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A geometric framework for time-dependent mechanical systems with unilateral constraints*

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The description of modern differential geometry for time-dependent Chetaev nonholonomic mechanical systems with unilateral constraints is studied. By using the structure of exact contact manifold, the geometric framework of time-dependent nonholonomic mechanical systems subject to unilateral nonholonomic constraints and unilateral holonomic constraints respectively is presented.

Keywords: analytical mechanics, time-dependent, unilateral constraint, differential geometry
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1. Introduction

In nature as well as in engineering, there are plenty of problems with unilateral constraints:[1] for example, the motion of a system restrained in a limited space, the motion of a system with gyro wheels, the motion of a robot round barriers, the motion of a system in vibration or under impact, and so on. The history of investigation of mechanical systems with unilateral constraints may be traced back to the beginning of the 20th century,[2] and in the 1930s and the 1940s of the 20th century, Levi-Civita and Amaldi,[3] and Suslov[4] studied the idea of unilateral constraint, the D'Alembert–Lagrange principle and the virtual work principle of systems with unilateral constraints. In 1958, Wang[5] introduced the virtual work principle of a particle and a system of particles with unilateral holonomic constraints and their simple applications. In 1988, Mei[6] reviewed the dynamics of systems with unilateral constraints. In 1993, Zhuravlev and Fufaev[7] published the first monograph on mechanics of systems with unilateral constraints. In recent years, some important results on researches of fundamental theories of systems with unilateral constraints have been obtained.[8−19] In Ref.[19] the tangent bundle geometry is used to describe time-independent dynamical systems subject respectively to unilateral nonholonomic constraints and unilateral holonomic constraints. In this paper, we further study the description of modern differential geometry for time-dependent Chetaev nonholonomic mechanical systems with unilateral constraints.

2. Preliminaries

Let Q be an n-dimensional differentiable manifold, TQ its tangent bundle, and $\tau : TQ \to Q$ the canonical tangent bundle projection. We will denote by $\{q^i; i = 1, \ldots, n\}$ a set of local coordinates in Q and by $\{q^i, v^i; i = 1, \ldots, n\}$ the associated coordinates in TQ.

The time-dependent velocity phase space is represented, in geometric terms, by $TQ \times R$. The natural coordinate on R will be denoted by t, and $\pi_1 : TQ \times R \to TQ$ and $\pi_2 : TQ \times R \to R$ are the two projections onto TQ and R, respectively. We will use as the fundamental structure the bundle over the space of events $Q \times R$ given by $\pi : TQ \times R \to Q \times R, \pi = \tau_1 \times \pi_2, \tau_1 = \tau \circ \pi_1$. The Liouville vector field will be denoted by Δ and the (1,1) tensor field representing

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the time-dependent vertical endomorphism in $TQ\times R$ by S^t. In terms of local coordinates they have the form,

$$\Delta = v^i\frac{\partial}{\partial v^i},\quad S^t = \frac{\partial}{\partial v^i}\otimes(\mathrm{d}q^i - v^i\mathrm{d}t). \tag{1}$$

We will use the adjoint operator S^{t*} instead of S^t when it acts on $\Lambda^1(TQ\times R)$. Obviously,

$$\begin{aligned} &S^t\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial v^i},\quad S^t\left(\frac{\partial}{\partial v^i}\right) = 0,\\ &S^{t*}(\mathrm{d}q^i) = S^{t*}(\mathrm{d}t) = 0,\\ &S^{t*}(\mathrm{d}v^i) = \mathrm{d}q^i - v^i\mathrm{d}t. \end{aligned} \tag{2}$$

Suppose that a time-dependent Lagrangian is given as a differentiable function L on $TQ\times R$, then we can construct an energy function $E_L\in C^\infty(TQ\times R)$, a semibasic 1-form $\Theta_L\in\Lambda^1(TQ\times R)$, and an exact 2-form $\Omega_L\in\Lambda^2(TQ\times R)$ by

$$E_L = \Delta(L) - L,\ \Theta_L = S^{t^*}(\mathrm{d}L) + L\mathrm{d}t,\ \Omega_L = -\mathrm{d}\Theta_L. \tag{3}$$

