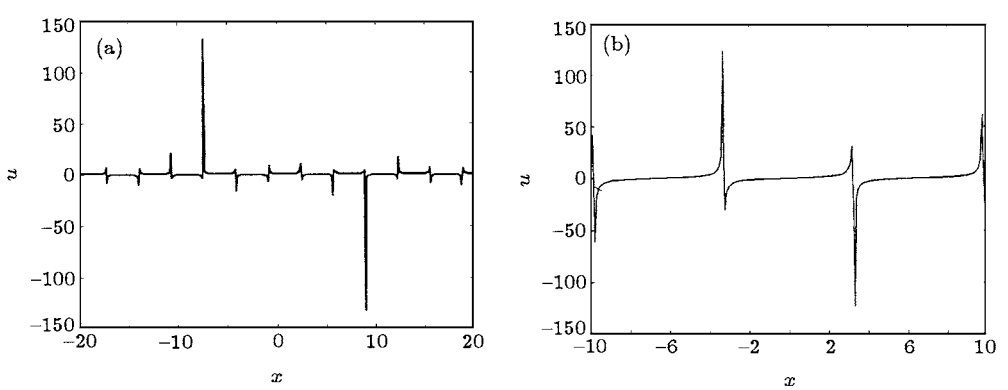


Fig.3. Case 2 solutions of nonlinear Klein–Gordon equation. The parameters are: (a) $\{k_1$=1, k_4=2, c_0=1, α= 2.76, β=1, $f = \mathrm{dn}(\xi)/(\sqrt{1-m^2}\mathrm{sn}(\xi) + \mathrm{cn}(\xi))\}$, so $m = \pm 0.4$; (b) $\{k_1$=1, k_4=2, c_0=1, α=–1.02, β=1, $f = \mathrm{sn}(\xi)/(1 + \mathrm{cn}(\xi))\}$, so $m = \pm 0.4$.

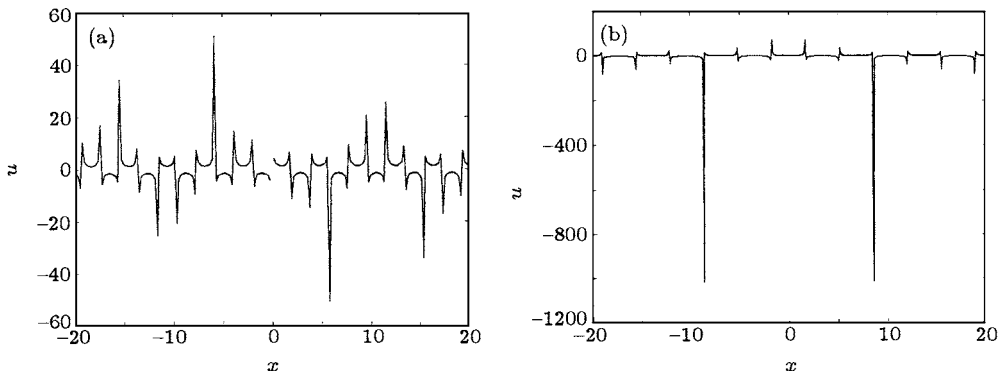


Fig.4. Case 3 solutions of nonlinear Klein–Gordon equation. The parameters are: (a) $\{k_1 = \sqrt{3},\ k_4{=}2,\ c_0{=}1,\ \alpha{=}1,\ \beta{=}1,\ f = \mathrm{sn}(\xi)/(1 + \mathrm{cn}(\xi))\}$, so m $=\pm 0.4687$; (b) $\{k_1{=}1,\ k_4{=}2,\ c_0{=}1,\ \alpha{=}{-}0.12,\ \beta{=}{-}1,\ f = \mathrm{cn}(\xi)/(1 + \mathrm{sn}(\xi))\}$, so m $=\pm 0.55668$.

4. Summary

The Klein–Gordon equation has been studied by use of modified mapping method.[18] The advantages of this method are (a) the range of j from $-n$ to n; (b) the indeterminate specific form of Jacobi elliptic function so that we can obtain more new exact solutions. We have obtained 96 kinds of solutions if we follow the choice of Jacobi elliptic function in Ref.[20] and the parameters conditions of these solutions. The latter is omitted in many studies except Ref.[22], as far as the authors' knowledge goes now. Actually, the specific forms of f are 32 kinds more than those in Ref.[20], for example $f = m\mathrm{sn}(\xi)/(1 \pm \mathrm{dn}(\xi))$, if we set $r = m^2/2$, $p = (m^2 - 2)/2$, $q = m^2/2$. We believe that it is very natural to apply this method to other equations that contain the higher dimension and derivative orders.

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