Vol 14 No 9, September 2005
1009-1963/2005/14(09)/1691-07

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

A method of moments for calculating dynamic responses beyond linear response theory*

Kang Yan-Mei(康艳梅)[a)b)], Xu Jian-Xue(徐健学)[b)], and Xie Yong(谢 勇)[b)†]

[a)] *Institute of Information & System Science, School of Science, Xi'an Jiaotong University, Xi'an 710049, China*
[b)] *Institute of Nonlinear Dynamics, School of Architectural Engineering & Mechanics, Xi'an Jiaotong University, Xi'an 710049, China*

(Received 18 March 2005; revised manuscript received 28 May 2005)

A method of moments for calculating the dynamic response of periodically driven overdamped nonlinear stochastic systems in the general response sense is proposed, which is a modification of the method of moments confined within linear response theory. The calculating experience suggests that the proposed technique is simple and efficient in implementation, and the comparison with stochastic simulation shows that the first three orders of susceptibilities calculated by the proposed technique have high accuracy. The dependence of the spectral amplification parameters at the first three harmonics on the noise intensity is also investigated, and another observed phenomenon of stochastic resonance in the systems induced by the location of a single periodic orbit is disclosed and explained.

Keywords: a method of moments, dynamic susceptibility, spectral amplification factor, stochastic resonance

PACC: 0250, 0200, 0540

1. Introduction

The interplay of noise and a periodic signal in certain nonlinear systems may cause the output signal-to-noise ratio and the spectral power amplification, varied as functions of the noise intensity, to exhibit a rise at a weaker noise level.[1,2] Such a counterintuitive phenomenon was termed as stochastic resonance (SR), which has been found to be of considerable importance in signal processing,[3,4] financial prediction[5] and neurophysiology,[6−9] etc (also see Ref.[10] for a review). Just due to the primary interest in the phenomenon, the dynamic response of nonlinear periodically driven stochastic systems has been the subject of intense investigations during the past twenty years.[11]

However, exact analytic expressions characterizing the amplification of external signals in stochastic nonlinear systems can be obtained only for a few special cases.[12−15] Although various approximation techniques such as two-state approaches,[16,17] the matrix continued fraction method,[18,19] the eigenfunction expansion method,[20] the operator splitting technique,[21] and the method of moments[22−25] were developed to calculate the dynamic response of nonlinear stochastic systems, these methods either are characterized by long computation time when higher order harmonics are involved, or are restricted by the narrow parameter range of adiabatic elimination and linear response theory in their applications.

The aim of this paper is to introduce an efficient technique to calculate the dynamic response of driven nonlinear stochastic systems in a general parameter range. The proposed technique is a modification of the method of moments within linear response theory, and it obtains the response information of high-order harmonics by solving a linear ordinary differential system rather than by solving several linear algebraic systems[23] or by using the technique of matrix continued fractions for several times.[18,19] The proposed technique can calculate nonlinear dynamic susceptibilities at any harmonics with an easily programmed code.

This paper is organized as follows. In the next sec-

*Project supported by BH foundation of XJTU and the major program of the National Natural Science Foundation of China (Grant No 10432010).

†Corresponding author. E-mail: yxie@mail.xjtu.edu.cn

tion, the proposed method of moments for calculating the asymptotic probability distribution of driven nonlinear stochastic systems is described. In Section 3, the accuracy of the proposed method is demonstrated by comparing the obtained susceptibilities and spectral amplification factors with those obtained from stochastic simulation of Langevin equations. In Section 4, a kind of stochastic resonance induced by a single local periodic orbit is discussed. Finally, discussions and conclusions are presented in Section 5.