We recall that if the Lagrangian L is regular, then the 2-form Ω_L has maximal rank. In this case $(TQ\times R, \Theta_L)$ is an exact contact manifold. We can then represent the dynamics by the vector field $X_L\in\chi(TQ\times R)$ satisfying the equation

$$i(X_L)\Omega_L = 0 \tag{4}$$

and the condition

$$i(X_L)\mathrm{d}t = 1. \tag{5}$$

The solution X_L is uniquely determined, which turns out to be a time-dependent second-order differential equation vector field (hereafter abbreviated as SODE). In local coordinates, X_L takes the form

$$\begin{aligned} &X_L = v^i\frac{\partial}{\partial q^i} + f^i(\boldsymbol{q},\boldsymbol{v},t)\frac{\partial}{\partial v^i} + \frac{\partial}{\partial t},\\ &f^i(\boldsymbol{q},\boldsymbol{v},t) = W^{ij}\left(\frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial q^k\partial v^j}v^k - \frac{\partial^2 L}{\partial t\partial v^j}\right), \end{aligned} \tag{6}$$

where $f^i(\boldsymbol{q},\boldsymbol{v},t)$ may be called the Lagrangian forces in contrast with the constraint forces and W^{kj} is the inverse matrix of the time-dependent Hessian matrix W_{ij}.

In the following we will denote by $\chi^v(TQ\times R)$ the set of all the vertical vector fields

$$\begin{aligned} &\chi^v(TQ\times R)\\ &=\left\{V\in\chi(TQ\times R)|S^t(V) = 0, i(V)\mathrm{d}t = 0\right\} \end{aligned} \tag{7}$$

by $\chi^2(TQ\times R)$ the set of the SODE vector fields

$$\begin{aligned} &\chi^2(TQ\times R)\\ &=\left\{D\in\chi(TQ\times R)|S^t(D) = 0, i(D)\mathrm{d}t = 1\right\}, \end{aligned} \tag{8}$$

and by $\Lambda^1_{\mathrm{sb}}(TQ\times R)$ the set of the semibasic 1-forms

$$\Lambda^1_{\mathrm{sb}}(TQ\times R) = \left\{\beta\in\Lambda^1(TQ\times R)|S^{t^*}(\beta) = 0\right\}. \tag{9}$$

We will denote by $\{\theta^i; i = 1,\cdots,n\}$ the set of local contact 1-forms with coordinate expression $\theta^i = \mathrm{d}q^i - v^i\mathrm{d}t$; they generate a subset of semibasic 1-forms that we will denote by $\Lambda^1_{\mathrm{c}}(TQ\times R)$. Therefore, a semibasic 1-form α belongs to $\Lambda^1_{\mathrm{c}}(TQ\times R)$ if $\alpha = \alpha_i\theta^i$ where α_i are local smooth functions.

3. Motion of nonholonomic systems with unilateral nonholonomic constraints

Now we assume that the system is subject to a Chetaev nonholonomic unilateral constraint and $g-1$ Chetaev nonholonomic bilateral constraints determined by a submanifold $\tilde{C}$ of $TQ\times R$ with boundary. In local coordinates, the constraints are defined by equations of the form

$$\begin{aligned} &\Phi^a = \Phi^a(\boldsymbol{q},\boldsymbol{v},t) = 0, (a = 1,\cdots,g-1)\\ &\Phi^g = \Phi^g(\boldsymbol{q},\boldsymbol{v},t)\geq 0, \end{aligned} \tag{10}$$

where we assume that the constraints (10) are nonintegrable in the sense of Frobenius. Obviously, the interior of $\tilde{C}$ (denoted by $\mathrm{Int}\tilde{C}$) is a submanifold of $TQ\times R$, and its boundary $\partial\tilde{C}$ is a submanifold of $TQ\times R$ of codimension 1 with respect to $\tilde{C}$, where the boundary $\partial\tilde{C}$ is assumed to be orientable.

Let us define 1-form on $\Lambda^1(TQ\times R)$ as follows:

$$\alpha_a = S^{t^*}(\mathrm{d}\Phi^a)(x),\quad x\in\tilde{C}, \tag{11}$$

$$\alpha_g = \begin{cases} 0 & \text{if}\quad x\in\mathrm{Int}\tilde{C},\\ S^{t^*}(\mathrm{d}\Phi^g)(x) & \text{if}\quad x\in\partial\tilde{C}. \end{cases} \tag{12}$$

