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Adaptive synchronization of a critical chaotic system*

Tu Li-Lan(涂俐兰)[a)b)] and Lu Jun-An(陆君安)[a)†]

[a)] *School of Mathematics & Statistics, Wuhan University, Wuhan 430072, China*

[b)] *College of Science, Wuhan University of Science and Technology, Wuhan 430081, China*

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This paper further investigates the synchronization problem of a new chaotic system with known or unknown system parameters. Based on the Lyapunov stability theory, a novel adaptive control law is derived for the synchronization of a new chaotic system with known or unknown system parameters. Theoretical analysis and numerical simulations show the effectiveness and feasibility of the proposed schemes.

Keywords: adaptive synchronization, a critical chaotic system, Lyapunov stability theory, Barbalat's lemma

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1. Introduction

Over the last four decades, chaos as a very interesting nonlinear phenomenon has been intensively investigated in science and technology communities.

In 1963, Lorenz found the first chaotic attractor—the Lorenz attractor.[1] In 1999, Chen and Ueta found another similar but not topologically equivalent chaotic attractor, called the Chen attractor.[2] In 2002, Lü and Chen found a critical chaotic system between the Lorenz and Chen systems, called the Lü system.[3] To bridge the gap between the Lorenz and Chen attractors, Lü *et al* proposed a unified chaotic system with the Lorenz and Chen systems as two extremes and the Lü system as a transition.[4] From the definition given by Vaněček and Čelikovský,[5] the general Lorenz system can be classified as follows: the Lorenz, Chen, and Lü systems satisfy the conditions $a_{12}a_{21} > 0$, $a_{12}a_{21} < 0$, and $a_{12}a_{21} = 0$, respectively, where a_{12} and a_{21} are the corresponding matrix elements of the linear part of the system. In fact, the unified system is a convex combination of the Lorenz and Chen systems. That is, for α=0, it becomes the Lorenz system; for α=1, it becomes the Chen system; for α=0.8, it becomes the Lü system.[4] Recently, Liu *et al* introduced a critical chaotic system,[6] which is a particular case of the so-called generalized Lorenz canonical form (GLCF) with $\tau = -1$.[7] Therefore, this critical chaotic system deserves further investigation.

This paper further investigates the adaptive synchronization[8−11] of this critical chaotic system. The remaining part is organized as follows. This chaotic system is briefly introduced in Section 2. The adaptive feedback synchronization criteria and corresponding numerical simulations for known and unknown parameters are given in Sections 3 and 4 respectively. Conclusions are finally drawn in Section 5.

2. System description

Liu *et al* proposed a critical chaotic system,[6] which is described by

$$\begin{cases} \dot{\hat{x}} = a\,(\hat{y} - \hat{x})\,, \\ \dot{\hat{y}} = b\hat{x} - k\hat{x}\hat{z}, \\ \dot{\hat{z}} = -c\hat{z} + h\hat{x}^2, \end{cases} \tag{1}$$

where a, b, c, k and h are positive parameters.

When $a = 10$, $b = 40$, $k = 1$, $c = 2.5$, and $h = 4$, system (1) has a chaotic attractor as shown in Fig.1. Obviously, system (1) is a particular case of the generalized Lorenz canonical form (GLCF) with $\tau = -1$.[7]

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†E-mail: JALu@whu.edu.cn

Notice that there is only one parameter in the GLCF. In particular, for $0 < \tau < +\infty$, the GLCF becomes the Lorenz system; for $\tau = 0$, the GLCF becomes the Lü system; for $-1 < \tau < 0$, the GLCF becomes the Chen system; and for $\tau = -1$, the GLCF becomes system (1).

