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Momentum-dependent symmetries and non-Noether conserved quantities for nonholonomic nonconservative Hamilton canonical systems*

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This paper investigates the momentum-dependent symmetries for nonholonomic nonconservative Hamilton canonical systems. The definition and determining equations of the momentum-dependent symmetries are presented, based on the invariance of differential equations under infinitesimal transformations with respect to the generalized coordinates and generalized momentums. The structure equation and the non-Noether conserved quantities of the systems are obtained. The inverse issues associated with the momentum-dependent symmetries are discussed. Finally, an example is discussed to further illustrate the applications.

Keywords: nonholonomic nonconservative Hamiltonian system, momentum-dependent symmetry, infinitesimal transformation, Lie group

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1. Introduction

The symmetric theorems are extremely useful tools for finding conserved quantities in mathematics, physics and mechanics. For a Lagrange system, that the most well-known types of symmetries are of course the Noether symmetries, which directly show that there is a close connection between the conserved quantities of a Lagrange dynamical system and those transformations with respect to coordinates and time that preserve the action integral.[1] Djukić and Vujanović made great progresses and give the Noether symmetries of the non-conservative mechanical systems.[2,3] Thereafter, studies of Noether symmetries have become very active in many areas.[4−9] Another method to obtain the conserved quantities of mechanical and physical systems is the Lie symmetry (also called the dynamical symmetry), in which the infinitesimal transformations with respect to space–time retains the equations of motion.[10−12] Now, the symmetry method has many applications in modern research fields.[13−25]

A theorem regarding the conserved quantities for second-order dynamical systems has been given by Hojman,[26] subsequently an ordinary theorem was obtained by González-Gascón[27] using geometric techniques. Expressed in generalized coordinates, the result may be stated as follows: Let the function $\eta_l(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ determine a symmetry operator $E = \eta_l\dfrac{\partial}{\partial q_l} + \dot{\eta}_l\dfrac{\partial}{\partial \dot{q}_l}$ for the equations of motion $\ddot{q}_l = \alpha_l(\boldsymbol{q},\dot{\boldsymbol{q}},t)$; if a function $\lambda(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ can be found that the quantity

$$\Omega = \frac{\partial \alpha_l}{\partial \dot{q}_l} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda$$

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vanishes, then a constant of the motion is given by

$$\phi = E(\ln\lambda) + \frac{\partial\eta_l}{\partial q_l} + \frac{\partial\dot{\eta}_l}{\partial\dot{q}_l}.$$

Recently, Lutzky applied the theorem to derive a certain conserved quantity for Lagrange systems which is associated with the presence of a velocity-dependent symmetry.[28] In addition, the generalized theorem shows that it is sufficient for Ω to be an invariant of the symmetry group in order for ϕ to be conserved.

In this paper, we present a method to directly derive the non-Noether conserved quantities for nonconservative nonholonomic Hamilton canonical systems which is associated with the presence of a momentum-dependent symmetries.

2. Momentum-dependent symmetries and non-Noether conserved quantities of the systems

The Hamiltonian $H(t,\boldsymbol{q},\boldsymbol{p})$ with n degrees of freedom, subject to nonpotential generalized force $Q_s(t,\boldsymbol{q},\boldsymbol{p})$ and nonholonomic constrained forces as

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0, \quad (\beta = 1,\cdots,g), \tag{1}$$

where

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p}), \quad \boldsymbol{q} = \{q_1,\cdots,q_2\}, \quad \boldsymbol{p} = \{p_1,\cdots,p_2\}$$

express generalized coordinates and generalized momentums respectively, has the generalized Hamilton canonical equations[29,30]

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s + \lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s} \quad (s = 1,\cdots,n). \tag{2}$$

Equation (2) can be put in the form

$$\dot{q}_s = h_s(t,\boldsymbol{q},\boldsymbol{p}), \quad \dot{p}_s = g_s(t,\boldsymbol{q},\boldsymbol{p}). \tag{3}$$

Let us introduce the infinitesimal transformations with respect to generalized coordinates and generalized momentums

$$\begin{aligned} q_s^*(t) &= q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\boldsymbol{p}), \\ p_s^*(t) &= p_s(t) + \varepsilon\eta_s(t,\boldsymbol{q},\boldsymbol{p}), \end{aligned} \tag{4}$$

here ε is a small parameter, and ξ_s and η_s are the generators of the infinitesimal transformations.

