Vol 15 No 1, January 2006
1009-1963/2006/15(01)/0019-03

Chinese Physics

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Lie symmetry and conserved quantity of a system of first-order differential equations*

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(Received 22 April 2005)

This paper focuses on studying the Lie symmetry and a conserved quantity of a system of first-order differential equations. The determining equations of the Lie symmetry for a system of first-order differential equations, from which a kind of conserved quantity is deduced, are presented. And their general conclusion is applied to a Hamilton system, a Birkhoff system and a generalized Hamilton system. Two examples are given to illustrate the application of the results.

Keywords: Lie symmetry, conserved quantity, differential equation, mechanical system
PACC: 0320

1. Introduction

Since Noether revealed the inner relation between symmetries and conserved quantities of dynamical systems in 1918,[1] people have made great progress with the subsequent research on both symmetries and conserved quantities,[2−5] and the application of Lie groups on differential equations has been made much account of.[6−8] The Lutzky conserved quantity[9,10] and the Hojman conserved quantity[5,11,12] can be directly deduced from the Lie symmetry. In this paper, the Lie symmetry and a conserved quantity of a system of first-order differential equations are studied, and the application of the conclusions to three kinds of mechanical systems is presented.

2. Lie symmetry and conserved quantity

A system of first-order differential equations can be written in the form

$$\dot{x}_\mu = f_\mu(t, \boldsymbol{x}) \qquad (\mu = 1, 2, \ldots, m). \tag{1}$$

Introduce infinitesimal transformations as

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{x}), \quad x^*_\mu(t^*) = x_\mu(t) + \varepsilon\xi_\mu(t, \boldsymbol{x})$$
$$(\mu = 1, 2, \ldots, m), \tag{2}$$

where ε is an infinitesimal parameter and ξ_0, ξ_μ are infinitesimal generators. Under the transformations (2), the determining equations of the Lie symmetry for Eq.(1) are

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu - f_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - X^{(0)}(f_\mu) = 0$$
$$(\mu = 1, 2, \ldots, m), \tag{3}$$

where

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial x_\mu}, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + f_\mu\frac{\partial}{\partial x_\mu}. \tag{4}$$

Proposition If the infinitesimal generators ξ_0, ξ_μ satisfy Eq.(3), and besides, there exists a function $\lambda = \lambda(t, \boldsymbol{x})$ satisfying

$$\frac{\partial f_\mu}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(0)}\left(\frac{\partial f_\mu}{\partial x_\mu}\right) = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X^{(0)}(\ln\lambda)], \tag{5}$$

then Eq.(1) has a conserved quantity as

$$I = \frac{\partial\xi_\mu}{\partial x_\mu} - f_\mu\frac{\partial\xi_0}{\partial x_\mu} - X^{(0)}(\ln\lambda) = \text{const.} \tag{6}$$

Proof Let

$$\Phi_\mu = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu - f_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - X^{(0)}(f_\mu).$$

*Project supported by the National Natural Science Foundation of China (Grant No 10272021) and the Doctoral Program Foundation of Institution of Higher Education of China (Grant No 20040007022).

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Differentiating Φ_μ with respect to x_μ and summing up for μ, we have

$$\frac{\partial \Phi_\mu}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu - \frac{\partial f_\mu}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - f_\mu\frac{\partial}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \frac{\partial}{\partial x_\mu}X^{(0)}(f_\mu). \quad \text{(A)}$$

