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An extended functional transformation method and its application in some evolution equations

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In this paper, an extended functional transformation is given to solve some nonlinear evolution equations. This function, in fact, is a solution of the famous KdV equation, so this transformation gives a transformation between KdV equation and other soliton equations. Then many new exact solutions can be given by virtue of the solutions of KdV equation.

Keywords: extended functional transformation, exact solution, KdV equation
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1. Introduction

The investigation of the travelling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Great progress has been made in finding new travelling wave solutions in the past decades.[1−33] Many methods have been proposed, such as inverse scattering method,[1,2] Darboux transformation,[3,4] Hirota bilinear method,[5,6] tanh function method and Jacobi elliptic function expansion method. Among them, the Jacobi elliptic function expansion method[7−26] provides an effective algorithm to obtain periodic wave solutions for many nonlinear evolution equations and its periodic solutions degenerate to the hyperbolic function solutions and trigonometric function solutions when the modulus $m \to 1$ and $m \to 0$ respectively.

In this paper we will discuss an extended functional transformation method, in which the function $\phi = \phi(\xi)$ is just a solution of the famous KdV equation. Then this method is used to get some new solutions for the Sine–Gordon equation

$$u_{xt} = \sin u \tag{1}$$

and the Caudrey–Dodd–Gibbon–Kawada equation

$$u_t + (u_{xxxx} + 30uu_x + 60u^3)_x = 0 \tag{2}$$

respectively. In Section 2 we will briefly introduce this extended functional transformation. In Section 3 we will give two examples to illustrate this method. Finally some conclusions are given in Section 4.

2. Extended functional transformation

For the well known KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3}$$

Ref.[7] introduced a transformation

$$w = u - 2\Gamma_x, \tag{4}$$

where $\Gamma = \dfrac{\phi_1}{\phi_2}$, from which two new classes of exact solutions are obtained:

Case 1 when $F(\eta)$ satisfy the following Airy equation:

$$\frac{\mathrm{d}F^2}{\mathrm{d}\eta^2} + \frac{\eta}{3}F = 0, \tag{5}$$

i.e. $F(\eta)$ is

$$F(\eta) = c_1\mathrm{Ai}(\eta) + c_2\mathrm{Bi}(\eta), \tag{6}$$

where $\mathrm{Ai}(\eta)$, $\mathrm{Bi}(\eta)$ are two Airy functions, while c_1, c_2 are two arbitrary constants, a new solution for KdV equation (3) is shown as

$$u(x,t) = \frac{x+1}{3(2t+1)} + 2(2t-1)^{-\frac{2}{3}}\frac{\mathrm{d}^2(\ln F)}{\mathrm{d}\eta^2}. \tag{7}$$

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Case 2 when $\xi = x + ct$, $c = 4\eta^2$, a new solution is given as

$$w = \frac{8\eta^2 \sinh(2\eta\xi)}{2\eta\xi + \sinh(2\eta\xi) - 16\xi^3(t+k)} - \frac{32\eta^2 \cosh^4(\eta\xi)}{(2\eta\xi + \sinh(2\eta\xi) - 16\eta^3(t+k))^2}. \quad (8)$$

Ref.[8] obtained some solutions for KdV equation (3)

$$u_1 = -\frac{1}{6}(c+c_2) + \frac{c_2}{2}\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right), \quad c_2 > 0;$$

$$u_2 = -\frac{1}{6}(c+c_2) + \frac{c_2}{2}\sec^2\left(\frac{\sqrt{-c_2}}{2}\xi\right), \quad c_2 < 0;$$

$$u_3 = -\frac{1}{6}c - \frac{1}{2\xi^2}, \quad c_2 = 0;$$

$$u_4 = -\frac{1}{6} + a_1\wp\left(\frac{\sqrt{-2c_1}}{2}\xi, g_2, g_3\right);$$

$$u_5 = -\frac{1}{6}(c+4c_2) + \frac{2c_2m^2}{2m^2-1}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \quad c_2 > 0, \quad c_0 = \frac{(1-m^2)c_2}{2m^2-1};$$