In local coordinates, we have

$$\alpha_a = \alpha_{aj}\theta^j = \frac{\partial\Phi^a}{\partial v^j}(\mathrm{d}q^j - v^j\mathrm{d}t),\quad (\boldsymbol{q},\boldsymbol{v},t)\in\tilde{C}, \tag{13}$$

$$\alpha_g = \begin{cases} 0 & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \mathrm{Int}\tilde{C}, \\ \alpha_{gj}\theta^j = \dfrac{\partial \Phi^g}{\partial v^j}(\mathrm{d}q^j - v^j \mathrm{d}t) & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}. \end{cases} \tag{14}$$

In order to obtain the vector field of dynamics of the system with unilateral constraints, we first consider the system of equations

$$i(X_L)\Omega_L = 0, \tag{15}$$

$$i(Z_a)\Omega_L = \alpha_a, \tag{16}$$

$$i(Z_g)\Omega_L = \alpha_g, \tag{17}$$

where Z_a and Z_g are respectively associated vertical vector fields of contact semibasic 1-forms α_a and α_g, which represent the modification that α_a and α_g introduce in the free Lagrangian dynamics. In local coordinates, we have

$$Z_a = -W^{ij}\alpha_{aj}\frac{\partial}{\partial v^i}, \tag{18}$$

$$Z_g = \begin{cases} 0 & \text{if} \quad (\boldsymbol{q}, \mathbf{v}, t) \in \mathrm{Int}\tilde{C}, \\ -W^{ij}\alpha_{gj}\dfrac{\partial}{\partial v^i} & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}. \end{cases} \tag{19}$$

Let $\lambda^A, A = 1, \cdots, g$ be a family of g functions in $TQ \times R$ and let Γ_λ denote the associated λ^A-dependent family of SODE vector fields defined by

$$\Gamma_\lambda = X_L - \lambda^a Z_a - \lambda^g Z_g,$$
$$\lambda^A \in C^\infty(TQ \times R),$$

then Γ_λ is a solution of the equations

$$\mathrm{i}(\Gamma_\lambda)\Omega_L = -\lambda^a\alpha_a - \lambda^g\alpha_g,$$
$$\mathrm{i}(\Gamma_\lambda)\mathrm{d}t = 1. \tag{20}$$

In local coordinates Γ_λ takes the form

$$\Gamma_\lambda = v^i\frac{\partial}{\partial q^i} + F^i(\boldsymbol{q}, \boldsymbol{v}, t, \lambda)\frac{\partial}{\partial v^i} + \frac{\partial}{\partial t}, \tag{21}$$

where

$$F^i(\boldsymbol{q}, \boldsymbol{v}, t, \lambda) = \begin{cases} f^i(\boldsymbol{q}, \boldsymbol{v}, t) + \lambda^a W^{ij}\alpha_{aj} & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \mathrm{Int}\tilde{C}, \\ f^i(\boldsymbol{q}, \boldsymbol{v}, t) + \lambda^a W^{ij}\alpha_{aj} + \lambda^g W^{ij}\alpha_{gj} & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}, \end{cases} \tag{22}$$

and its integral curves are the solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q^i = \begin{cases} f^i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \lambda^a W^{ij}\alpha_{aj} & \text{if} \quad (\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \mathrm{Int}\tilde{C}, \\ f^i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \lambda^a W^{ij}\alpha_{aj} + \lambda^g W^{ij}\alpha_{gj} & \text{if} \quad (\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \in \partial\tilde{C}. \end{cases} \tag{23}$$

In the set $\{\Gamma_\lambda, \lambda^A \in C^\infty(TQ \times R), A = 1, \cdots, g\}$ there is a unique vector field $\Gamma \in \chi^2(TQ \times R)$ that is tangent to the submanifold $\tilde{C} \subset TQ \times R$ with boundary. It is determined by the following conditions:

$$\Gamma_\lambda(\Phi^a) = 0, (a = 1, \cdots, g-1) \quad \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \mathrm{Int}\tilde{C}, \tag{24}$$

$$\Gamma_\lambda(\Phi^A) = 0, (A = 1, \cdots, g) \quad \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}, \tag{25}$$

i.e.

$$\begin{aligned} &X_L(\Phi^a) - \lambda^b Z_b(\Phi^a) = 0, (a, b = 1, \ldots, g-1) \quad \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \mathrm{Int}\tilde{C}, \\ &X_L(\Phi^A) - \lambda^B Z_B(\Phi^A) = 0, (A, B = 1, \ldots, g) \quad \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}. \end{aligned} \tag{26}$$

Suppose the matrix

$$M = \begin{cases} (Z_b(\Phi^a)) & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \mathrm{Int}\tilde{C}, \\ (Z_B(\Phi^A)) & \text{if} \quad (\boldsymbol{q}, \boldsymbol{v}, t) \in \partial\tilde{C}, \end{cases}$$

is non-singular, and λ can be solved from Eqs.(26). Therefore, the vector field Γ of dynamics for the time-dependent system with unilateral constraints is obtained.

