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Upper bound for the time derivative of entropy for a dynamical system driven by coloured cross-correlated white noises*

Xie Wen-Xian(谢文贤)[a)†], Xu Wei(徐 伟)[a)],
Cai Li(蔡 力)[b)], and Jin Yan-Fei(靳艳飞)[a)]

[a)] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China*

[b)] *College of Astronautics, Northwestern Polytechnical University, Xi'an 710072, China*

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It is shown how the cross-correlation time and strength of coloured cross-correlated white noises can set an upper bound for the time derivative of entropy in a nonequilibrium system. The value of upper bound can be calculated directly based on the Schwartz inequality principle and the Fokker–Planck equation of the dynamical system driven by coloured cross-correlated white noises. The present calculations can be used to interpret the interplay of the dissipative constant and cross-correlation time and strength of coloured cross-correlated white noises on the upper bound.

Keywords: information entropy, coloured cross-correlation, Fokker–Planck equation
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1. Introduction

In recent years, study of the information entropy and related quantities has become an important issue for the understanding of noise driven dynamical systems in details, such as Refs.[1–9]. In the context, Shannon's information measure

$$S = -\int P(x,t)\ln P(x,t)\mathrm{d}x \tag{1}$$

is a very important tool for the study of nonequilibrium and stationary states of a Brownian particle. Here S is called information entropy, and $P(x,t)$ is the continuous probability distribution function in phase space.

In general, the multiplicative noise and additive noise of a nonlinear system are taken as different origins. However, noises in certain situation may have a common origin. The cross-correlated noises were considered for stochastic resonance[10−13] and computation of escape rate,[14] a two-dimensional hydrodynamical problem,[15] etc. Bag *et al*[1,2] have studied the upper bound of the time derivative of information entropy for dynamical systems driven by external Ornstein–Uhlenbeck noise and white cross-correlated noise processes. The calculation shows that the upper bound exhibits extremal nature with the variation of system parameters such as noise correlation time, dissipative strength.

This paper extends the calculation of upper bound[1] to the dissipative dynamical system driven by coloured cross-correlated white noises (i.e. the coupling of the multiplicative and additive white noises is coloured with nonzero cross-correlation time τ), which is also thermodynamically open in the sense that they can be described by classical stochastic processes with the help of the standard Langevin equation.[10] We shall show how the properties of coloured cross-correlated white noises influence the upper bound of the rate of entropy change in a nonequilibrium system. Due to the introduction of coloured cross-correlated white noises, the nonzero cross-correlation time τ is expected to influence upper bound remarkably.

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†Corresponding author. E-mail: wenxianxie@mail.nwpu.edu.cn

2. The Fokker–Planck equation of the dynamical system driven by coloured cross-correlated white noises

We consider a simple dynamical system driven by both multiplicative and additive white noises with coloured cross-correlation time. The relevant Langevin equation is as follows:

$$\dot{X} = -\gamma X - X\zeta(t) + \eta(t), \tag{2}$$

where $\zeta(t)$ and $\eta(t)$ are multiplicative and additive white noises, respectively. The two noise terms are characterized by their mean and variance

$$\begin{aligned} \langle\zeta(t)\rangle &= \langle\eta(t)\rangle = 0, \\ \langle\zeta(t)\zeta(t')\rangle &= 2D'\delta(t-t'), \\ \langle\eta(t)\eta(t')\rangle &= 2D\delta(t-t'), \\ \langle\zeta(t)\eta(t')\rangle &= \langle\eta(t)\zeta(t')\rangle = \frac{\lambda\sqrt{DD'}}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right] \end{aligned} \tag{3}$$

where D' and D correspond to intensities of multiplicative and additive white noises respectively, τ is the cross-correlation time of the coupling between the multiplicative and additive white noises, λ denotes the cross-correlation strength between these two noises with the limit $0 \le \lambda \le 1$.