For any a, b, $c \in R$ and $k, h \in R \backslash \{0\}$, system (1) is equivalent to

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx - xz, \\ \dot{z} = -cz + x^2, \end{cases} \tag{2}$$

under the following linear transform of coordinates:

$$\begin{cases} x = \hat{x}\sqrt{hk}, \\ y = \hat{y}\sqrt{hk}, \\ z = k\hat{z}. \end{cases} \tag{3}$$

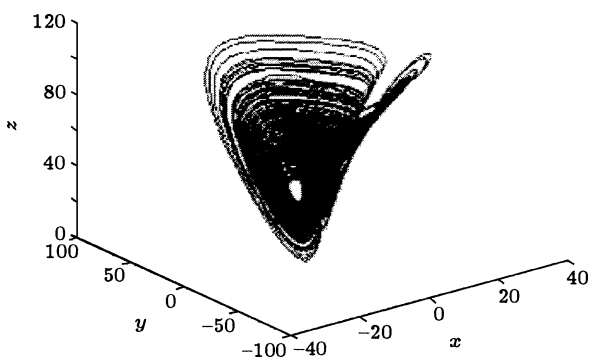


Fig.1. Critical chaotic attractor.

When $k = h = 1$, system (2) is equivalent to system (1).[7]

In the following, we investigate the adaptive synchronization of system (2). Assume that system (2) is the drive system, and the response system is given by

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + u_1, \\ \dot{y}_1 = bx_1 - x_1 z_1 + u_2, \\ \dot{z}_1 = -cz_1 + x_1^2 + u_3, \end{cases} \tag{4}$$

where u_1, u_2, and u_3 are three controllers. The control goal is to design and implement appropriate controllers u_1, u_2, and u_3 for the response system in which the controlled response system (4) and the drive system (2) are synchronous, i.e.

$$\lim_{t\to+\infty}(x_1 - x) = 0, \lim_{t\to+\infty}(y_1 - y) = 0,$$
$$\lim_{t\to+\infty}(z_1 - z) = 0.$$

Define the state errors between the response system and drive system as follows:

$$e_x = x_1 - x, \quad e_y = y_1 - y, \quad \text{and} \quad e_z = z_1 - z.$$

Suppose that M_x and M_z are the upper bounds of $|x|$, $|x_1|$ and $|z|$, $|z_1|$ respectively.

3. Adaptive feedback synchronization with known parameters

Theorem The response system (4) can synchronize the drive system (2) if $u_1 = 0$, $u_3 = 0$, and $u_2 = -\lambda e_y$, where $\dot{\lambda} = \alpha e_y^2$, $\lambda(0) = 0$, and $\alpha > 0$.

Proof According to systems (2) and (4), the error dynamics is

$$\begin{cases} \dot{e}_x = a(e_y - e_x), \\ \dot{e}_y = be_x - x_1 z_1 + xz - \lambda e_y, \\ \dot{e}_z = -ce_z + (x_1 + x)e_x. \end{cases} \tag{5}$$

Construct a Lyapunov function as follows:

$$V(e, \lambda) = \frac{1}{2}\left(\frac{2M_x^2}{ac}e_x^2 + e_y^2 + e_z^2 + \frac{1}{\alpha}(\lambda - \lambda^*)^2\right),$$

where λ^* is a constant to be determined. Its time derivative along with the system (5) becomes

$$\begin{aligned} \dot{V}(e, \lambda) =& \frac{2M_x^2}{ac}e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + \frac{1}{\alpha}\dot{\lambda}(\lambda - \lambda^*) \\ =& \frac{2M_x^2}{c}e_x e_y - \frac{2M_x^2}{c}e_x^2 + be_x e_y \\ & - (z_1 e_x + xe_z)e_y \\ & - \lambda^* e_y^2 - ce_z^2 + (x_1 + x)e_x e_z \\ \le& -\frac{2M_x^2}{c}e_x^2 - \lambda^* e_y^2 - ce_z^2 \\ & + \left(\frac{2M_x^2}{c} + b + M_z\right)|e_x|\,|e_y| \\ & + 2M_x|e_x|\,|e_z| + M_x|e_y|\,|e_z| \\ =& -\boldsymbol{e}^{\mathrm{T}} P \boldsymbol{e}, \end{aligned}$$

where $\boldsymbol{e} = [|e_x|, |e_y|, |e_z|]^{\mathrm{T}}$ and

$$P=\begin{bmatrix} \dfrac{2M_x^2}{c} & -\dfrac{1}{2}\left(\dfrac{2M_x^2}{c}+b+M_z\right) & -M_x \\ -\dfrac{1}{2}\left(\dfrac{2M_x^2}{c}+b+M_z\right) & \lambda^* & -\dfrac{M_x}{2} \\ -M_x & -\dfrac{M_x}{2} & c \end{bmatrix}.$$