2. Asymptotic probability distribution calculation using the method of moments

Consider an ensemble of biased overdamped oscillators characterized by a coordinate, whose Langevin equation reads

$$\dot{x} = x - x^3 + h + \varepsilon(t) + \xi(t), \tag{1}$$

where h is an external bias. The oscillators are driven by Gaussian delta-correlated white noise $\xi(t)$ of intensity D and an external field $\varepsilon(t)$. The probability density $p(x,t)$ of Eq.(1) is governed by the Fokker–Planck (FP) equation[26]

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial p}{\partial x} - (x - x^3 + h + \varepsilon(t))p\right). \tag{2}$$

Here $p(x,t)$ satisfies the natural boundary conditions, i.e. $\lim\limits_{x\to\pm\infty} p^{(n)}(x,t) = 0$. Both the angular frequency Ω and the amplitude ε of the harmonic driving force $\varepsilon(t) = \varepsilon\cos(\Omega t)$ are assumed to be arbitrary, since our discussion aims to exceed the parameter range of adiabatic elimination and linear response theory.

Now multiplication of FP equation (2) by some function $f(x)$ and integration over x from $-\infty$ to $+\infty$ yield the following evolutionary equation for $\langle f\rangle(t) = \int f(x)p(x,t)\mathrm{d}x$ (supposing it exists):

$$\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}t} = D\left\langle\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right\rangle + \left\langle(x - x^3 + h)\frac{\mathrm{d}f}{\mathrm{d}x}\right\rangle + \varepsilon(t)\left\langle\frac{\mathrm{d}f}{\mathrm{d}x}\right\rangle. \tag{3}$$

In deriving Eq.(3) the technique of integration by parts together with the natural boundary conditions is used.

By Floquet's theorem, the long-time solution of FP equation (2) is time-periodic with the period of the external field,[11] i.e.

$$p(x, t + 2\pi/\Omega) = p(x,t). \tag{4}$$

So instead of decomposing the solution into Fourier series of time argument as usual,[18,19,22–25] we seek it in the form

$$p(x,t) = \sum_{n=0}^{\infty} C_n(t)p_0(x)x^n, \tag{5}$$

where $p_0(x) = Z\exp(-(x^4/4 - x^2/2 - hx)/D)$ denotes the stationary solution of FP equation (2) without external driving, with Z being the normalized constant. Here $C_n(t)(n \geq 0)$ are unknown periodically-varying coefficients, namely they satisfy

$$C_n(t) = C_n(t + 2\pi/\Omega). \tag{6}$$

Conservation of the norm of the distribution $p_{\mathrm{as}}(x,t)$ implies

$$\sum_{n=0}^{\infty} C_n(t)\langle x^n\rangle_0 = 1. \tag{7}$$

Hereafter we have $\langle x^n\rangle_0 = \int x^n p_0(x)\mathrm{d}x$. In order to find the unknown coefficients $C_n(t)$, we require Eq.(3) to be satisfied for all moments, i.e. we should take

$$f(x) = x^m, \quad m = 0,\ 1,\ 2,\ \ldots. \tag{8}$$

In the simplest case $m = 0$ (i.e. $f(x) = 1$), substitution of Eq.(5) into Eq.(3) gives

$$\sum_{n=0}^{\infty} \dot{C}_n(t)\langle x^n\rangle_0 = 0. \tag{9}$$

It is clear that Eq.(9) is a direct result implied by Eq.(7), thus only the latter will be considered in the following. Similarly, for $m \geq 1$ one obtains

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\mathrm{d}C_n(t)}{\mathrm{d}t}\langle x^{n+m}\rangle_0 \\ &= \sum_{n=0}^{\infty} C_n(t)\big(-Dmn\langle x^{m+n-2}\rangle_0 \\ &\quad + \varepsilon(t)m\langle x^{m+n-1}\rangle_0\big).\end{aligned} \tag{10}$$