$$u_6 = -\frac{1}{6}(c+4c_2) + \frac{2c_2m^2}{m^2+1}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right), \quad c_2 < 0, \quad c_0 = \frac{2m^2c_2^2}{m^2+1};$$

where $\wp\left(\frac{\sqrt{-2c_1}}{2}\xi, g_2, g_3\right)$, $g_2 = \frac{2c_1}{a_1}$, $g_3 = \frac{2c_0}{a_1}$ is called invariants of Weierstrass elliptic function. Some other solutions of KdV equation have been studied in other papers.[29,30]

Now we briefly introduce the extended functional transformation method. For the given nonlinear evolution equations with two variables, x, t,

$$H(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \quad (9)$$

we introduce a travelling wave transformation

$$u(x,t) = U(\xi), \xi = x + ct, \quad (10)$$

then Eq.(9) can be transformed into an ordinary differential equation

$$H(U, U', U'', \ldots) = 0, \quad (11)$$

where "$'$" denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$. We seek the following polynomial solutions of Eq.(11):

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i\phi(\xi)^i, \quad (12)$$

where a_i are constants to be determined later. Substituting Eq.(12) into Eq.(11) and balancing the highest order derivative term and the nonlinear term, we can get the value of n easily. When $n = \frac{p}{q}$, we make the transformation

$$U(\xi) = V(\xi)^{\frac{p}{q}}, \quad (13)$$

when n is a negative integer, we make a transformation

$$U(\xi) = V(\xi)^m, \quad (14)$$

then the substitution of Eq.(13) or Eq.(14) into Eq.(11) and under the transformation Eq.(12), we can determine a positive integer n once again.

If $\phi = \phi(\xi)$ can be chosen as tanh function, then we obtain the hyperbolic function solutions of Eq.(9). If $\phi(\xi)$ can be chosen as the Jacobi elliptic function $\mathrm{sn}\xi$, $\mathrm{cn}\xi$, $\mathrm{dn}\xi$, then we obtain the periodic wave solution of Eq.(9).

On the other hand, under the transformation Eq.(10), KdV equation can be changed into

$$cU' + 6UU' + U''' = 0. \quad (15)$$

Integrating twice and letting α be the integrating constant, we have

$$U'' + 3U^2 + cU + \alpha = 0, \quad (16)$$

$$(U')^2 = -2\alpha U - cU^2 - 2U^3. \quad (17)$$

Let $\phi = \phi(\xi)$ be a solution of KdV equation, i.e. $\phi(\xi)$ satisfy Eqs.(16) and (17). The derivative with respect to the variable ξ become the derivative with respect to the variable ϕ, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \longrightarrow \mp\sqrt{-2\alpha\phi - c\phi^2 - 2\phi^3}\frac{\mathrm{d}}{\mathrm{d}\phi};$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \longrightarrow (-\alpha - c\phi - 6\phi^2)\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^2 + (-2\alpha\phi - c\phi^2 - 2\phi^3)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}.$$

Using the wave transformation Eqs.(10) and (12), (16), (17), Eq.(11) can be changed into a polynomial in terms of $\phi(\xi)^k$, $(k = 0, 1, \ldots)$. Balance the highest powers and the highest derivative terms, we can determine the constant n uniquely.

Setting the coefficients of all terms of ϕ^j to zeros, we obtain an algebra system, from which the constants a_i can be found explicitly. So all kinds of exact solutions of Eq.(9) can be given from the solutions of KdV equation.

