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Higher-order squeezing of quantum field and higher-order uncertainty relations in non-degenerate four-wave mixing*

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It is found that the field of the combined mode of the probe wave and the phase conjugate wave in the process of non-degenerate four-wave mixing exhibits higher-order squeezing to all even orders. The higher-order squeezed parameter and squeezed limit due to the modulation frequency are investigated. The smaller the modulation frequency is, the stronger the degree of higher-order squeezing becomes. Furthermore, the higher-order uncertainty relations in the process of non-degenerate four-wave mixing are presented for the first time. The product of higher-order noise moments is related to even order number N and the length L of the medium.

Keywords: higher-order squeezing, higher-order uncertainty relations, quantum fields, non-degenerate four-wave mixing
PACC: 4250, 0365, 4265, 4265F

1. Introduction

Squeezed states[1−4] play a prominent role in modern quantum optics. They have drawn a great deal of interest in connection with low-noise communications, high-precession interferometric measurement , quantum teleportation with continuous variables.[5−8]

The generation of squeezed state requires a nonlinear phase-dependent interaction. The first observation of the second-order squeezing of quantum field was achieved by Slusher *et al* in 1985 at Bell laboratories in the process of four-wave mixing in atomic sodium.[9] This was soon followed by demonstration of squeezing in four-wave mixing in an optical fibre by Levenson *et al.*[10] And a number of articles[11−14] and books[15,16] on the theory of second-order squeezing in four-wave mixing have been published and can be consulted for more details.

With the development of techniques for making higher-order correlation measurement in quantum optics, the new concept of higher-order squeezing of the single-mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985.[17−19] Lately Li and Shan have calculated the higher-order squeezing in the process of degenerate four-wave mixing.[20] As a natural generalization of Hong and Mandel's work, we established the quantum theory of higher-order squeezing of the quantum fields in two-mode squeezed states in 2004.[21] The above theory can be applied to non-degenerate two-photon nonlinear optical processes, in which the two-mode output field exhibits higher-order squeezing. In this paper we confine our discussion only to the process of non-degenerate four-wave mixing (NDFWM). The properties of this higher-order squeezing in NDFWM are investigated and the higher-order uncertainty relations among higher-order noise moments of quadrature components in NDFWM are given for

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the first time.

2. Definition of higher-order squeezing of two mode quantum fields

The real two mode output field $\hat{E}$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$,

$$\hat{E} = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi). \tag{1}$$

Here, $\hat{E}_1$ and $\hat{E}_2$ are canonical conjugates, and obey the commutation relations

$$[\hat{E}_1, \hat{E}_2] = 2\mathrm{i}C_0, \tag{2}$$

where C_0 is the commutator.

Then the field is squeezed to the Nth-order in $\hat{E}_1 (N = 2, 4, 6, \cdots)$ if there exists a phase angle ϕ such that $\langle(\triangle\hat{E}_1)^N\rangle$ is smaller than its value in a completely two-mode coherent state of the field, viz

$$\langle(\triangle\hat{E}_1)^N\rangle < \langle(\triangle\hat{E}_1)^N\rangle_{\text{two-mode coh.s.}} \tag{3}$$

This is the definition of higher-order squeezing of two mode quantum fields.

3. Scheme for generation of higher-order squeezing via NDFWM

The scheme for generation of higher-order squeezing via NDFWM is shown in Fig.1.

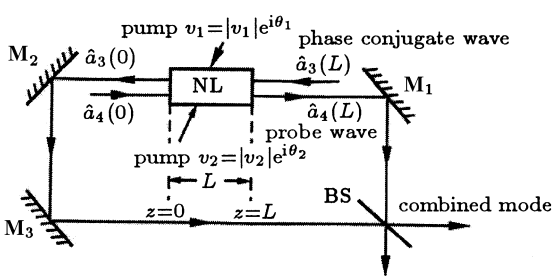


Fig.1. Schematic for generation of higher-order squeezing via NDFWM. M_1, M_2, M_3 are mirrors, BS is the 50%–50% beam splitter.