From Eqs.(7) and (10), one derives the following linear ordinary differential system with periodic boundaries

$$\begin{cases} \sum\limits_{n=1}^{\infty} \frac{dC_n(t)}{dt} \left(\langle x^{n+m} \rangle_0 - \langle x^n \rangle_0 \langle x^m \rangle_0 \right) \\ = \sum\limits_{n=1}^{\infty} C_n(t) \left[-Dmn \left\langle x^{m+n-2} \right\rangle_0 + \varepsilon(t) m \left(\left\langle x^{m+n-1} \right\rangle_0 - \left\langle x^{m-1} \right\rangle_0 \langle x^n \rangle_0 \right) \right] + \varepsilon(t) m \left\langle x^{m-1} \right\rangle_0, \\ C_n(0) = C_n(2\pi/\Omega) \\ C_0(t) = 1 - \sum\limits_{n=1}^{\infty} C_n(t) \langle x^n \rangle_0 \end{cases} . \quad (11)$$

For the purpose of numerical calculation, one has to truncate Eq.(5) at the Nth power of x with N being some positive integer. We denote $A = (a_{m,n})$, $B = (b_{m,n})$ and $E = (e_{m,n})$ with $a_{m,n} = \langle x^{n+m} \rangle_0 - \langle x^n \rangle_0 \langle x^m \rangle_0$, $b_{m,n} = Dmn \left\langle x^{m+n-2} \right\rangle_0$ and $e_{m,n} = m \left(\left\langle x^{m+n-1} \right\rangle_0 - \left\langle x^{m-1} \right\rangle_0 \langle x^n \rangle_0 \right)$ for $1 \le m, n \le N$, and take $C = (C_1, C_2, \ldots, C_N)'$ and $G = \left(1, 2\langle x \rangle_0, \ldots, N \left\langle x^{N-1} \right\rangle_0\right)'$, then Eqs.(11) turn into a finite-dimensional linear ordinary differential system with periodic boundaries

$$\begin{aligned} &\dot{C} = A^{-1}(\varepsilon(t)E - B)C + \varepsilon(t)A^{-1}G, \\ &C_n(0) = C_n(2\pi/\Omega). \end{aligned} \quad (12)$$

Here the positiveness of A can be proven with a technique used in Ref.[3].

To solve the coefficients $C(t)$ with less storage requirements and less computational time, the simple Euler method is used to integrate Eqs.(11) from an initial guess until they become time-periodic. Our numerical results have indicated that one can arrived at the asymptotic process in not more than 5 driving periods for noise intensity not in the zero limit. So the calculation is very fast.

Supposing that $C(t)$ has been solved from the linear system (12), then the asymptotic probability density can be calculated using Eq.(5), and then the asymptotic ensemble average $(EX)(t)$ can be calculated according to

$$(EX)(t_l) = \langle x \rangle_0 + \sum_{n=1}^{N} C_n(t_l) \left(\left\langle x^{n+1} \right\rangle_0 - \langle x^n \rangle_0 \langle x \rangle_0 \right), \quad 1 \le l \le L \quad (13)$$

with $t_l = \dfrac{2\pi l \Delta t}{\Omega L}$ and $\Delta t = T/L$ being the integration step-length. Then, applying Fourier transform to the sample data (13), one arrives at the Fourier coefficients of the kth-order harmonic

$$a_k = \frac{2}{T} \int_0^T EX(t) \cos\left(\frac{2k\pi t}{T}\right) \mathrm{d}t \approx 2\langle x \rangle_0 + \frac{2\Delta t}{T} \sum_{l=1}^{L} \left[\sum_{n=1}^{N} C_n(t_l) \left(\left\langle x^{n+1} \right\rangle_0 - \langle x^n \rangle_0 \langle x \rangle_0 \right) \right] \cos\left(\frac{2k\pi t_l}{T}\right), \quad (14)$$

$$b_k = \frac{2}{T} \int_0^T EX(t) \sin\left(\frac{2k\pi t}{T}\right) \mathrm{d}t \approx \frac{2\Delta t}{T} \sum_{l=1}^{L} \left[\sum_{n=1}^{N} C_n(t_l) \left(\left\langle x^{n+1} \right\rangle_0 - \langle x^n \rangle_0 \langle x \rangle_0 \right) \right] \sin\left(\frac{2k\pi t_l}{T}\right). \quad (15)$$