Equation (3) shows that the coupling between multiplicative and additive noises is coloured cross-correlation.[10−12] The result of Ref.[10] is important for treatment of our problem. Furthermore, within small noise approximation,[2] we can write the Fokker–Planck equation corresponding to Langevin Eq.(2) and Eq.(3) as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + B\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{4}$$

where the drift term is

$$A(x) = -\gamma x + D'x - \frac{\lambda\sqrt{DD'}}{1+\gamma\tau}, \tag{5}$$

and

$$B = D'x_{\rm e}^2 - \frac{2\lambda\sqrt{DD'}}{1+\gamma\tau}x_{\rm e} + D, \tag{6}$$

where $x_{\rm e}$ is a solution of $A(x) = -\gamma x + D'x - \frac{\lambda\sqrt{DD'}}{1+\gamma\tau} = 0$, i.e.

$$x_{\rm e} = \frac{\lambda\sqrt{DD'}}{(1+\gamma\tau)(D'-\gamma)}. \tag{7}$$

If we put Eq.(7) in Eq.(6), the value of diffusion coefficient B is

$$B = \frac{[D'l^2 + 2l^2\Gamma + D\Gamma^2]}{\Gamma^2}, \tag{8}$$

where

$$l = -\frac{\lambda\sqrt{DD'}}{(1+\gamma\tau)}, \tag{9}$$

$$\Gamma = \gamma - D'. \tag{10}$$

3. The upper bound for dynamical system (2) and (3)

The Fokker–Planck Eq.(4) can be rewritten as the general form of continuity equation

$$\frac{\partial P(x,t)}{\partial t} = -\nabla_x \cdot j, \tag{11}$$

where ∇_x denotes the phase space divergence and j is

$$j = A(x)P(x,t) - B\frac{\partial P(x,t)}{\partial x}. \tag{12}$$

Bag[1] has used Eq.(11) and Schwartz inequality and defined the upper bound for the rate of evolution of entropy

$$\frac{{\rm d}S}{{\rm d}t} \le U_{\rm B}(t),$$

$$U_{\rm B}(t) = \left(\int {\rm d}x \frac{j^2}{P(x,t)}\right)^{1/2}\left(\int {\rm d}x \frac{(\nabla_x P(x,t))^2}{P(x,t)}\right)^{1/2}. \tag{13}$$

Since the Fokker–Planck Eq.(4) has the same form and significance as the Eq.(37) of Ref.[1], hence the upper bound for the rate of entropy change can be calculated similarly and the final expression for the upper bound $U_{\rm B}(t)$ is given by

$$\frac{{\rm d}S}{{\rm d}t} \le U_{\rm B}(t),$$

where

$$U_{\rm B}(t) = \frac{(2\beta(t)^2\Gamma^2\sigma(t) - 4\beta(t)\Gamma l\sigma(t) + 2l^2\sigma(t) + \Gamma^2\sigma(t)^2 + 4B^2 - 4B\Gamma\sigma(t))^{1/2}}{\sigma(t)}. \tag{14}$$

The time evolution of $\sigma(t)$ and $\beta(t)$ can be written as

$$\sigma(t) = \frac{2B}{\Gamma}[1 - \exp(-2\Gamma t)] + \sigma(0)\exp(-2\Gamma t), \quad (15)$$

and

$$\beta(t) = \frac{l}{\Gamma}[1 - \exp(-\Gamma t)] + \beta(0)\exp(-\Gamma t). \quad (16)$$

The initial conditions for $\sigma(0)$ and $\beta(0)$ can be set by the Eq.(25) of Ref.[1].

Equations (14)–(16) depict the interplay of the dissipative constant γ of Eqs.(2) and (3), the cross-correlation time τ and cross-correlation strength λ of coloured cross-correlated multiplicative and additive white noises. In the limit $\tau \to 0$, i.e., in case of white cross-correlated between multiplicative and additive white noises, the details have discussed in Ref.[1]. The interest of this paper is the general case of nonzero cross-correlation time.

4. Effects of noises and dissipative constant γ on the upper bound

By combining the results of Eqs.(8)–(10) and (14), we obtain the schematic representations of the upper bound vs time t, dissipative constant γ, cross-correlation time τ and cross-correlation strength λ shown respectively. In the following figures, we choose $D' = 1.0$, $D = 1.0$, $\sigma(0) = 0.0$ and $\beta(0) = 1.0$ accordingly.

Figures 1–3 show the time dependence of the upper bound $U_{\rm B}(t)$, the rate of entropy change decreases monotonically, with different values of the dissipative constant γ, cross-correlation time τ, cross-correlation strength λ. In Fig.1, $U_{\rm B}(t)$ with the weak dissipative constant such as $\gamma = 0.5$, depicted by curve a, at first the value is below that with the strong dissipative constant such as $\gamma = 3.0$, depicted by curve b. But the change of $U_{\rm B}(t)$ is quite the reverse as soon as time increases up to a critical value and later. The role of cross-correlation time τ on $U_{\rm B}(t)$ is investigated in Fig.2. It shows that $U_{\rm B}(t)$ becomes larger while larger τ is chosen. Next, we consider the effect of cross-correlation strength λ of coloured cross-correlated multiplicative and additive noises on the time dependence of $U_{\rm B}(t)$, shown in Fig.3. It shows that $U_{\rm B}(t)$ becomes smaller while larger λ is chosen.