It is shown in the figure that two strong, classical pump waves of complex amplitude ($v_1 = |v_1|e^{i\theta_1}$ and $v_2 = |v_2|e^{i\theta_2}$) with the same frequency Ω are incident on a nonlinear crystal possessing a third-order ($\chi^{(3)}$) nonlinearity. The length of the medium is L. $\hat{a}_4$ is the annihilation operator of the probe wave with the frequency ω_4, $\hat{a}_3$ is the annihilation operator of the phase conjugate wave with the frequency ω_3, and

$$\Omega = \frac{\omega_3 + \omega_4}{2}. \tag{4}$$

The effective Hamiltonian of this interaction system has the form of

$$\begin{aligned}\hat{H} =& \hbar\omega_3\hat{a}_3^\dagger\hat{a}_3 + \hbar\omega_4\hat{a}_4^\dagger\hat{a}_4 \\ &+ \hbar g_0(v_1 v_2 \hat{a}_3^\dagger\hat{a}_4^\dagger e^{-2\mathrm{i}\Omega t} + \mathrm{H.C}),\end{aligned} \tag{5}$$

where g_0 is the coupling const, t is the propagation time of light in NL crystal.

By solving the Heisenberg equation of motion we obtain the output mode

$$\begin{aligned}\hat{a}_3(t) =& [\mu\hat{a}_3(L) + \nu\hat{a}_4^\dagger(0)]e^{-\mathrm{i}\omega_3 t} \\ & (z = L - ct \quad \text{for} \quad \hat{a}_3),\end{aligned} \tag{6}$$

$$\begin{aligned}\hat{a}_4(t) =& [\mu\hat{a}_4(0) + \nu\hat{a}_3^\dagger(L)]e^{-\mathrm{i}\omega_4 t} \\ & (z = ct \quad \text{for} \quad \hat{a}_4),\end{aligned} \tag{7}$$

where

$$\left.\begin{aligned}\mu &= \sec|k|L, \\ \nu &= -\mathrm{i}e^{\mathrm{i}(\theta_1+\theta_2)}\tan|k|L, \\ |k| &= \frac{g_0|v_1||v_2|}{c}.\end{aligned}\right\} \tag{8}$$

4. Combined mode and its quadrature components

It can be verified that the field of either $\hat{a}_3(0)$ or $\hat{a}_4(L)$ mode does not exhibit higher-order squeezing.

We consider the field of the combined mode of $\hat{a}_3(t)$ and $\hat{a}_4(t)$

$$\begin{aligned}\hat{E}(t) =& \lambda\sqrt{\frac{\omega_3}{2}}\hat{a}_3(t) - \mathrm{i}\lambda\sqrt{\frac{\omega_4}{2}}\hat{a}_4(t) + \mathrm{H.C} \\ =& \lambda\sqrt{\frac{\Omega}{2}}\lambda_3\hat{a}_3(t) - \mathrm{i}\lambda\sqrt{\frac{\Omega}{2}}\lambda_4\hat{a}_4(t) + \mathrm{H.C},\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\lambda_3 = \sqrt{\frac{\omega_3}{\Omega}}, \quad \lambda_4 = \sqrt{\frac{\omega_4}{\Omega}}, \\ &\lambda = \sqrt{\frac{\hbar}{V\varepsilon_0}} = \sqrt{\frac{1}{V\varepsilon_0}},\end{aligned} \tag{10}$$

V is the mode volume and $-\mathrm{i}$ denotes the phase delay. The units are chosen so that $\hbar = c = 1$.

$\hat{E}(t)$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$,

$$\hat{E}(t) = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi), \tag{11}$$

where

$$\Omega = \frac{\omega_3 + \omega_4}{2}, \tag{12}$$

and ϕ is an arbitrary phase angle that may be chosen at will.