Using Eqs.(14) and (15), the asymptotic ensemble average $(EX)(t)$ can be expanded into

$$(EX)(t) = \langle x \rangle_0 (\Omega, \varepsilon) + \sum_{k=1}^{\infty} \chi^{(k)}(\Omega, \varepsilon) \varepsilon^k \mathrm{e}^{\mathrm{i}k\Omega}, \quad (16)$$

where $\chi^{(k)}(\Omega, \varepsilon) = (a_k - \mathrm{i}b_k)/\varepsilon^k$ is called the dynamic susceptibility of the kth harmonic, and it depends on the driving amplitude and frequency in the general response sense. For simplicity, we denote it as $\chi^{(k)}$. It is clear that the square of the module of $\chi^{(k)}$, namely $|\chi^{(k)}|^2$ is the spectral amplification factor of the kth harmonic, which is often used as a quantifier of SR.

3. Accuracy analysis of the proposed method of moments

To verify the accuracy of the proposed method of moments, a comparative analysis technique is adopted. That is to say, the dynamic susceptibilities and the spectral amplification factors derived from the method we proposed are compared with those derived from stochastic simulations in the general response sense. A standard Runge–Kutta procedure with the Box–Muller algorithm to generate the Gaussian random number[27] is employed to integrate Eq.(1). A

time history of 5000 periods (the transient evolution has been discarded) is averaged to give the ensemble average $(EX)(t)$, and then the simulative dynamic susceptibility and the spectral amplification factor are calculated according to Eqs.(14) and (15). Since the first three harmonics are of primary importance in the context of SR, we only check the susceptibility and the amplification factor for these harmonics to verify the correctness of the modified moment method.

Figure 1 shows the dependence of susceptibilities on frequency for the first three harmonics obtained from the proposed method and the stochastic simulation. Figure 2 shows the dependence of the spectral amplification factors on noise intensity for the first three harmonics, where the spectral amplification factors for the first two harmonics are calculated from both the proposed method and the stochastic simulation. Seen from the figures, a very good agreement is clear between the results obtained from both methods. Since the spectral amplification factor for the third-order harmonic obtained from stochastic simulations is very sensitive to the background noise so that regular results are not easy to be captured, only the results obtained from the proposed method are shown in Fig.2(c). From Fig.2(c), however, one sees that for $\varepsilon = 0.3$ the third-order spectral amplification factor attains two extremes at different noise intensities, namely, SR also occurs twice at the third harmonic, which has the same tendency as that reported within the linear response theory.[23] This indirectly shows that the numerical results for the third harmonic are correct.