Suppose a time-dependent Lagrangian L takes the form

$$L(\boldsymbol{q}, \boldsymbol{v}, t) = \frac{1}{2}g_{ij}v^i v^j - V,$$
$$g_{ij} = g_{ij}(\boldsymbol{q}, t), V = V(\boldsymbol{q}, t).$$

Since $W_{ij} = g_{ij}$, so the condition of it being non-

singular is satisfied automatically.

4. Motion of nonholonomic systems with unilateral holonomic constraints

Suppose the system is subject to d ideal holonomic unilateral constraint and $g-1$ Chetaev nonholonomic bilateral constraints determined by a submanifold $\tilde{C}$ of $TQ\times R$ with boundary. In local coordinates, we have

$$\Phi^a = \Phi^a(\boldsymbol{q},\boldsymbol{v},t) = 0, (a=1,\cdots,g), \tag{27}$$

$$\Psi^b = \Psi^b(\boldsymbol{q},t) \geq 0, (b=1,\cdots,d), \tag{28}$$

where Φ^a is non-integrable in the sense of Frobenius.

Let us define the 1-form on $\Lambda^1(TQ\times R)$ as follows:

$$\alpha_a = S^{t^*}(\mathrm{d}\Phi^a)(x) \quad x\in\tilde{C}, \tag{29}$$

$$\beta_b = \begin{cases} 0 & \text{if} \quad x\in \mathrm{Int}\tilde{C}, \\ \pi^*(\mathrm{d}\Psi^b)(x) & \text{if} \quad x\in\partial\tilde{C}, \end{cases} \tag{30}$$

where $\pi^*: T^*(Q\times R)\to T^*(TQ\times R)$ is the induced mapping of π. In local coordinates, we have,

$$\alpha_a = \alpha_{aj}\mathrm{d}q^j = \frac{\partial\Phi^a}{\partial v^j}\mathrm{d}q^j \quad (\boldsymbol{q},\boldsymbol{v},t)\in\tilde{C}, \tag{31}$$

$$\beta_b = \begin{cases} 0 & \text{if} \quad (\boldsymbol{q},\boldsymbol{v},t)\in \mathrm{Int}\tilde{C}, \\ \beta_{bj}\mathrm{d}q^j = \dfrac{\partial\Psi^b}{\partial q^j}\mathrm{d}q^j & \text{if} \quad (\boldsymbol{q},t)\in\partial\tilde{C}. \end{cases} \tag{32}$$

We consider the system expressed by the following equations:

$$i(X_L)\Omega_L = 0, \tag{33}$$

$$i(Z_{\alpha a})\Omega_L = \alpha_a, \tag{34}$$

$$i(Z_{\beta b})\Omega_L = \beta_b. \tag{35}$$

The family of SODE vector fields

$$\left\{\bar{\Gamma}(\lambda,\mu)\in\chi^2(TQ\times R)|\lambda,\mu\in C^\infty(TQ\times R)\right\}$$

is defined by

$$\bar{\Gamma}(\lambda,\mu) = X_L - \lambda^a Z_{\alpha a} - \mu^b Z_{\beta b} \tag{36}$$

and satisfies the following equations:

$$i(\bar{\Gamma})\Omega_L = -\lambda^a\alpha_a - \mu^b\beta_b, \tag{37}$$

$$i(\bar{\Gamma})\mathrm{d}t = 1. \tag{38}$$

Therefore, in coordinates $\bar{\Gamma}$ takes the form

$$\bar{\Gamma} = v^i\frac{\partial}{\partial q^i} + F^i(\boldsymbol{q},\boldsymbol{v},t,\lambda,\mu)\frac{\partial}{\partial v^i} + \frac{\partial}{\partial t}, \tag{39}$$

where

$$F^i = \begin{cases} f^i(\boldsymbol{q},\boldsymbol{v},t) + \lambda^a W^{ij}\alpha_{aj} & \text{if} \quad (\boldsymbol{q},\boldsymbol{v},t)\in\mathrm{Int}\tilde{C}, \\ f^i(\boldsymbol{q},\boldsymbol{v},t) + \lambda^a W^{ij}\alpha_{aj} + \mu^b W^{ij}\beta_{bj} & \text{if} \quad (\boldsymbol{q},\boldsymbol{v},t)\in\partial\tilde{C}, \end{cases} \tag{40}$$