$\hat{E}_1$ can be expressed in terms of initial modes $\hat{a}_3(L)$ and $\hat{a}_4(0)$,

$$\hat{E}_1 = g\hat{a}_3(L) + h\hat{a}_4(0) + g^*\hat{a}_3^\dagger(L) + h^*\hat{a}_4^\dagger(0), \tag{13}$$

where

$$g = \lambda\sqrt{\frac{\Omega}{2}}[\lambda_3\mu e^{-i\phi} + \lambda_4\nu^* e^{i(\phi+\pi/2)}]e^{i\epsilon t}, \tag{14}$$

$$h = \lambda\sqrt{\frac{\Omega}{2}}[\lambda_4\mu e^{-i(\phi+\pi/2)} + \lambda_3\nu^* e^{i\phi}]e^{-i\epsilon t}, \tag{15}$$

$$\epsilon = \Omega - \omega_3 = \omega_4 - \Omega, \tag{16}$$

ϵ is the modulation frequency.

Now we introduce a new operator $\hat{B}$, defined by

$$\hat{B} = g\hat{a}_3(L) + h\hat{a}_4(0). \tag{17}$$

The adjoint of $\hat{B}$ is $\hat{B}^\dagger$,

$$\hat{B}^\dagger = g^*\hat{a}_3^\dagger(L) + h^*\hat{a}_4^\dagger(0). \tag{18}$$

Then we can express $\hat{E}_1$ in terms of $\hat{B}$ and $\hat{B}^\dagger$ operators in the form

$$\hat{E}_1 = \hat{B} + \hat{B}^\dagger. \tag{19}$$

We call $\hat{B}$ the combined two-mode squeezed annihilation operator in NDFWM. The operator $\hat{B}$ can be used to calculate the higher-order noise moment $\langle(\Delta\hat{E}_1)^N\rangle$ in NDFWM.

5. Higher-order noise moment $\langle(\Delta\hat{E}_1)^N\rangle$ and higher-order squeezing

The calculation of the higher-order noise moment $\langle(\Delta\hat{E}_1)^N\rangle$ is very complex . However, Applying the above new combined two-mode squeezed annihilation operator $\hat{B}$, we can calculate the higher-order noise moment in NDFWM easily.

By using the Campbell–Baker–Hausdorff formula, we express the higher-order noise moment of ΔE_1 in the following relation:[17–19]

$$\langle(\Delta\hat{E}_1)^N\rangle = \langle::(\Delta\hat{E}_1)^N::\rangle + \frac{N^{(2)}}{1!}\left(\frac{1}{2}C_0\right)\langle::(\Delta\hat{E}_1)^{N-2}::\rangle + \frac{N^{(4)}}{2!}\left(\frac{1}{2}C_0\right)^2\langle::(\Delta\hat{E}_1)^{N-4}::\rangle$$
$$+\cdots+\begin{cases}(N-1)!!C_0^{N/2} & (N \text{ is even}),\\ \dfrac{N!}{3!2^{(1/2)N-3/2}\left(\frac{1}{2}N-\frac{3}{2}\right)!}\times\dfrac{C_0^{(1/2)N-3/2}}{\left(\frac{1}{2}N-\frac{3}{2}\right)!}\langle::(\Delta\hat{E}_1)^3::\rangle & (N \text{ is odd}),\end{cases} \tag{20}$$

where

$$N^{(r)} = N(N-1)\cdots(N-r+1),$$
$$C_0 = \frac{1}{2i}[\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^\dagger], \tag{21}$$

and :: :: denotes normal ordering with respect to $\hat{B}$ and $\hat{B}^\dagger$.