3. Application

3.1. Exact solutions of the Sine–Gordon equation

Consider the Sine–Gordon equation

$$u_{xt} = \sin u. \tag{18}$$

Let $v = \mathrm{e}^{\mathrm{i}u}$, $v = v(x,t) = V(\xi)$, $\xi = x + ct$, then Eq.(18) is transformed into

$$2cVV'' - 2c(V')^2 - V^3 + V = 0. \tag{19}$$

Substitute transformation (12), (16) and (17) into Eq.(18) and balance the highest derivative term and the nonlinear terms, then the constant n is determined to be $n = 1$. So we have

$$V = a_0 + a_1\phi, \tag{20}$$

where a_0, a_1 are constants to be determined later. Using Eqs.(11), (16) and (17) again, we can change Eq.(19) into a polynomial of $\phi(\xi)$. Setting the coefficients of all ϕ^i to zero, we have a set of algebra equations

$$\begin{aligned}
&a_0 - a_0^3 - 2c\alpha a_1 = 0,\\
&a_1 - 3a_0^2a_1 - 2c^2a_0a_1 - 2c\alpha a_1^2 + 4c\alpha = 0,\\
&6ca_0a_1 + 3a_0a_1^2 + 2c^2a_1^2 - 2c^2 = 0,\\
&6ca_1^2 + a_1^3 - 4c = 0,
\end{aligned}$$

from which we get $c = \mp\dfrac{1}{2}$, $a_0 = 0$, $a_1 = \pm 1$. So the new solutions of the Sine–Gordon equation are listed as follows:

$$\begin{aligned}
u_1 =& -i\ln\left[a_1\frac{x+1}{3(2t+1)} + 2(2t-1)^{-\frac{2}{3}}\frac{\mathrm{d}^2(\ln F)}{\mathrm{d}\eta^2}\right],\\
u_2 =& -i\ln\left[a_1\frac{8\eta^2\sinh(2\eta\xi)}{2\eta\xi + \sinh(2\eta\xi) - 16\xi^3(t+k)}\right.\\
&\left. - \frac{32\eta^2\cosh^4(\eta\xi)}{(2\eta\xi + \sinh(2\eta\xi) - 16\eta^3(t+k))^2}\right],\\
u_3 =& -i\ln\left[a_1 - \frac{1}{6}(c+a_1) + \frac{a_1}{2}\operatorname{sech}^2\left(\frac{\sqrt{a_1}}{2}\xi\right)\right],\\
& a_1 > 0;\\
u_4 =& -i\ln\left[-\frac{1}{6}(c+a_1) + \frac{a_1}{2}\sec^2\left(\frac{\sqrt{-a_1}}{2}\xi\right)\right],\\
& a_1 < 0;\\
u_5 =& -i\ln\left[-\frac{1}{6}(c+4a_1)\right.\\
&\left. + \frac{2a_1m^2}{2m^2-1}\operatorname{cn}^2\left(\sqrt{\frac{a_1}{2m^2-1}}\xi\right)\right],\\
& a_1 > 0,\ c_0 = \frac{(1-m^2)c_2}{2m^2-1};\\
u_6 =& -i\ln\left[-\frac{1}{6}(c+4a_1)\right.\\
&\left. + \frac{2a_1m^2}{m^2+1}\operatorname{sn}^2\left(\sqrt{-\frac{a_1}{m^2+1}}\xi\right)\right],\\
& a_1 < 0,\ c_0 = \frac{2m^2c_2^2}{m^2+1}.
\end{aligned}$$

In Ref.[6], the one soliton solution and N-soliton solutions are given, correspondingly we obtain the N-soliton solutions for the Sine–Gordon equation.

Case 1 two soliton solutions

$$u(x,t) = -i\ln[a_1(1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_1+\xi_2+A_{12}})_{xx}];$$

Case 2 three soliton solutions

$$\begin{aligned}
u(x,t) =& -i\ln[a_1(1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_3}\\
&+ \mathrm{e}^{\xi_1+\xi_2+A_{12}} + \mathrm{e}^{\xi_1+\xi_3+A_{13}}\\
&+ \mathrm{e}^{\xi_2+\xi_3+A_{23}}\\
&+ \mathrm{e}^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}+A_{123}})_{xx}];
\end{aligned}$$