and its integral curves are the solutions of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q^i = \begin{cases} f^i(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \lambda^a W^{ij}\alpha_{aj} & \text{if} \quad (\boldsymbol{q},\dot{\boldsymbol{q}},t)\in\mathrm{Int}\tilde{C}, \\ f^i(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \lambda^a W^{ij}\alpha_{aj} + \mu^b W^{ij}\beta_{bj} & \text{if} \quad (\boldsymbol{q},\dot{\boldsymbol{q}},t)\in\partial\tilde{C}, \end{cases} \tag{41}$$

which agree with the following equations by the classical approach:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^i} - \frac{\partial L}{\partial q^i} = \begin{cases} \lambda^a\dfrac{\partial\Phi^a}{\partial\dot{q}^i} & \text{if} \quad \Psi^b(\boldsymbol{q},\dot{\boldsymbol{q}},t) > 0, \\ \lambda^a\dfrac{\partial\Phi^a}{\partial\dot{q}^i} + \mu^b\dfrac{\partial\Psi^b}{\partial\dot{q}^i} & \text{if} \quad \Psi^b(\boldsymbol{q},\dot{\boldsymbol{q}},t) = 0. \end{cases} \tag{42}$$

The Lagrange multipliers λ^a and μ^b in the classical approach can be regarded as the "intensity" of the modification of the vertical vector fields in the geometrical approach. In order to determine λ^a and μ^b, it needs to be complemented with some conditions, such as we may assume

$$\dot{\Psi}^b = \frac{\partial\Psi^b}{\partial q^i}v^i + \frac{\partial\Psi^b}{\partial t} = 0 \quad \text{if} \quad \Psi^b(\boldsymbol{q},t) = 0, \tag{43}$$

then in the set $\left\{\bar{\Gamma}|\lambda^a,\mu^b\in C^\infty(TQ\times R)\right\}$ there is a unique vector field $\Gamma\in\chi^2(TQ\times R)$ that satisfies the following equations:

$$\begin{aligned}&\Gamma(\Phi^a)=0,(a=1,\cdots,g),\\&\Gamma(\dot\Psi^b)=0,(b=1,\cdots,d)\quad\text{if}\quad(\boldsymbol{q},\boldsymbol{v},t)\in\partial\tilde C,\end{aligned}\tag{44}$$

i.e.

$$X_L(\Phi^a)-\lambda^c Z_{\alpha c}(\Phi^a)=0\quad\text{if}\quad(\boldsymbol{q},\boldsymbol{v},t)\in\mathrm{Int}\tilde C,\tag{45}$$

$$\left.\begin{aligned}&X_L(\Phi^a)-\lambda^c Z_{\alpha c}(\Phi^a)-\mu^e Z_{\beta e}(\Phi^a)=0\\&X_L(\dot\Psi^b)-\lambda^c Z_{\alpha c}(\dot\Psi^b)-\mu^e Z_{\beta e}(\dot\Psi^b)=0\\&(c,a=1,\cdots,g;e,b=1,\cdots,d)\end{aligned}\right\}\quad\text{if}\quad(\boldsymbol{q},\boldsymbol{v},t)\in\partial\tilde C.\tag{46}$$

These are the algebraic equations about λ^a and μ^b, if the coefficient matrix of the equations is non-singular, λ^a and μ^b can be solved out, and therefore, the vector field Γ of the unilateral constrained dynamics can be obtained.

5. Example

Suppose the Lagrangian of a mechanical system is

$$L=\frac{1}{2}\left[\left(\dot q^1\right)^2+\left(\dot q^2\right)^2+\left(\dot q^3\right)^2\right],\tag{47}$$

and the constraints for this system are defined by equations of the form

$$\Phi^1=\dot q^1+at\dot q^2-aq^2+t=0,\tag{48}$$

$$\Phi^2=\dot q^3-at\dot q^1+aq^1\geq 0.\tag{49}$$

In this case the 2-form Ω_L and the 1-forms α_1, α_2 are

$$\begin{aligned}\Omega_L=&\mathrm{d}q^1\wedge\mathrm{d}\dot q^1+\mathrm{d}q^2\wedge\mathrm{d}\dot q^2+\mathrm{d}q^3\wedge\mathrm{d}\dot q^3\\&+\dot q^1\mathrm{d}\dot q^1\wedge\mathrm{d}t+\dot q^2\mathrm{d}\dot q^2\wedge\mathrm{d}t+\dot q^3\mathrm{d}\dot q^3\wedge\mathrm{d}t.\end{aligned}\tag{50}$$

