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Integrals of generalized Hamilton systems with additional terms*

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Two kinds of integrals of generalized Hamilton systems with additional terms are discussed. One kind is the integral deduced by Poisson method; the other is Hojman integral obtained by Lie symmetry.

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1. Introduction

The theory of symmetries and conserved quantities is an important research field in mathematics and physics, especially in analytical mechanics. One can obtain Hojman conserved quantities[1−3] and Lutzky conserved quantities[4,5] directly by Lie symmetries. Noether conserved quantities[6−8] can be deduced indirectly by Lie symmetries. To form a generalized Hamilton system is to construct a Hamilton system by replacing canonical conjugate variables with noncanonical variables, which are usually physical variables of this system. The generalized Hamilton system has made good progress[9−12] since 1950s. Lie symmetries of generalized Hamilton system and Hojman conserved quantities led by it have been discussed in Ref.[13]. Some differential equations of motion in practical problems, for example, differential equation of rotation of a rigid body about a fixed point under moments of force, usually cannot be expressed in a generalized Hamilton system. This problem can be solved by adding an additional term of moment of force on the right-hand side of the generalized Hamilton system. In this paper, the integrals of the generalized Hamilton systems with additional terms are discussed; this includes integrals led by the Poisson method and Hojman integrals led by Lie symmetries.

2. Differential equation of motion

We shall study the following equations:

$$\dot{x}_i = J_{ij}\frac{\partial H}{\partial x_j} + F_i, \qquad (i,j=1,\cdots,m) \tag{1}$$

where $J_{ij} = J_{ij}(\boldsymbol{x})$. They satisfy $J_{ij} = -J_{ji}$ and the following equations:

$$J_{il}\frac{\partial J_{jk}}{\partial x_l} + J_{jl}\frac{\partial J_{ki}}{\partial x_l} + J_{kl}\frac{\partial J_{ij}}{\partial x_l} = 0, \quad (i,j,l=1,\cdots,m). \tag{2}$$

Function $H = H(t,\boldsymbol{x})$ is Hamiltonian. Functions $F_i = F_i(t,\boldsymbol{x})$ are additional terms in generalized Hamilton system. If conditions $F_i = 0$ hold, then Eqs.(1) degenerate into usual Hamilton systems.

3. Integration method of Poisson

Let

$$F_i = K_{ij}\frac{\partial H}{\partial x_j}, \quad (K_{ij}) = \begin{pmatrix} K_{11} & & 0 \\ & \ddots & \\ 0 & & K_{mm} \end{pmatrix}, \tag{3}$$

$$L_{ij} = J_{ij} + K_{ij},$$

then Eqs.(1) can be expressed in the following form:

$$\dot{x}_i = L_{ij}\frac{\partial H}{\partial x_j}. \tag{4}$$

Define a derivative of function $A(\boldsymbol{x})$ with respect to t as a product $A \circ H$, i.e.

$$\dot{A} = \frac{\partial A}{\partial x_i}\dot{x}_i = \frac{\partial A}{\partial x_i}L_{ij}\frac{\partial H}{\partial x_j} \triangleq A \circ H. \tag{5}$$

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Generally speaking, this product is not a Lie algebra product, but possible it is a Lie admissible algebra product. If function $I = I(t, \boldsymbol{x})$ is an integral of equations (4), then we have

$$\frac{\partial I}{\partial t} + I \circ H = 0. \tag{6}$$

On the other hand, if function I satisfies Eq.(6), then it is an integral of Eqs.(4). Equation (6) is called generalized Poisson condition about first integral of Eqs.(4). Integrals of Eqs.(4) can be obtained by Eq.(6). We can easily prove the two following conclusions.

(1) If there is an integral I containing time t in Eqs.(4) and functions L_{ij} and H do not contain time t, then functions $\dfrac{\partial I}{\partial t}$, $\dfrac{\partial^2 I}{\partial t^2}, \ldots$ are all integrals of this system.

(2) If there is an integral I containing x_ρ in Eqs.(4) and functions L_{ij} and H do not contain x_ρ, then functions $\dfrac{\partial I}{\partial x_\rho}$, $\dfrac{\partial^2 I}{\partial x_\rho^2}, \cdots$ are all integrals of this system.

Generally speaking, if functions I_1 and I_2 are integrals of Eqs.(4), then $I_1 \circ I_2$ is not an integral of Eqs.(4).

4. Hojman method

Expanding Eqs.(1), we have

$$\dot{x}_i = \alpha_i(t, \boldsymbol{x}). \tag{7}$$

Taking the following infinitesimal transformations:

$$t^* = t, \quad x_i^*(t^*) = x_i(t) + \varepsilon \xi_i(t, \boldsymbol{x}). \tag{8}$$

Under the infinitesimal transformations (8), the determining equations of Lie symmetry of Eqs.(4) are

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_i = \frac{\partial \alpha_i}{\partial x_j}\xi_j, \tag{9}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \alpha_i \frac{\partial}{\partial x_i}. \tag{10}$$

If there is a function $\mu = \mu(t, \boldsymbol{x})$ such that

$$\frac{\partial \alpha_i}{\partial x_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{11}$$

then there exist following Hojman conserved quantities for Eqs.(1)

$$I_{\mathrm{H}} = \frac{1}{\mu} \frac{\partial}{\partial x_i}(\mu \xi_i) = \text{const.} \tag{12}$$

If additional terms are vanished, that is to say

$$F_i = 0, \qquad (i = 1, \cdots, m)$$

then Eqs.(12) are degenerated into the results of Ref.[13].