Substituting the following three relations:

$$\frac{\partial}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_\mu}{\partial x_\mu} + \frac{\partial \xi_\mu}{\partial x_\nu}\frac{\partial f_\nu}{\partial x_\mu},$$

$$\frac{\partial}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial x_\mu} + \frac{\partial \xi_0}{\partial x_\nu}\frac{\partial f_\nu}{\partial x_\mu},$$

$$\frac{\partial}{\partial x_\mu}X^{(0)}(f_\mu) = X^{(0)}\left(\frac{\partial f_\mu}{\partial x_\mu}\right) + \frac{\partial f_\mu}{\partial t}\frac{\partial \xi_0}{\partial x_\mu} + \frac{\partial f_\mu}{\partial x_\nu}\frac{\partial \xi_\nu}{\partial x_\mu}$$

into formula (A), we obtain

$$\frac{\partial \Phi_\mu}{\partial x_\mu} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_\mu}{\partial x_\mu} - f_\mu\frac{\partial \xi_0}{\partial x_\mu}\right) - X^{(0)}\left(\frac{\partial f_\mu}{\partial x_\mu}\right) - \frac{\partial f_\mu}{\partial x_\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0. \quad \text{(B)}$$

Substituting relation (5) into formula (B), we get

$$\frac{\partial \Phi_\mu}{\partial x_\mu} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial \xi_\mu}{\partial x_\mu} - f_\mu\frac{\partial \xi_0}{\partial x_\mu} - X^{(0)}(\ln\lambda)\right],$$

and then the system (1) has the integral (6).

Corollary 1 If

$$\frac{\partial f_\mu}{\partial x_\mu} = 0, \quad (7)$$

Eq.(1) has a conserved quantity as

$$I = \frac{\partial \xi_\mu}{\partial x_\mu} - f_\mu\frac{\partial \xi_0}{\partial x_\mu} = \text{const.} \quad (8)$$

Corollary 2 If

$$\frac{\partial f_\mu}{\partial x_\mu} = 0, \quad \xi_0 = 0, \quad (9)$$

Eq.(1) has a conserved quantity as

$$I = \frac{\partial \xi_\mu}{\partial x_\mu} = \text{const.} \quad (10)$$

3. Application to mechanical systems

For a Hamilton system

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \quad (s = 1, 2, \ldots, n), \quad (11)$$

we have

$$\frac{\partial f_\mu}{\partial x_\mu} = \frac{\partial}{\partial q_s}\frac{\partial H}{\partial p_s} + \frac{\partial}{\partial p_s}\left(-\frac{\partial H}{\partial q_s}\right) = 0, \quad (12)$$

where

$$x_s = q_s, \quad x_{n+s} = p_s \quad (s = 1, 2, \ldots, n; \quad m = 2n). \quad (13)$$

According to corollary 1, the system has the conserved quantity (8) which subsequently becomes

$$I = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} - \frac{\partial H}{\partial p_s}\frac{\partial \xi_0}{\partial q_s} + \frac{\partial H}{\partial q_s}\frac{\partial \xi_0}{\partial p_s} = \text{const}, \quad (14)$$

where $\eta_s = \xi_{n+s}$. Particularly, if $\xi_0 = 0$, relation (14) will give

$$I = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} = \text{const}, \quad (15)$$

which is a Hojman conserved quantity of the Hamilton system.

For a Birkhoff system

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) \quad (\mu = 1, 2, \ldots, 2n), \quad (16)$$

let

$$x_\mu = a^\mu, \quad f_\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right), \quad (17)$$

and then the system (16) can be written in the form of Eq.(1). According to corollary 2, if relation (9) holds, the system will have the conserved quantity (10) which was given in proposition 4 of Ref.[13].

A generalized Hamilton system

$$\dot{x}_\mu = J_{\mu\nu}\frac{\partial H}{\partial x_\nu} \quad (\mu = 1, 2, \ldots, m) \quad (18)$$

can certainly be written in the form of Eq.(1). If

$$\frac{\partial}{\partial x_\mu}\left(J_{\mu\nu}\frac{\partial H}{\partial x_\nu}\right) = 0, \quad (19)$$

according to corollary 1, the system (18) will have the conserved quantity (8), where

$$f_\mu = J_{\mu\nu}\frac{\partial H}{\partial x_\nu}, \quad (20)$$

and subsequently, will become relation (10) in the special case of $\xi_0 = 0$.