We take the initial quantum state as $|\alpha\rangle_4|0\rangle_3$, which is a product of the coherent state $|\alpha\rangle_4$ for $\hat{a}_4(0)$ mode and the vacuum state $|0\rangle_3$ for $\hat{a}_3(L)$ mode. Since $|\alpha\rangle_4|0\rangle_3$ is the eigenstate of $\hat{B}$, we obtain

$$\langle::(\Delta\hat{E}_1)^N::\rangle = \langle::(\Delta\hat{B}+\Delta\hat{B}^\dagger)^N::\rangle$$
$$= \sum_{\gamma=0}^{N}\begin{bmatrix}N\\ \gamma\end{bmatrix}\ {}_3\langle 0|_4\langle\alpha|::(\Delta\hat{B}^\dagger)^\gamma(\Delta\hat{B})^{N-\gamma}::|\alpha\rangle_4|0\rangle_3 = 0. \tag{22}$$

Then from Eq.(20) we acquire

$$\langle(\Delta\hat{E}_1)^N\rangle = (N-1)!!C_0^{N/2}, \tag{23}$$

and

$$\begin{aligned} C_0 =&[\hat{B},\hat{B}^\dagger] = |g|^2+|h|^2, \\ =&\frac{\Omega}{2}\lambda^2\{(\lambda_3^2+\lambda_4^2)(|\mu|^2+|\nu|^2) \\ &+2\lambda_3\lambda_4[\mu^*\nu^*\mathrm{e}^{\mathrm{i}(2\phi+\frac{\pi}{2})}+\mu\nu\mathrm{e}^{-\mathrm{i}(2\phi+\frac{\pi}{2})}]\}. \end{aligned} \tag{24}$$

where

$$\lambda_3^2+\lambda_4^2=2, \qquad \lambda_3\lambda_4=\sqrt{1-\frac{\epsilon^2}{\Omega^2}}.$$

Substituting Eqs.(8), (10), (24) into Eq.(23), we have the Nth-order moment of $\Delta\hat{E}_1$,

$$\begin{aligned} &\langle(\Delta\hat{E}_1)^N\rangle \\ =&(N-1)!!\lambda^N\Omega^{N/2}[\sec^2|k|L+\tan^2|k|L \\ &-2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}\sec|k|L\tan|k|L\cos(2\phi-\theta_1-\theta_2)]^{N/2}. \end{aligned} \tag{25}$$

If ϕ is chosen to satisfy

$$2\phi-\theta_1-\theta_2=0, \quad \text{or} \quad \cos(2\phi-\theta_1-\theta_2)=1,$$

then the above Eq.(25) leads to the result

$$\begin{aligned} \langle(\Delta\hat{E}_1)^N\rangle =&(N-1)!!\lambda^N\Omega^{N/2}[\sec^2|k|L+\tan^2|k|L \\ &-2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}\sec|k|L\tan|k|L]^{N/2}. \end{aligned} \tag{26}$$

When $0<|k|L<\pi$, for most L the right-hand side is less than $(N-1)!!\lambda^N\Omega^{N/2}$, which is the corresponding Nth-order moment for two-mode coherent states. It follows that the field of the combined mode of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders.

6. Higher-order squeezed parameter in NDFWM

The squeeze parameter q_N for measuring the degree of the Nth-order squeezing in NDFWM is

$$q_N = \frac{\langle(\Delta\hat{E}_1)^N\rangle - \langle(\Delta\hat{E}_1)^N\rangle_{\text{two-mode coh.s}}}{\langle(\Delta\hat{E}_1)^N\rangle_{\text{two-mode coh.s}}} \tag{27}$$

$$= \left[\sec^2|k|L+\tan^2|k|L - 2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}\sec|k|L\tan|k|L\right]^{N/2} - 1. \tag{28}$$

The numerical results of the higher-order squeeze parameter q_N versus $|k|L$ are illustrated in Figs.2(a), (b), (c).