If Eq.(3) is form invariant under the infinitesimal transformation (4) of one-parameter Lie group, the infinitesimal generators $\xi_s(t,\boldsymbol{q},\boldsymbol{p})$ and $\eta_s(t,\boldsymbol{q},\boldsymbol{p})$ satisfy the determining equations

$$\dot{\xi}_s = E^0(h_s), \ \dot{\eta}_s = E^0(g_s), \tag{5}$$

and restricting equations

$$E^0(\tilde{f}_\beta(t,\boldsymbol{q},\boldsymbol{p})) = 0, \tag{6}$$

where the E^0 is the symmetry operator,[31] given by

$$E^0 = \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}, \tag{7}$$

and its first extended form

$$E^1 = E^0 + \dot{\xi}_s\frac{\partial}{\partial\dot{q}_s} + \dot{\eta}_s\frac{\partial}{\partial\dot{p}_s} \tag{8}$$

and the vector field

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + h_s\frac{\partial}{\partial q_s} + g_s\frac{\partial}{\partial p_s} \tag{9}$$

represents the total time derivative along the trajectories of Eq.(2), then for any function ϕ

$$\dot{\phi} = \frac{\partial\phi}{\partial t} + h_s\frac{\partial\phi}{\partial q_s} + g_s\frac{\partial\phi}{\partial p_s}. \tag{10}$$

Based on the invariance of the differential equations under the infinitesimal transformation (4), if the Eq.(3) is invariant under the transformation of continuous Lie group (4), the invariance is called the momentum-dependent symmetry of the corresponding nonholonomic nonconservative Hamiltonian systems.

As the transformation (4) is called velocity-dependent symmetry in state space in Refs.[28, 29, 32] when

$$t^* = t, \quad q_s^*(t) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{4a}$$

the present authors call the symmetry (4) the momentum-dependent symmetry in phase space.

Equation (5) may be regarded as a criterion of the momentum-dependent symmetry, i.e.,

Criterion 1 The symmetry is momentum-dependent associated with the holonomic nonconservative system (4) if the infinitesimal generators ξ_s and η_s satisfy the determining Eq.(5).

Proof The invariance of Eq.(3) under the infinitesimal transformation (4) can be represented as

$$\begin{aligned} E^1(\dot{q}_s - h_s(t,\boldsymbol{q},\boldsymbol{p})) = 0, \\ E^1(\dot{p}_s - g_s(t,\boldsymbol{q},\boldsymbol{p})) = 0. \end{aligned} \tag{11}$$

Using formulae (3), (7) and (8) in the (11) leads to the Eq.(5).

Criterion 2 The symmetry is momentum-dependent

associated with the nonholonomic nonconservative system (1) and (2) if the infinitesimal generators ξ_s and η_s satisfy the determining Eq.(5) and restricting Eq.(6).

Proof The invariance of Eqs.(3) and (1) under the infinitesimal transformation (4) can be represented in the forms (11) and (6). Using formulae (3), (7) and (8) in the (11) leads to the Eq.(5), and also leads directly to the Eq.(6).

In order to obtain the conserved quantity associated with the momentum-dependent symmetry, we give:

Theorem 1 If the infinitesimal transformation (4) of the Eqs.(1) and (3) is the momentum-dependent, the system possesses the conserved quantities in the form

$$\phi = E^0\left[G\right] + \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} \tag{12}$$

if and only if a gauge function G can be found such that the quantity

$$\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s} + \dot{G} = \psi. \tag{13}$$

We call Eq.(13) the structural equation of the momentum-dependent symmetry for nonholonomic nonconserved systems, and form (12) is a non-Noether conserved quantity associated with the system.

We should point out that the non-Noether conserved quantities associated with the symmetries could be obtained by taking different ψ.

Proof For the theorem, we show that $E^0\left(\psi\right) = 0$ implies $\dot{\phi} = 0$. This is accomplished by explicitly differentiating ϕ to demonstrate that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[E^0\left(G\right)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial \dot{q}_s}\right]. \tag{14}$$

It is straightforward to show that for any function$f = f\left(t, \boldsymbol{q}, \boldsymbol{p}\right)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_s} = \frac{\partial \dot{f}}{\partial q_s} - \frac{\partial f}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial f}{\partial p_k}\frac{\partial g_k}{\partial q_s}, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial p_s} = \frac{\partial \dot{f}}{\partial p_s} - \frac{\partial f}{\partial q_k}\frac{\partial h_k}{\partial p_s} - \frac{\partial f}{\partial p_k}\frac{\partial g_k}{\partial p_s}, \tag{16}$$

$$E^0\left(\frac{\partial f}{\partial q_s}\right) = \frac{\partial}{\partial q_s}E^0(f) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial q_s} - \frac{\partial f}{\partial p_k}\frac{\partial \eta_k}{\partial q_s}, \tag{17}$$

$$E^0\left(\frac{\partial f}{\partial p_s}\right) = \frac{\partial}{\partial p_s}E^0\left(f\right) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial p_s} - \frac{\partial f}{\partial p_k}\frac{\partial \eta_k}{\partial p_s}. \tag{18}$$