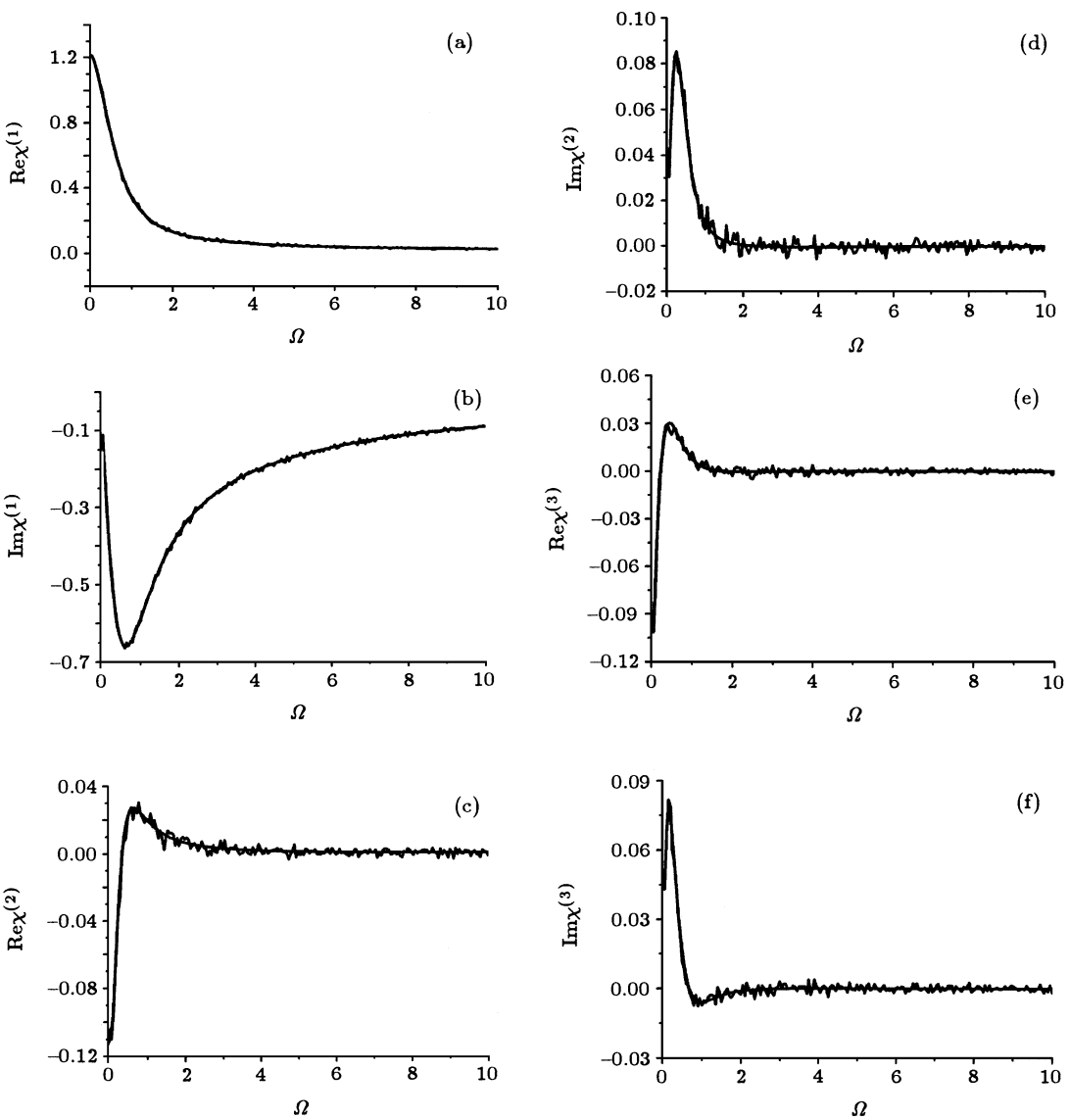


Fig.1. Dependence of susceptibility on frequency at the first (a), (b), second(c), (d), and third (e), (f) harmonics. Plots (a), (c), (e) correspond to the real, and (b), (d), (f) to the imaginary part of susceptibility. The results are obtained from stochastic simulations (coarse curves) and the modified moment method (smooth curves) with D=0.5, ε=1.0, h=0.2.