To make the origin of the error system (5) asymptotically stable, the symmetrical matrix P should be positive definite, that is, P must satisfy the following conditions:

(i) $\dfrac{2M_x^2}{c}>0$,

(ii) $\dfrac{2M_x^2\lambda^*}{c}-\left(\dfrac{M_x^2}{c}+\dfrac{b+M_z}{2}\right)^2>0$,

(iii) $M_x^2\lambda^*-\dfrac{M_x^4}{2c}-M_x^2\left(\dfrac{M_x^2}{c}+\dfrac{b+M_z}{2}\right)$
$-c\left(\dfrac{M_x^2}{c}+\dfrac{b+M_z}{2}\right)^2>0.$

Obviously, the condition (i) holds. And if $\lambda^*>\dfrac{5M_x^2}{2c}+\dfrac{3\,(b+M_z)}{2}+c\left(\dfrac{b+M_z}{2M_x}\right)^2$, the conditions (ii) and (iii) hold. That is, P is positive definite and $\dot{V}$ is negative semidefinite. Thus $e_x\,(t)\,,e_y\,(t)\,,e_z\,(t)\in L_\infty$ and one obtains

$$\begin{aligned}&\int_0^t\lambda_{\min}\,(P)\,\left(e_x^2+e_y^2+e_z^2\right)\mathrm{d}t\\ &\leq\int_0^t e^{\mathrm{T}}Pe\mathrm{d}t\leq-\int_0^t\dot{V}\mathrm{d}t\\ &=V\,(e\,(0)\,,\lambda\,(0))-V\,(e\,(t)\,,\lambda\,(t))\leq V\,(e\,(0)\,,\lambda\,(0))\,,\end{aligned}$$

where $\lambda_{\min}\,(P)$ is the minimum eigenvalue of the matrix P. It follows that $e_x\,(t)\,,e_y\,(t)\,,e_z\,(t)\in L_2$. From the system (5), one has $\dot{e}_x\,(t)\,,\dot{e}_y\,(t)\,,\dot{e}_z\,(t)\in L_\infty$. Moreover, according to Barbalat's lemma,[12] one has $e_x\,(t)\to 0$, $e_y\,(t)\to 0$, and $e_z\,(t)\to 0\,(t\to+\infty)$. Therefore, the system (4) synchronizes the system (2). *QED*

Figures 2 and 3 display the synchronous errors and the feedback gain λ. In all simulations, the parameters are $a=10$, $b=40$, $c=2.5$, and $\alpha=2$, and the initial conditions are $(x=2.2, y=2.4, z=38)$ and $(x_1=30, y_1=30, z_1=20, \lambda=0)$. Figure 2 shows the synchronous errors of e_x, e_y, and e_z, respectively.

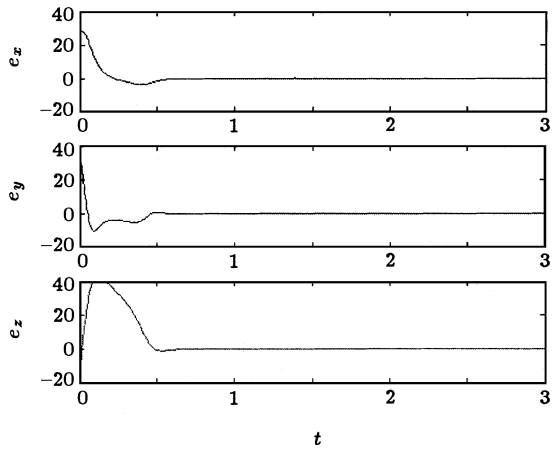


Fig.2. Synchronous errors versus time t.