The condition that the generators ξ_s and η_s define a symmetry for $\dot{q}_s = h_s\left(t, \boldsymbol{q}.\boldsymbol{p}\right)$, $\dot{p}_s = g_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right)$ are given by $\dot{\xi}_s = E^0(h_s)$, $\dot{\eta}_s = E^0(g_s)$, and the ξ_s and η_s satisfy form (6); using this, we can show that if E^0 is a symmetry operator, then

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0\left(f\right) = E^0\left(\dot{f}\right). \tag{19}$$

Differentiating ϕ we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = E^0(\dot{G}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right), \tag{20}$$

here we have used (19).

From Eqs.(15) and (16) we can show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_s}{\partial q_l} + \frac{\partial \eta_s}{\partial p_s}\right)\\ =&\frac{\partial \dot{\xi}_s}{\partial q_s} - \frac{\partial \xi_s}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial \xi_s}{\partial p_k}\frac{\partial g_k}{\partial q_s}\\ &+ \frac{\partial \dot{\eta}_s}{\partial p_s} - \frac{\partial \eta_s}{\partial q_k}\frac{\partial h_k}{\partial p_s} - \frac{\partial \eta_s}{\partial p_k}\frac{\partial g_k}{\partial p_s}.\end{aligned} \tag{21}$$

Furthermore, Eqs.(17) and (18) allow us to show that

$$\begin{aligned}&E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s}\right)\\ =&\frac{\partial E^0(h_s)}{\partial q_s} - \frac{\partial \xi_s}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial \eta_s}{\partial q_k}\frac{\partial h_k}{\partial p_s}\\ &+ \frac{\partial E^0(g_s)}{\partial p_s} - \frac{\partial \xi_s}{\partial p_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial \eta_s}{\partial p_k}\frac{\partial g_k}{\partial p_s}.\end{aligned} \tag{22}$$

By the determining Eq.(5) in Eqs.(21) and (22) we can derive that the right-hand sides of them is equal, so that we may put

$$E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right]. \tag{23}$$

Substituting Eq.(23) into Eq.(14) yields

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s} + \dot{G}\right) = E^0(\psi). \tag{24}$$

Consequently Eq.(12) leads to the conclusion that ϕ is a conserved quantity when $E^0(\psi) = 0$.

Now, we show have to apply the theorem which directly derives the conserved quantities for nonholonomic nonconservative systems which is associated with the presence of momentum-dependent symmetries. In addition, we derive the theorem by showing that it is sufficient for ψ to be an invariant of the symmetry group in order for ϕ to be conserved.

3. Inverse issues of the momentum-dependent symmetries for the systems

Finding the momentum-dependant symmetries from the given conserved quantities is the inverse problem, which can be solved by the following steps.

Firstly, suppose that there exists the conserved quantity in the nonholonomic Hamiltonian system

$$\phi = \phi(\boldsymbol{q}, \boldsymbol{p}, t) = \text{const}, \tag{25}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial q_s}\dot{q}_s + \frac{\partial\phi}{\partial p_s}\dot{p}_s = 0, \tag{26}$$

i.e.,

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial q_s}\frac{\partial H}{\partial p_s} + \frac{\partial\phi}{\partial p_s}\dot{p}_s = 0. \tag{27}$$

Multiplying (2) by ξ_s then summing up for s leads to

$$\left(-\dot{p}_s - \frac{\partial H}{\partial q_s} + Q_s\right)\xi_s = 0. \tag{28}$$

Adding Eqs.(27) and (28), and letting the coefficients of the $\dot{p}_s$ to be zero yields

$$\xi_s = \frac{\partial\phi}{\partial p_s}. \tag{29}$$

Let the conserved quantity in Eq.(11) be equal to equation (25), i.e.

$$E^0(G) + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial\eta_s}{\partial p_s} = \phi, \tag{30}$$

then we can obtain generators ξ_s and η_s from (29) and (30) when the gauge function G is obtained by form (13).

Secondly, we determine the momentum-dependent symmetry by substituting generators ξ_s and η_s into Eqs.(5) and (6) hold. We call the symmetry the momentum-dependent symmetry of the nonholonomic nonconservative system (1) and (3) when the infinitesimal transformation generators ξ_s and η_s are determined by (29) and (30) and satisfy the determining equation (5) and restricting equations (6).