Case 3 N-soliton solutions

$$u(x,t) = -i\ln\left[a_1\left(\ln\left(\sum_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j\xi_j + \sum_{1\le j<l}^{n}\mu_j\mu_lA_{jl}}\right)\right)_{xx}\right].$$

3.2. Exact solutions of Caudrey–Dodd–Gibbon–Kawada equation

Consider the CDGK equation

$$u_t + (u_{xxxx} + 30uu_x + 60u^3)_x = 0. \tag{21}$$

Let $u(x,t) = U(\xi)$, $\xi = x + ct$, then Eq.(21) is transformed into

$$cU + U^{(4)} + 30UU'' + 60U^3 = 0. \tag{22}$$

Substituting the transformation Eq.(12) into Eq.(21), using Eqs.(16) and (17) and balancing the highest derivative terms and the nonlinear terms, we obtain $n = 1$. So

$$U(\xi) = a_0 + a_1\phi, \tag{23}$$

where $\phi = \phi(\xi)$ satisfies Eqs.(16) and (17). Substituting Eq.(12) into Eq.(22) and setting the coefficients of ϕ^i to zero, we have a set of algebra equations as follows:

$$a_0 c + 60a_0^3 - 30a_0 a_1 \alpha + a_1 \alpha c = 0;$$

$$a_1 c + a_1 c^2 - 30a_1 a_0 c + 180a_0^2 a_1 - 30a_1^2 \alpha = 0;$$

$$15a_1 c - 30a_1^2 c + 90a_0 a_1 = 0;$$

$$2a_1^2 - 3a_1 + 1 = 0,$$

from which we have $a_0 = 0$, $a_1 = \frac{1}{2}$, $c = -1$, $\alpha = 0$. Then new solutions of the CDGK equation are given as

$$u_1(x,t) = \frac{x+1}{6(2t+1)} + (2t-1)^{-\frac{2}{3}} \frac{\mathrm{d}^2(\ln F)}{\mathrm{d}\eta^2};$$

$$u_2(x,t) = \frac{4\eta^2 \sinh(2\eta\xi)}{2\eta\xi + \sinh(2\eta\xi) - 16\xi^3(t+k)} - \frac{16\eta^2 \cosh^4(\eta\xi)}{(2\eta\xi + \sinh(2\eta\xi) - 16\eta^3(t+k))^2};$$

$$u_3(x,t) = -\frac{1}{12}(c + a_1) + \frac{a_1}{4}\mathrm{sech}^2\left(\frac{\sqrt{a_1}}{2}\xi\right), \quad a_1 > 0;$$

$$u_4(x,t) = -\frac{1}{12}(c + 4a_1) + \frac{a_1 m^2}{2m^2 - 1}\mathrm{cn}^2\left(\sqrt{\frac{a_1}{2m^2-1}}\xi\right), \quad a_1 > 0, \quad c_0 = \frac{(1-m^2)a_1}{2m^2-1}.$$

Its N-soliton solution can be written as

$$u(x,t) = \frac{1}{2}\left(\ln\left(\sum_{\mu=0,1} \mathrm{e}^{\sum_{j=1}^{n} \mu_j \xi_j + \sum_{1 \le j < l}^{n} \mu_j \mu_l A_{jl}}\right)\right)_{xx}$$

4. Conclusion

In this paper, we introduce the functional transformation $U(\xi) = \sum_{i=0}^{n} \phi^i$, where $\phi = \phi(\xi)$ is a solution of the famous KdV equation. By virtue of the solutions of KdV equation we obtain some new solutions for the Sine–Gordon equation and CDGK equation. We find that a) this method is useless for some soliton equations such as Burgers equation and Hirota equation, b) if we let $\phi = \phi(\xi)$ satisfy other soliton equations, for example the Burgers equation, then we can obtain a new transformation between the Burgers equation and other soliton equations, and hence many new solutions can be derived from the solutions of the Burgers equation.

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