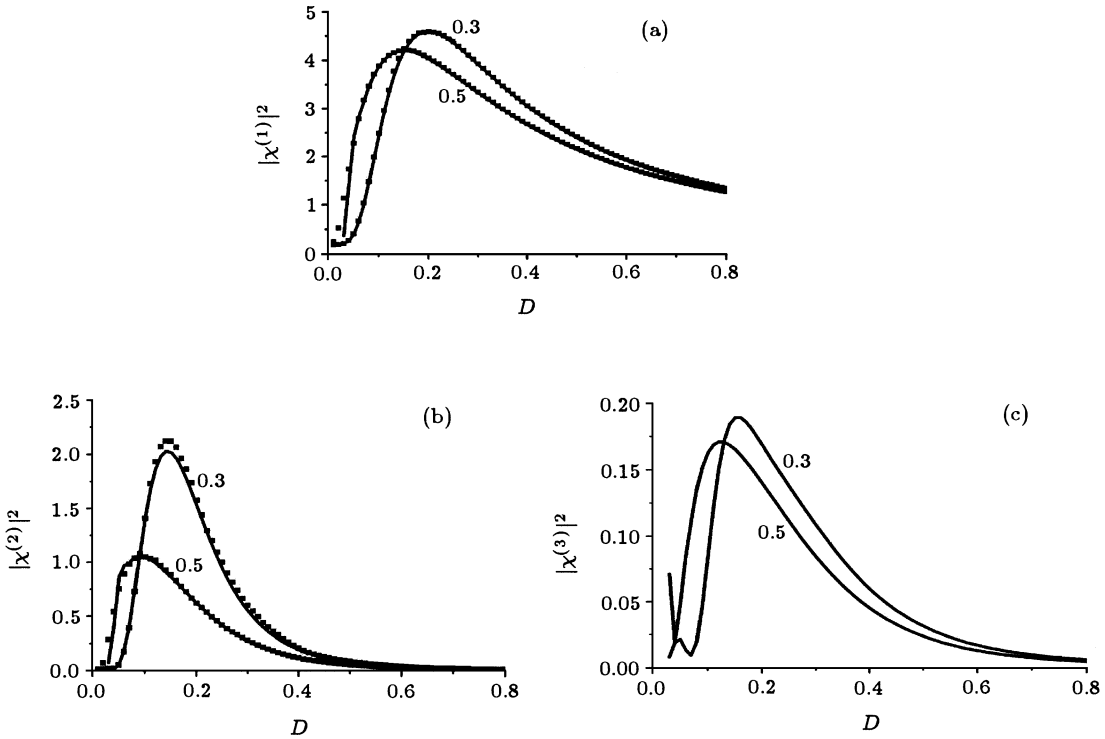


Fig.2. Dependence of the spectral amplification factor on noise intensity at the first (a), second (b) and third(c) harmonics. The results are obtained from the stochastic simulations (squares) and the modified moment method (full curve) with Ω=0.1, h=0.2 and ε labelled on curves.

In addition, we also check the validity of the spectral amplification factor for the first two harmonics for more stronger driving such as $\varepsilon = 0.8$, $\Omega = 0.1$ and $h = 0.2$, and we find that there is a little discrepancy between the curve obtained from the proposed method and that obtained from the stochastic simulation in the zero noise limit (the figures are omitted). By a scrutiny into the calculation, we find that the discrepancy is due to the truncation order N. Although increasing N can avoid the discrepancy theoretically, too large a truncation order will make the matrix A become ill-conditioned.

4. Stochastic resonance induced by a single local periodic orbit

Figure 3 shows the deterministic bifurcation diagram of Eq.(1) on the Poincaré′ section $\{x(\bar{t}) : \bar{t} = 2i\pi/\Omega, i \in Z^+, i \gg 1\}$, where $i \gg 1$ denotes that the transient evolution has been discarded. From Fig.3, one observes there is a critical amplitude $\varepsilon_c \approx 0.198$. That is to say, for $\Omega = 0.1$ and $h = 0.2$, Eq.(1) with $D = 0$ has two stable periodic orbits in the case of $0 < \varepsilon < \varepsilon_c$, while it has only one periodic orbit in the case of $\varepsilon > \varepsilon_c$. So for all the driving signals considered in Fig.2(a), Eq.(1) with $D = 0$ is monostable, but why does SR still occur? To explain this, we check the evolution of Eq.(1) under different noise intensities. At first, the three equilibrium points of Eq.(1) in the case of $\varepsilon = 0$ and $D = 0$ are marked as $x_1 \approx 1.09$, $x_0 \approx -0.209$, and $x_2 \approx -0.8789$. Here $x_{1,2}$ are stable and x_0 is unstable. In the absence of noise, the long-time solution of Eq.(1) locally oscillates around x_1 whether for $\varepsilon = 0.3$ or for $\varepsilon = 0.5$. Although at the weak noise level the noise adds some disturbance to the periodic oscillation, there occur noise-induced global jumps that make the oscillation around x_2 become possible as the noise intensity increases, as shown in Fig.4. At a more intense noise level, the noise-induced jump occurs more frequently and the system (1) completely loses in disorder. For a better demonstration of this, let us turn to Fig.5. The probability distributions under different noise intensities show that there is some suitable noise intensity that protrudes the probability peak near x_2. This further supports that there exists some noise intensity, at which the noise-induced jump becomes coherent with the external driving. From these observations, one knows that the resonance-like behaviour in Fig.2(a) is similar to the conventional SR of the standard overdamped bistable system, but obviously it is a new phenomenon of SR, because the noise induced-jump is caused by the asymmetric location of a single peri-