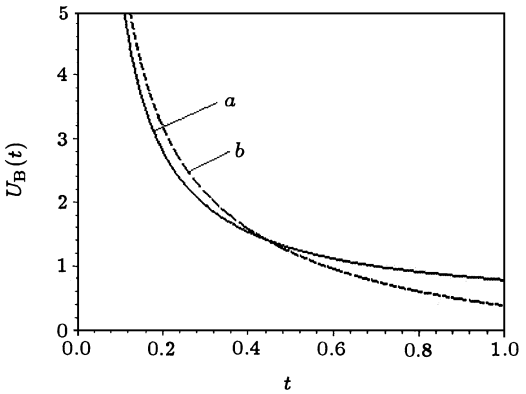


Fig.1. Plot of upper bound for the time derivative of entropy $U_{\rm B}(t)$ vs time t for Eq.(14) where $\lambda = 0.5$, $\tau = 1.0$, curve a for $\gamma = 0.5$ curve b for $\gamma = 3.0$ (units are arbitrary).

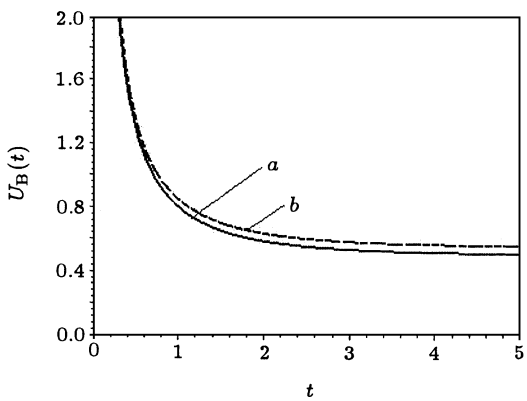


Fig.2. Plot of upper bound for the time derivative of entropy $U_{\rm B}(t)$ vs time t for Eq.(14) where $\lambda = 0.5$, $\gamma = 0.5$, curve a for $\tau = 0.5$ curve b for $\tau = 4.0$ (units are arbitrary).

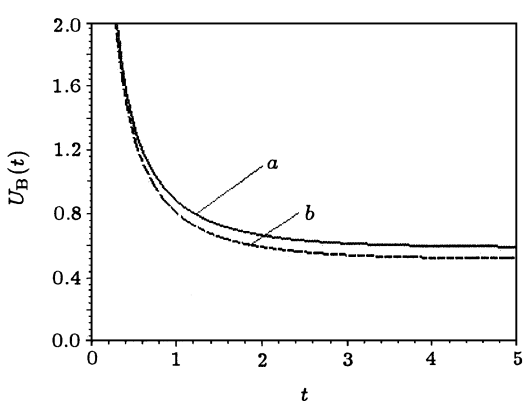


Fig.3. Plot of upper bound for the time derivative of entropy $U_{\rm B}(t)$ vs time t for Eq.(14) where $\tau = 3.0$, $\gamma = 0.5$, curve a for $\lambda = 0.2$ curve b for $\lambda = 0.9$ (units are arbitrary).

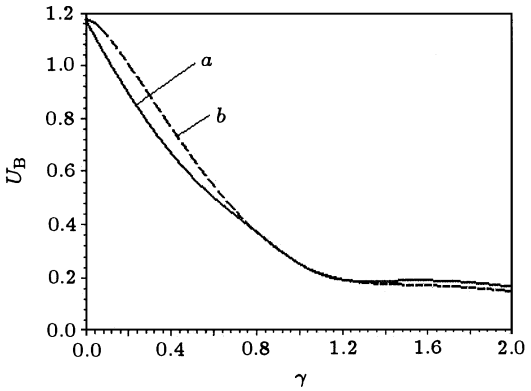


Fig.4. Plot of U_B vs dissipative constant γ at $t = 2$ for Eq.(14) where $\lambda = 0.5$, curve a for $\tau = 0.2$ curve b for $\tau = 8.0$ (units are arbitrary).

In Fig.4 we plot the variation of U_B at $t = 2$ vs dissipative constant γ with different cross-correlation time τ. Where γ increases, U_B decreases monotonically to a minimum at first and then slowly increases a maximum and coasts slowly later for $\tau = 0.2$. However, for larger τ, such as $\tau = 8.0$, U_B monotonically decreases fast at first and then settle down slowly. τ accelerates the decreasing effect of γ on U_B.