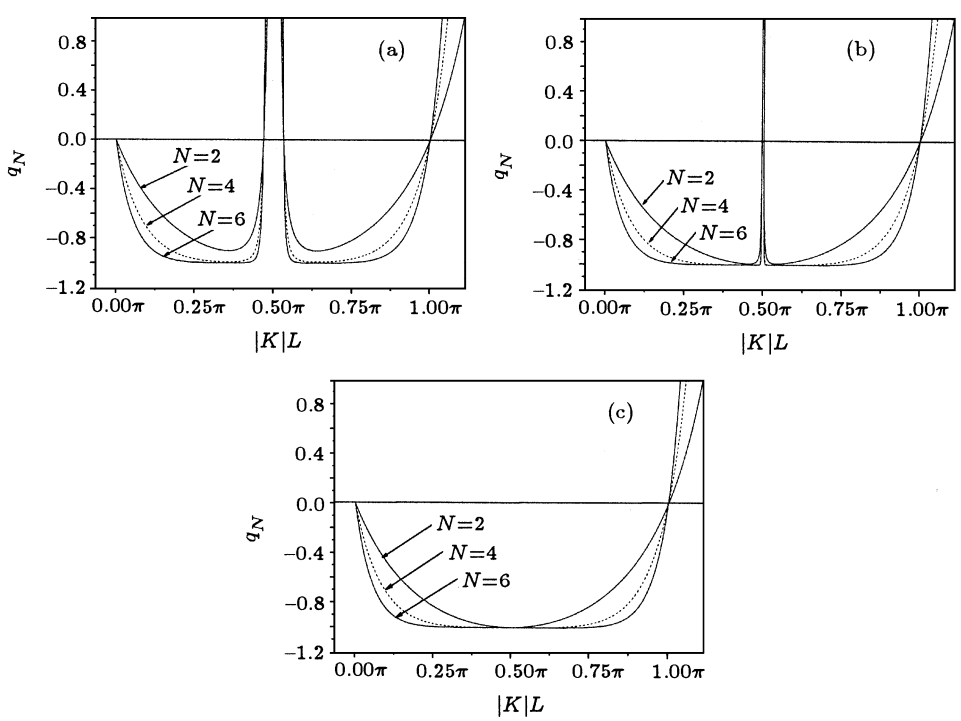


Fig.2. Higher-order squeeze parameter q_N versus $|k|L$ for (a) $\frac{\epsilon}{\Omega}$=0.1, (b) $\frac{\epsilon}{\Omega}$ = 0.01, (c) $\frac{\epsilon}{\Omega}$=0.

From Eq.(28) and Figs.2 (a), (b), (c), it is found that

(1) For different values of $\frac{\epsilon}{\Omega}$, the higher-order squeeze parameter q_N is a function of $|k|L$ and even order number N.

(2) When $0 < |k|L < \pi$, q_N is negative for most L and has minimum values $q_{N\min}$. When $\epsilon = 0$, q_N is negative for all L and $q_{N\min}$ may approach the maximum negative value -1.

(3) $|q_N|$ increases with N, this gives out the conclusion that the degree of higher-order squeezing is greater than that of the second order.

(4) The smaller the modulation frequency ϵ is, the larger the degree of higher-order squeezing becomes.

7. Higher-order uncertainty relations in NDFWM

We now derive and discuss the higher-order uncertainty relations among higher-order noise moments of quadrature components $\hat{E}_1$ and $\hat{E}_2$ in NDFWM.

$\hat{E}_2$ can be regarded as a particular case of $\hat{E}_1$ if ϕ is replaced by $\phi + \pi/2$. Then if ϕ is chosen to satisfy $2\phi - \theta_1 - \theta_2 = 0$, from Eq.(25) it follows that

$$\langle(\Delta\hat{E}_2)^N\rangle = (N-1)!!\lambda^N\Omega^{N/2}\left[\sec^2|k|L + \tan^2|k|L + 2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}\sec|k|L\tan|k|L\right]^{N/2}. \quad (29)$$

When $0 < |k|L < \pi$, the right-hand side is greater than $(N-1)!!\lambda^N\Omega^{N/2}$ for most L.