5. Example

Example 1 Differential equations of rotation of rigid body about a fixed point are

$$\begin{aligned} A_1\dot{\omega}_1 &= (A_2 - A_3)\omega_2\omega_3 + M_1, \\ A_2\dot{\omega}_2 &= (A_3 - A_1)\omega_3\omega_1 + M_2, \\ A_3\dot{\omega}_3 &= (A_1 - A_2)\omega_1\omega_2 + M_3, \end{aligned} \tag{13}$$

where A_1, A_2, A_3 are principal moments of inertia, ω_1, ω_2, ω_3 are projections of angular velocity on its principal body axis, and M_1, M_2, M_3 are external moments of force. Furthermore, we suppose

$$M_1 = kA_2\omega_2, \quad M_2 = -kA_1\omega_1, \quad M_3 = 0, \tag{14}$$

where k is a constant. Find integrals of the system by the Poisson method.

Let

$$x_1 = A_1\omega_1, \quad x_2 = A_2\omega_2, \quad x_3 = A_3\omega_3, \tag{15}$$

then Eqs.(13) can be expressed in the following form:

$$\begin{aligned} \dot{x}_1 &= \frac{A_2 - A_3}{A_2A_3} x_2x_3 + kx_2, \\ \dot{x}_2 &= \frac{A_3 - A_1}{A_3A_1} x_3x_1 - kx_1, \\ \dot{x}_3 &= \frac{A_1 - A_2}{A_1A_2} x_1x_2. \end{aligned} \tag{16}$$

The generalized Poisson condition Eq.(6) gives

$$\begin{aligned} &\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_1}\left\{\frac{A_2 - A_3}{A_2A_3} x_2x_3 + kx_2\right\} \\ &+ \frac{\partial I}{\partial x_2}\left\{\frac{A_3 - A_1}{A_3A_1} x_3x_1 - kx_1\right\} \\ &+ \frac{\partial I}{\partial x_3}\left\{\frac{A_1 - A_2}{A_1A_2} x_1x_2\right\} = 0. \end{aligned} \tag{17}$$

Equation (17) is equivalent to the following equation:

$$\begin{aligned} \frac{\mathrm{d}t}{1} &= \frac{\mathrm{d}x_1}{\dfrac{A_2 - A_3}{A_2A_3} x_2x_3 + kx_2} \\ &= \frac{\mathrm{d}x_2}{\dfrac{A_3 - A_1}{A_3A_1} x_3x_1 - kx_1} \\ &= \frac{\mathrm{d}x_3}{\dfrac{A_1 - A_2}{A_1A_2} x_1x_2}. \end{aligned}$$

Furthermore we can obtain

$$x_1 \mathrm{d}x_1 + x_2 \mathrm{d}x_2 + x_3 \mathrm{d}x_3 = 0.$$

After integration, we have

$$x_1^2 + x_2^2 + x_3^2 = \text{const.} \tag{18}$$

This is the integral of energy.

Example 2 Find integral of following equations by the Hojman method.

$$\begin{aligned} \dot{x}_1 &= x_2 + k x_2^2, \\ \dot{x}_2 &= -x_1 + k x_1 x_2, \\ \dot{x}_3 &= -4 x_1 x_2, \end{aligned} \tag{19}$$

where k is a constant.

As the determining equations of Lie symmetries, Eqs.(9) result in the following equations:

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 &= \xi_2 + 2k x_2 \xi_2, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 &= -\xi_1 + k x_1 \xi_2 + k x_2 \xi_1, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 &= -4\xi_1 x_2 - 4\xi_2 x_1. \end{aligned} \tag{20}$$

We can find their solution

$$\xi_1 = \xi_2 = 0, \quad \xi_3 = 1, \tag{21}$$

$$\xi_1 = \xi_2 = 0, \quad \xi_3 = \frac{1}{2}(x_1^2 - x_2^2 + x_3)^2. \tag{22}$$

According to Eq.(11), we have

$$-k x_1 + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0.$$

Its solution is

$$\mu = \exp\left(\int k x_1 \mathrm{d}t \right). \tag{23}$$

Substituting Eqs.(22) and (23) into Eq.(12), we obtain the Hojman integral

$$I_{\mathrm{H}} = x_1^2 - x_2^2 + x_3 = \text{const.} \tag{24}$$

6. Conclusions

We have studied the integral of generalized Hamilton system with additional terms on its right hand. The Poisson integral and Hojman integral of the system are given in Section 3 and Section 4 respectively. These results generalize the work of Ref.[13].

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