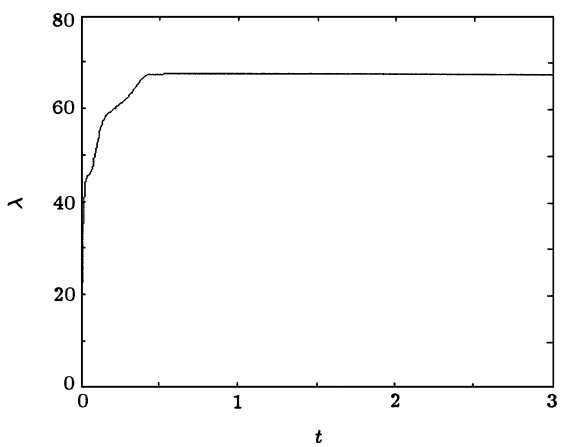


Fig.3. Feedback gain λ versus time t.

4. Adaptive feedback synchronization with unknown parameters

This section discusses the adaptive synchronization of the response system (4) with unknown parameters.

Assume that parameters a, b and c of the system (4) are unknown. Then the system (4) can be rewritten as follows:

$$\begin{cases} \dot{x}_1 = \hat{a}\,(y_1 - x_1) + u_1, \\ \dot{y}_1 = \hat{b}x_1 - x_1 z_1 + u_2, \\ \dot{z}_1 = -\hat{c}z_1 + x_1^2 + u_3, \end{cases} \tag{6}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are estimated values of a, b and c, respectively. Thus the error dynamics is given by

$$\begin{cases} \dot{e}_x = ae_y - ae_x + (\hat{a} - a)(y_1 - x_1) + u_1, \\ \dot{e}_y = be_x - z_1 e_x - xe_z + \left(\hat{b} - b\right) x_1 + u_2, \\ \dot{e}_z = -ce_z + (x_1 + x)\,e_x + (-\hat{c} + c)\,z_1 + u_3. \end{cases} \tag{7}$$

Denote $\tilde{a} = \hat{a} - a$, $\tilde{b} = \hat{b} - b$, $\tilde{c} = \hat{c} - c$ and define the active control functions u_1, u_2, u_3 as follows:

$$\begin{cases} u_1 = (z_1 - a - b)\,e_y, \\ u_2 = xe_z - e_y, \\ u_3 = -(x_1 + x)\,e_x. \end{cases} \tag{8}$$

Construct a quadratic Lyapunov function as follows:

$$V\left(e, \tilde{a}, \tilde{b}, \tilde{c}\right) = \frac{1}{2}\left(e_x^2 + e_y^2 + e_z^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2\right).$$

According to the Eqs.(7), and (8), the differential of $V\left(e, \tilde{a}, \tilde{b}, \tilde{c}\right)$ is

$$\begin{aligned} \dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{c}\right) =& e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} \\ =& -ae_x^2 - e_y^2 - ce_z^2 + \tilde{a}\,(y_1 - x_1)\,e_x \\ &+ \tilde{b}x_1 e_y - \tilde{c}z_1 e_z + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}. \end{aligned} \tag{9}$$

Define the updating laws of the estimated parameters as follows:

$$\begin{cases} \tilde{a}\,(y_1 - x_1)\,e_x + \tilde{a}\dot{\tilde{a}} = 0, \\ \tilde{b}x_1 e_y + \tilde{b}\dot{\tilde{b}} = 0, \\ -\tilde{c}z_1 e_z + \tilde{c}\dot{\tilde{c}} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} \dot{\hat{a}} = -(y_1 - x_1)\,e_x, \\ \dot{\hat{b}} = -x_1 e_y, \\ \dot{\hat{c}} = z_1 e_z. \end{cases}$$

From Eqs.(9) and (10), one has

$$\dot{V}\left(e, \hat{a}, \hat{b}, \hat{c}\right) = -ae_x^2 - e_y^2 - ce_z^2. \tag{11}$$