From Eqs.(26) and (29), we obtain

$$\langle(\Delta\hat{E}_1)^N\rangle \cdot \langle(\Delta\hat{E}_2)^N\rangle = [(N-1)!!]^2\lambda^{2N}\Omega^N \times \left[1 + 4\frac{\epsilon^2}{\Omega^2}\sec^2|k|L\tan^2|k|L\right]^{N/2}. \quad (30)$$

This shows that the product of the two higher-order noise moments is related to the even order number N and the length L of the medium . For certain given values of N and L, the product is a constant. So Eq.(30) shows that $\langle(\Delta\hat{E}_1)^N\rangle$ and $\langle(\Delta\hat{E}_2)^N\rangle$ can not be made arbitrarily small simultaneously. We call Eq.(30) the higher-order uncertainty relations in NDFWM, which are the generalization of the ordinary second order uncertainty relations in NDFWM.

In the degenerate case $\omega_4 = \omega_3 = \Omega$ and $\epsilon = 0$, then from Eqs.(26),(28) and (30) we obtain

$$\langle(\Delta\hat{E}_1)^N\rangle = (N-1)!!\lambda^N\Omega^{N/2}[\sec|k|L - \tan|k|L]^N, \quad (31)$$

$$q_N = [\sec|k|L - \tan|k|L]^N - 1, \quad (32)$$

$$\langle(\Delta\hat{E}_1)^N\rangle \cdot \langle(\Delta\hat{E}_2)^N\rangle = [(N-1)!!]^2\lambda^{2N}\Omega^N. \quad (33)$$

When $N = 2$, we obtain

$$\langle(\Delta\hat{E}_1)^2\rangle = \lambda^2\Omega[\sec|k|L - \tan|k|L]^2, \quad (34)$$

$$q_2 = [\sec|k|L - \tan|k|L]^2 - 1, \quad (35)$$

$$\langle(\Delta\hat{E}_1)^2\rangle \cdot \langle\langle(\Delta\hat{E}_2)^2\rangle = \lambda^4\Omega^2 \quad (36)$$

These results are in agreement with the conclusions in the previous references.[20,10]

8. Conclusions

The higher-order squeezing of the quantum electromagnetic field and the higher-order uncertainty relations in the process of NDFWM have been studied. The major new results of this paper are as follows.

(1) We define a new combined two-mode squeezed annihilation operator $\hat{B}$ in the process of NDFWM ,which is the linear combination of the input modes to the four-wave mixer. By applying this operator $\hat{B}$, we can calculate the higher-order noise moment $\langle(\Delta\hat{E}_1)^N\rangle$ in NDFWM easily.

(2) We have studied the quantum theory of the higher-order squeezing of the quantum field in the process of NDFWM. It is found that if ϕ is chosen to satisfy $2\phi - \theta_1 - \theta_2 = 0$, and when $0 < |k|L < \pi$, the two-mode output field of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders in most cases.

(3)We have calculated the squeeze parameter q_N for measuring the degree of the Nth order squeezing. When $0 < |k|L < \pi$, q_N is negative for most L and has minimum values $q_{N\min}$. The smaller the modulation frequency ϵ is, the larger the degree of higher-order squeezing becomes. When $\epsilon = 0$, q_N is negative for all L and $q_{N\min}$ may approach the maximum negative value -1.

(4) $|q_N|$ increases with N, this gives out the conclusion that the degree of higher-order squeezing in the process of NDFWM is greater than that of the second order squeezing. This confirms that the higher-order squeezing of quantum fields is particularly attractive for applications from the point of view of noise reduction.

(5) We have presented the higher-order uncertainty relations among higher-order noise moments of quadrature components in the process of NDFWM. The product of the two higher-order noise moments

is related to even order number N and the length L of the medium. This higher-order uncertainty relations are the generalization of ordinary second order uncertainty relations in the process of NDFWM, and enable people to have a deeper understanding of the indeterminacy principle in quantum mechanics.

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