4. Illustrative examples

Example 1 A generalized Hamilton system is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \quad \dot{x}_3 = -4x_1x_2. \quad (21)$$

The determining equations (3) of the Lie symmetry for the system give

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 - x_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \xi_2,$$
$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 + x_1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = -\xi_1,$$
$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 + 4x_1x_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = -4\xi_1x_2 - 4\xi_2x_1,$$

which have a solution as

$$\xi_1 = \xi_2 = \xi_3 = 0, \quad \xi_0 = (x_1\sin t + x_2\cos t)^2. \quad (22)$$

It is obvious that relation (7) holds, hence the conserved quantity (8) gives

$$\begin{aligned} I =& -x_2\frac{\partial\xi_0}{\partial x_1} + x_1\frac{\partial\xi_0}{\partial x_2} \\ =& 2(x_1\sin t + x_2\cos t)(x_1\cos t - x_2\sin t) \\ =& \text{const.} \end{aligned} \quad (23)$$

Example 2 The Birkhoff functions and the Birkhoffian of a second-order Birkhoff system are

$$R_1 = a^2, \quad R_2 = 0,$$
$$B = \frac{1}{2m}(a^2)^2 + \frac{1}{m}a^2\theta_1(t) + \frac{1}{2m}\theta_1^2(t) + \theta_0(t), \quad (24)$$

where m is a constant and $\theta_0(t)$, $\theta_1(t)$ are known functions with respect to time t. The Birkhoff equations of the system can be written in the form

$$\dot{a}^1 = \frac{1}{m}a^2 + \frac{1}{m}\theta_1(t), \quad \dot{a}^2 = 0. \quad (25)$$

We can easily verify that relation (7) holds. The determining equations (3) of the Lie symmetry give

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 - \frac{1}{m}(a^2+\theta_1)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \frac{1}{m}(\xi_2 + \dot{\theta}_1\xi_0), \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = 0,$$

which have a solution as

$$\xi_0 = (a^1 - \frac{t}{m}a^2 - \int\frac{\theta_1}{m}\mathrm{d}t)^2,$$
$$\xi_1 = \frac{\theta_1}{m}(a^1 - \frac{t}{m}a^2 - \int\frac{\theta_1}{m}\mathrm{d}t)^2, \quad \xi_2 = 0. \quad (26)$$

Hence the conserved quantity (8) gives

$$\begin{aligned} I =& \frac{\partial\xi_1}{\partial a^1} - \frac{1}{m}(a^2+\theta_1)\frac{\partial\xi_0}{\partial a^1} \\ =& -\frac{2a^2}{m}(a^1 - \frac{t}{m}a^2 - \int\frac{\theta_1}{m}\mathrm{d}t) = \text{const.} \end{aligned} \quad (27)$$

5. Conclusion

The condition (5) under which the Lie symmetry can lead to a conserved quantity for a system of first-order differential equations and the form of the conserved quantity (6) are presented in this paper. These results can be applied to studying the Lie symmetries and the conserved quantities of the Hamilton system, the Birkhoff system and the generalized Hamilton system. We are looking forward to generalizing the results to second-order differential equations.

References

[1] Noether E 1918 *Nachr König Gesell Wissen Göttingen Math. Phys.* **Kl** 235

[2] Li Z P 1993 *Classical and Quantal Dynamics of Constrained Systems and Their Symmetrical Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)

[3] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) (in Chinese)

[4] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[5] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[6] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[7] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)

[8] Ibragimov N H 1995 *CRC Handbook of Lie Group Analysis of Differential Equations* (Boca Raton, FL: CRC Press)

[9] Lutzky M 1979 *Phys. Lett.* A **72** 86

[10] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255

[11] Hojman S A 1992 *J. Phys.* A *Math. Gen.* **25** L291

[12] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)

[13] Zhang Y 2002 *Acta Phys. Sin.* **51** 461 (in Chinese)