4. An example

The Hamiltonian system

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V \tag{31}$$

is subject to nonholonomic constraint

$$f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0, \tag{32}$$

where V is the potential energy of the system , b is a constant. Let us study the non-Noether conserved quantities of the system.

Equation (2) yields

$$\begin{aligned} &\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \\ &\dot{p}_1 = \lambda, \quad \dot{p}_2 = \lambda bt, \end{aligned} \tag{33}$$

in which λ is a coefficient. Using Eqs.(2) and (33), one obtains

$$\lambda = -\frac{1}{1 + b^2t^2}. \tag{34}$$

Equation (33) can be put in the form

$$\begin{aligned} &\dot{q}_1 = p_1 = h_1, \quad \dot{q}_2 = p_2 = h_2, \\ &\dot{p}_1 = -\frac{1}{1 + b^2t^2} = g_1, \\ &\dot{p}_2 = -\frac{bt}{1 + b^2t^2} = g_2. \end{aligned} \tag{35}$$

The determining equations of the momentum-dependent symmetry in this case are

$$\dot{\xi}_1 = \eta_1, \quad \dot{\xi}_2 = \eta_2, \quad \dot{\eta}_1 = 0, \quad \dot{\eta}_2 = 0, \tag{36}$$

Equation (36) has the following solutions,

$$\begin{aligned} &\xi_1 = c_1t + c_3, \quad \xi_2 = c_2t + c_4, \\ &\eta_1 = c_1, \quad \eta_2 = c_2, \end{aligned} \tag{37}$$

and restricting equations,

$$\begin{aligned} &E^0(\dot{q}_1 + bt\dot{q}_2 - bq_2 + t) \\ =&E^0(p_1 + btp_2 - bq_2 + t) \\ =&c_1 + btc_2 - b(c_2t + c_4) = 0, \end{aligned} \tag{38}$$

then one has $c_1 = c_4$.

Substituting Eqs.(34) and (36) into Eq.(13) yields

$$\dot{G} = \psi. \tag{39}$$

In the Eq.(36), we take $\psi = c_5$ ($E^0(c_5) = 0$), then the gauge function is

$$\begin{aligned} G =&\exp\left(p_1 + \frac{1}{b}\arctan(bt)\right) \\ &+ \exp\left(p_2 + \frac{1}{2b}\ln(1 + b^2t^2)\right) + c_5t. \end{aligned} \tag{40}$$

Substituting Eqs.(37) and (40) into Eq.(12) leads to the non-Noether conserved quantity

$$\begin{aligned} \phi =&c_1\exp\left(p_1 + \frac{1}{b}\arctan(bt)\right) \\ &+ c_2\exp\left(p_2 + \frac{1}{2b}\ln(1 + b^2t^2)\right) = \text{const}. \end{aligned} \tag{41}$$

5. Connections between the momentum-dependent symmetries and both the Lie symmetries and Noether symmetries

Now, we show the connections between the momentum-dependent symmetries and both the Lie symmetries and Noether symmetries.

Firstly, Lie symmetries can be called that the Noether-type Lie symmetries when the symmetries also preserve the action integral of the systems, and the systems possess the Noether conserved quantities; and Lie symmetries can be called the non-Noether-type Lie symmetries (also called non-Noether symmetries) when the Lie symmetries can not retain the action integral of the systems, and the systems possess the non-Noether-type conserved quantities. Where, the non-Noether symmetries in phase space are called the momentum-dependent symmetries by authors, and the systems possess the non-Noether conserved quantities in phase space. In phase space, Lie symmetries is also called the momentum-dependent symmetries and possess non-Noether conserved quantities when the transformation do not contain one with respect to time and can not protect the action integral of the systems.

Secondly, the momentum-dependent symmetries of the nonholonomic nonconservative Hamiltonian canonical systems can lead directly to the non-Noether conserved quantities; the momentum-dependent symmetries and Noether symmetries of dynamical systems are two different symmetries; above all, their definition are different; next, the gauge function of the momentum-dependent symmetries is determined by both Eq.(13) and via choosing invariant ψ of the symmetry group before the conserved quantities are given with (12); nevertheless, the conserved quantities of the Noether symmetries are obtained with Noether theorem. The momentum-dependent symmetries and Lie symmetries of the dynamical systems are also different, the gauge function and the conserved quantities of the Lie symmetries are obtained using structural equation and the formula of Noether conserved quantities, respectively.

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