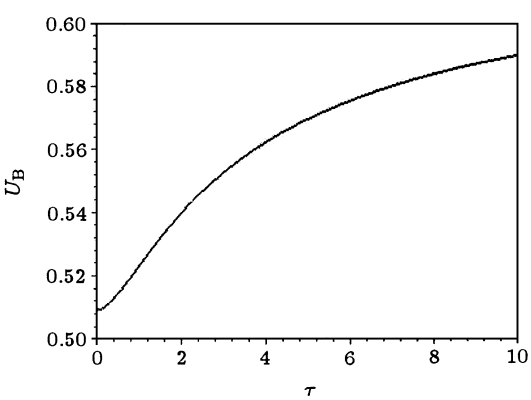


Fig.5. Plot of U_B vs cross-correlation time τ at $t = 4$ for Eq.(14) where $\gamma = 0.5$ and $\lambda = 0.5$ (units are arbitrary).

Furthermore, we discuss the relation between U_B and τ at $t = 4$, shown in Fig.5. U_B increases monotonically with increasing τ. So the dissipative constant γ always weakens U_B and τ enhances U_B by virtue of nonzero coloured cross-correlation time. For interpreting the variation of U_B with the strength of cross-correlation λ, we plot U_B vs λ at $t = 4$ in Fig.6. U_B at first decreases and then increases after passing through a minimum.

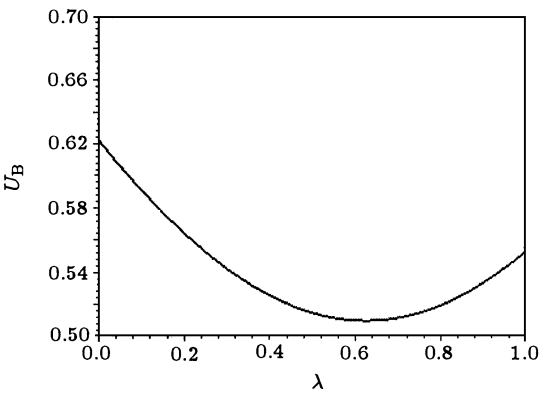


Fig.6. Plot of U_B vs cross-correlation strength λ at $t = 4$ for Eq.(14) where $\gamma = 0.5$ and $\tau = 0.5$ (units are arbitrary).

5. Conclusions

In this paper we extend the calculation of the upper bound for the time derivative of entropy[1] to a dynamical system driven by coloured cross-correlated multiplicative and additive white noises based on the new results about Fokker–Planck equation corresponding to Refs.[2,10]. The dissipative constant weakens the upper bound and cross-correlation time of coloured cross-correlated white noises always enhances the upper bound monotonically, however, the cross-correlation strength of coloured cross-correlated white noises at first decreases and then increases the upper bound after passing through a minimum. Moreover, we reveal how the interplay of the dissipative constant and the intrinsic characteristic of coloured cross-correlated white noises can set on the upper bound, shown explicitly in Figs.1–3. In short, the import of coloured cross-correlation influences the upper bound for the time derivative of entropy distinctly.

References

[1] Bag B C 2002 *Phys. Rev.* E **65** 046118
[2] Bag B C, Banik S K and Ray D S 2001 *Phys. Rev.* E **64** 026110
[3] Goswami G, Mukherjee B and Bag B C 2005 *Chem. Phys.* **312** 47
[4] Bag B C 2003 *J. Chem. Phys.* **119** 4988
[5] Bag B C 2002 *Phys. Rev.* E **66** 026122
[6] Daems D and Nicolis G 1999 *Phys. Rev.* E **59** 4000
[7] Li C X and Fang M F 2003 *Chin. Phys.* **12** 294
[8] Hu Y Q 2003 *Acta Phys. Sin.* **52** 1379 (in Chinese)
[9] Xin X S 2003 *Acta Phys. Sin.* **52** 2969 (in Chinese)
[10] Luo X Q and Zhu S Q 2003 *Phys. Rev.* E **67** 021104
[11] Gammaitoni L, Hanggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 224
[12] Jia Y and Li J R 1996 *Phys. Rev.* E **53** 5786
[13] Luo X Q and Zhu S Q 2004 *Chin. Phys.* **13** 1201
[14] Majee P and Bag B C 2004 *J. Phys.* A: *Math. Gen.* **37** 3353
[15] Fedchenia I I 1998 *J. Stat. Phys.* **52** 1005