Since V is a positive and decreasing function, $\dot{V}$ is negative. It follows that the equilibrium point $\left(e_x, e_y, e_z, \tilde{a}, \tilde{b}, \tilde{c}\right)$=(0, 0, 0, 0, 0, 0) of the systems (7) and (10) is uniformly stable, i.e. e_x, e_y, e_z, $\tilde{a}$, $\tilde{b}$, $\tilde{c} \in L_\infty$. According to Eq.(11), one obtains

$$\begin{aligned} &\int_0^t \left(ae_x^2 + e_y^2 + ce_z^2\right) \le -\int_0^t \dot{V}\,(e, \alpha)\mathrm{d}t \\ &= V\,(e\,(0), \alpha\,(0)) - V\,(e\,(t), \alpha\,(t)) \le V\,(e\,(0), \alpha\,(0))\,, \end{aligned}$$

where $\alpha = \left(\tilde{a}, \tilde{b}, \tilde{c}\right)$. Obviously, $e_x\,(t), e_y\,(t), e_z\,(t) \in L_2$. From Barbalat's lemma, under any initial condition, the expression (7) implies $\dot{e}_x\,(t), \dot{e}_y\,(t), \dot{e}_z\,(t) \in L_\infty$. Thus one has $e_x\,(t), e_y\,(t), e_z\,(t) \to 0\,(t \to \infty)$. Therefore, the two systems can realize synchronization under the feedback controllers.

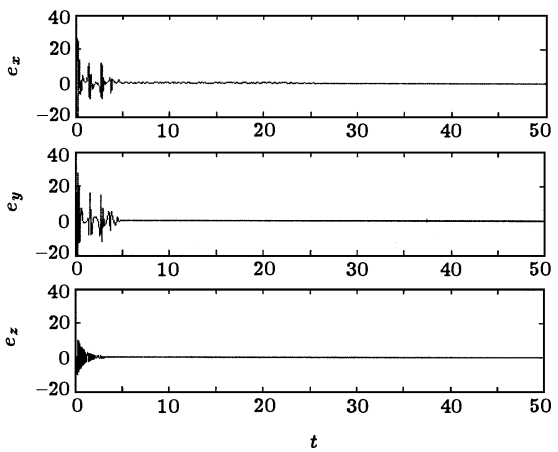


Fig.4. Synchronous errors versus time t.

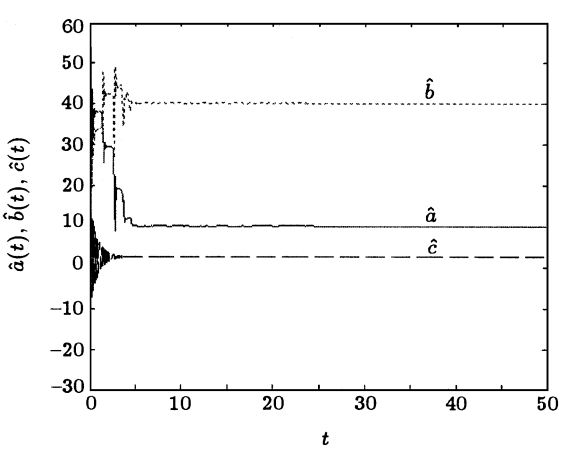


Fig.5. The parameters identification results versus time t.

Figures 4 and 5 show the numerical simulations for the adaptive synchronization of the systems (2) and (6). Here, $a = 10$, $b = 40$, and 2.5. The initial values of the drive and response systems are given by ($x = 2$, $y = 2$, $z = 7.6$), and ($x_1 = 30$, $y_1 = 30$,

$z_1 = 20$) respectively. The initial values for the parameters are ($\hat{a}(0) = 0$, $\hat{b}(0) = 0$, $\hat{c}(0) = 0$). Figure 4 shows the synchronous errors e_x, e_y, e_z respectively. Figure 5 displays the parameter identification results of $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$ denoted by the solid, dotted, and dashed line respectively.

5. Conclusions

This paper has further studied the adaptive synchronization of a critical chaotic system. Two novel adaptive feedback synchronization criteria are given. Theoretical analysis and numerical simulations show the effectiveness and feasibility of the proposed methods.

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