odic orbit. In other words, it is the noise that makes the small local periodic orbit change into a large global periodic orbit. This is somewhat similar to the monostable SR reported previously in some sense.[28]

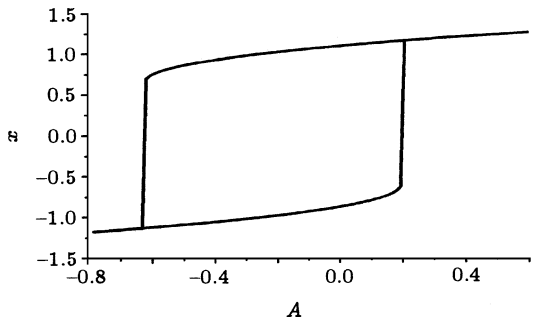


Fig.3. The bifurcation graph on Poincaré section via amplitude with h=0.2 and Ω=0.1.

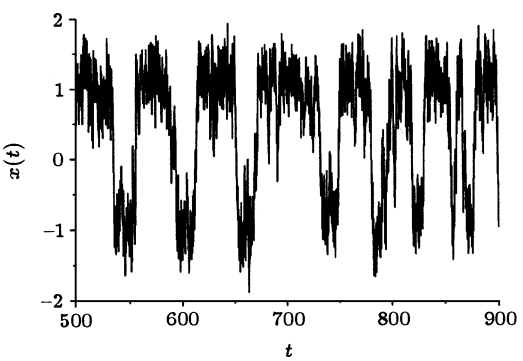


Fig.4. The time history of system (1) with parameters h=0.2, ε=0.3 and D=0.18.

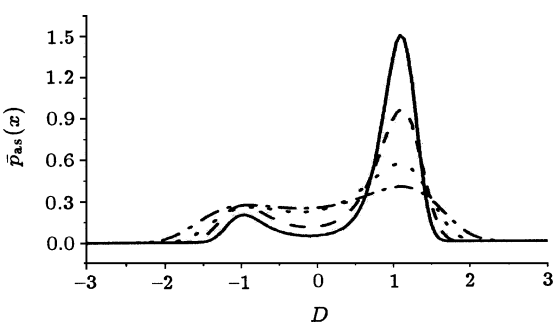


Fig.5. The time–averaged long-term probability distribution of system (1) calculated from formula (5) with parameters h=0.2, ε=0.3 and D is 0.1 (solid curve), 0.2 (dashed curve), 0.5 (dotted curve), 1.0 (dot-dashed curve), respectively.

5. Discussions and conclusions

It has been demonstrated that the modified method of moments is a useful tool in the investigation of the nonlinear dynamic response of periodically driven stochastic systems. First-, second-, and third-order susceptibilities of a biased overdamped nonlinear stochastic oscillator are calculated from the modified method and from stochastic simulations beyond linear response theory. Good agreement is found between the results obtained using these two methods.

The dependence of the spectral amplification of the first three harmonics on the noise intensity is also investigated. By comparing the spectral amplifications of the first two harmonics obtained by the method of moments and those obtained stochastic simulations, it is shown that the proposed method has a good performance except for the case when the driving amplitude is too large, but the noise intensity is too weak. Although the method of matrix continued fractions can avoid this defect, it does not greatly decrease the usefulness of the proposed method, because the proposed method is more efficient in calculating the nonlinear response properties of high-order harmonics for noise intensity beyond the zero limit. Moreover, we suggest that this defect could be overcome by using orthogonal polynomial expansion rather than the power series in Eq.(5).