$$\alpha_1=\mathrm{d}q^1+at\mathrm{d}q^2-\left(\dot q^1+at\dot q^2\right)\mathrm{d}t,\tag{51}$$

$$\alpha_2=\begin{cases}0&\text{if}\quad\Phi^2>0,\\-at\mathrm{d}q^1+\mathrm{d}q^3+\left(at\dot q^1-\dot q^3\right)\mathrm{d}t&\text{if}\quad\Phi^2=0,\end{cases}\tag{52}$$

and the vector fields X_L and Z_1, Z_2 turn out to be

$$X_L=\frac{\partial}{\partial t}+\dot q^k\frac{\partial}{\partial q^k},\tag{53}$$

$$Z_1=-\frac{\partial}{\partial\dot q^1}-at\frac{\partial}{\partial\dot q^2},\tag{54}$$

$$Z_2=\begin{cases}0&\text{if}\quad\Phi^2>0,\\at\dfrac{\partial}{\partial\dot q^1}-\dfrac{\partial}{\partial\dot q^3}&\text{if}\quad\Phi^2=0.\end{cases}\tag{55}$$

Thus, the λ^A-dependent vector field Γ_λ takes the form

$$\Gamma_\lambda=\frac{\partial}{\partial t}+\dot q^k\frac{\partial}{\partial q^k}+\lambda^1\frac{\partial}{\partial\dot q^1}+\lambda^1at\frac{\partial}{\partial\dot q^2}\quad\text{if}\quad\Phi^2>0,\tag{56}$$

$$\begin{aligned}\Gamma_\lambda=&\frac{\partial}{\partial t}+\dot q^k\frac{\partial}{\partial q^k}+\lambda^1\frac{\partial}{\partial\dot q^1}+\lambda^1at\frac{\partial}{\partial\dot q^2}\\&-\lambda^2at\frac{\partial}{\partial\dot q^1}+\lambda^2\frac{\partial}{\partial\dot q^3}\quad\text{if}\quad\Phi^2=0,\end{aligned}\tag{57}$$

and its integral curves satisfy

$$\ddot q^1=\lambda^1,\quad\ddot q^2=at\lambda^1,\quad\ddot q^3=0\quad\text{if}\quad\Phi^2>0,\tag{58}$$

$$\ddot q^1=\lambda^1-at\lambda^2,\quad\ddot q^2=at\lambda^1,\quad\ddot q^3=\lambda^2\quad\text{if}\quad\Phi^2=0.\tag{59}$$

The functions λ^1 and λ^2 are to be obtained by imposing the conditions

$$\Gamma_\lambda\left(\Phi^1\right)=0\quad\text{if}\quad\Phi^2>0,\tag{60}$$

$$\Gamma_\lambda\left(\Phi^1\right)=0,\Gamma_\lambda\left(\Phi^2\right)=0\quad\text{if}\quad\Phi^2=0,\tag{61}$$

thus we obtain

$$\lambda^1=-\frac{1}{1+a^2t^2},\quad\lambda^2=0\quad\text{if}\quad\Phi^2>0,\tag{62}$$

$$\begin{aligned}\lambda^1=&-\frac{1+a^2t^2}{1+a^2t^2+a^4t^4},\\\lambda^2=&-\frac{at}{1+a^2t^2+a^4t^4}\quad\text{if}\quad\Phi^2=0.\end{aligned}\tag{63}$$

Hence, the dynamics for the time-dependent system with unilateral constraint is given by

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} - \frac{1}{1+a^2t^2}\frac{\partial}{\partial \dot{q}^1} - \frac{at}{1+a^2t^2}\frac{\partial}{\partial \dot{q}^2} \quad \text{if} \quad \Phi^2 > 0, \tag{64}$$

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} - \frac{1}{1+a^2t^2+a^4t^4} \times \left[\frac{\partial}{\partial \dot{q}^1} + at\left(1+a^2t^2\right)\frac{\partial}{\partial \dot{q}^2} + at\frac{\partial}{\partial \dot{q}^3}\right] \quad \text{if} \quad \Phi^2 = 0. \tag{65}$$

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