When the external field is strong enough to make the bistable system turn into a monostable system, the SR-type behaviour still might be observed. By means of the asymptotic probability distribution and the time evolution of the stochastic systems, the occurrence of this different phenomenon of SR is related to a kind of noise-induced jumps, which is due to the asymmetry in the location of the single periodic orbit.

Finally, we emphasize that the modified method of moments can be directly applied to the case when the periodic driving force is not harmonic. This has not been treated by the method of matrix continued fractions.

References

[1] Benzi R, Sutera A and Vupiani A 1981 *J. Phys.* A: *Math. Gen.* **14** L453

[2] Benzi R, Parisi G, Sutera A and Vulpani A 1983 *SIAM J. Appl. Math.* **43** 565

[3] Xu B H, Duan F B, Bao R H and Li J L 2002 *Chaos Solitons & Fractals* **13** 633

[4] Zhu H J, Li R and Wen X D 2003 *Acta Phys. Sin.* **52** 2404 (in Chinese)

[5] Mao X M, Sun K and Ouyang Q 2002 *Chin. Phys.* **11** 1106

[6] Wang J F, Liu F, Wang J Y, Cheng G and Wang W 1997 *Acta Phys. Sin.* **46** 2305 (in Chinese)
[7] Xie Y, Xu J X, Kang Y M, Hu S J and Duan F B 2004 *Chin. Phys.* **13** 1396
[8] Xie Y, Xu J X and Hu S J 2004 *Chaos Soliton & Fractals* **21** 177
[9] Zhang G J and Xu J X 2005 *Acta Phys. Sin.* **54** 557 (in Chinese)
[10] Gammaitoni L, Hänggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
[11] Jung P 1993 *Phys. Rep. (Review Section of Physics Letters)* **234** 175
[12] Melnikov V I 1993 *Phys. Rev.* E **48** 2481
[13] Berdichevsky V and Gitterman M 1996 *J. Phys.* A: *Math. Gen.* **29** L447
[14] Berdichevsky V and Gitterman M 1998*J. Phys.* A: *Math. Gen.* **31** 9773
[15] Jin Y F, Xu W, Li W and Xu M 2005 *Acta Phys. Sin.* **54** 2562 (in Chinese)
[16] McNamara B and Wiesenfeld K 1989 *Phys. Rev.* E **39** 4854
[17] Dhara A K and Banerjee S R 2001 *J. Stat. Phys.* **105** 771
[18] Jung P and Hänggi P 1991 *Phys. Rev.* A **44** 8032
[19] Jung P 1989 *Z. Phys.* B-*Condens. Matter* **76** 521
[20] Hu G, Nicolis G and Nicolis C 1990 *Phys. Rev.* A **42** 2030
[21] Casado-Pascual J, Gomez-Ordóñez J and Morillo M 2002 *Fluct. Noise Lett.* **2** L125
[22] Dykman M I, Luchinsky G G, Mannella R, McClintock P V E, Sosckin S M and Stein N D 1996 *Phys. Rev.* E **54** 2336
[23] Evstigneev M, Pankov V and Prince R H 2001 *J. Phys.* A: *Math. Gen.* **34** 2595
[24] Evstigneev M, Pankov V and Prince R H 2002 *Phys. Rev. Lett.* **88** 240201
[25] Kang Y M, Xu J X and Xie Y 2003 *Phys. Rev.* E **68** 036123
[26] Risken H 1984 *The Fokker–Planck Equation Methods of Solution and Applications* (Berlin: Springer)
[27] Fox R F, Gatland I R, Roy R and Vemuri G 1988 *Phys. Rev.* A **38** 5938
[28] Kang Y M, Xu J X and Xie Y 2003 *Acta Phys. Sin.* **52** 2712 (in Chinese)