Vol 14 No 8, August 2005
1009-1963/2005/14(08)/1526-07

Chinese Physics

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Phase synchronization and anti-phase synchronization of chaos for degenerate optical parametric oscillator

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(Received 2 February 2005; revised manuscript received 25 March 2005)

We have investigated chaotic synchronization in the generalized sense for the degenerate optical parametric oscillator (DOPO). The numerical results show that two unidirectional coupling DOPOs in chaos can be completely phase synchronization or anti-phase synchronization with a suitable coupling coefficient under which the maximum condition Lyapunov exponent (MCLE) is negative. Phase synchronization and anti-phase synchronization of chaos can be realized through positive and negative coupling. On the other hand, the different synchronization states depend on the coupling types used in the DOPO systems.

Keywords: phase synchronization, anti-phase synchronization, chaos, maximum condition Lyapunov exponent

PACC: 0545, 4265

1. Introduction

Synchronization of chaos[1] has aroused much interest due to its potential applications. In particular, the use of chaotic synchronization in secure communication systems and laser dynamics have been investigated by several authors.[2−9] Recently, within the context of chaotic synchronization, different ways to achieve synchronization in nonlinear dynamic systems have been carried out. Synchronization of chaos in erbium-doped fibre dual-ring laser has been investigated via mutual coupling and unidirectional coupling.[10,11] A model consisting of three semiconductor waveguide lasers is considered;[12] these lasers are coupled by means of their overlapping evanescent fields. Synchronization between two chaotic diode resonators has been experimentally carried out by applying a generalization of the occasional proportional feedback scheme.[13] Synchronization of chaotic intensity fluctuations of three modulated pump or loss is investigated experimentally, numerically, and analytically.[14] Generalized synchronization of chaos has been investigated in a class of chaotic synchronized motions for drive-response systems,[15] and other authors have investigated the generalized synchronization of chaos for different YAG lasers.[16] Generalized synchronization can be weaker than phase synchronization depending on parameter misfits. Generalized synchronization does not always lead to phase synchronization.[17]

The optical parametric oscillator (OPO) has seen in recent years a renewal of interest as broadly tunable source of highly coherent radiation. Bistability, self-pulsing, Hoff bifurcations, and chaos of the DOPO were investigated for the first time by Pettiaux, Ruo-Ding Li and Paul Mandel,[18] and the pioneering work of Oppo *et al* has stimulated intense activity on spatiotemporal effect, local structure, and patterns in the mean field models of the continuous-wave DOPOs.[19−24] In this paper, we investigate the phase synchronization and anti-phase synchronization of chaos in two different DOPOs. We find that two different DOPOs can reach phase synchronization or anti-phase synchronization with each other through positive coupling and negative coupling only if the MCLE is negative.

In Section 2 this paper will briefly analyse chaotic behaviour of the DOPO. In Section 3 this paper will put forward a scheme for phase synchronization in the

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DOPOs; the main idea is to couple the DOPOs in two different types. In Section 4 this paper will investigate anti-phase synchronization by positive coupling and negative coupling, and show that anti-phase synchronization is a kind of generalized synchronization.

2. Chaotic behaviour of the DOPO

The dynamical equations of the DOPO are shown as follows:[18]

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = -(1 + \mathrm{i}\Delta_1)A_1 + A_1^* A_0, \qquad (1)$$

$$\frac{\mathrm{d}A_0}{\mathrm{d}t} = -(\gamma + \mathrm{i}\Delta_0)A_0 + E_A - A_1^2. \qquad (2)$$

where A_1 and A_0 are the complex amplitude of the subharmonic mode and fundamental mode of the field respectively, Δ_1 and Δ_0 are the detuning parameters, γ is the reduced decay rate of the fundamental mode, E_A is the input field amplitude which is chosen as real and positive. These are nonlinear complex equations, Pettiaux *et al* have discussed its nonlinear phenomena thoroughly.[18] Eqs.(1) and (2) with parameters $\gamma = \Delta_0$=1.0, Δ_1=–5.0 are numerically examined for different input field amplitude E_A. Figures 1(a)–(c) are the spectrum of the Lyapunov exponents, hyperchaotic attractors, and periodic orbits. From Fig.1 we know the system has hyperchaotic orbits when the input field amplitude is E_A=10.8 (the Lyapunov exponents are 0.25, 0.15, –2.86, and –3.23 respectively). The system has periodic orbits when E_A=8.0.

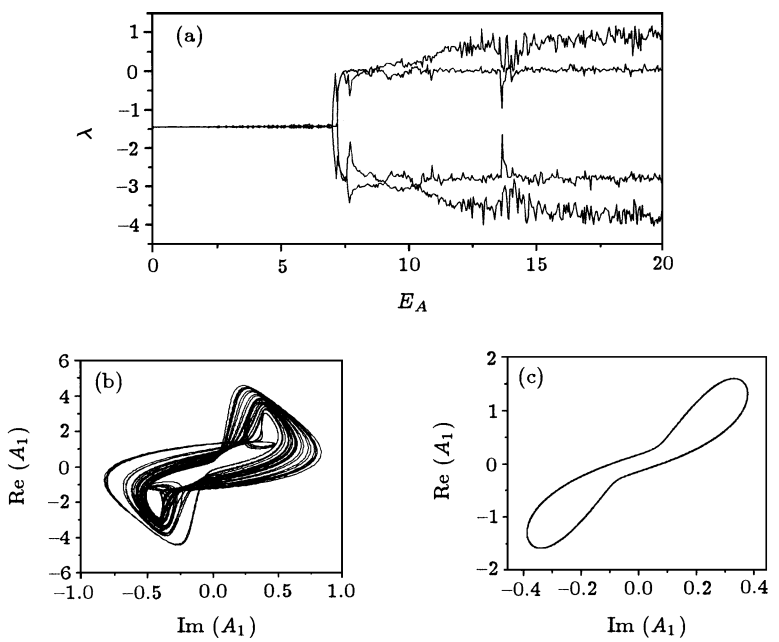


Fig.1. (a) Spectrum of the Lyapunov exponents as a function of the input field amplitude $0 < E_A < 20.0$. (b) Hyperchaotic attractors when E_A=10.8. (c) Periodic orbits when E_A=8.0. The parameters are $\gamma = \Delta_0$=1.0, and Δ_1=–5.0, the initial values are A_1^0=0.1+i0.1, and A_0^0=0.1+i0.1.

3. A unidirectional coupling scheme for phase synchronization

In order to obtain chaotic synchronization, we put forward the scheme shown in Fig.2. In the scheme, system A is a driving system, system B and C are two driven systems driven by a signal from system A through positive coupling or negative coupling at the same time (corresponding to "±" in the circle in Fig.2), and system A is not affected by system B and C.

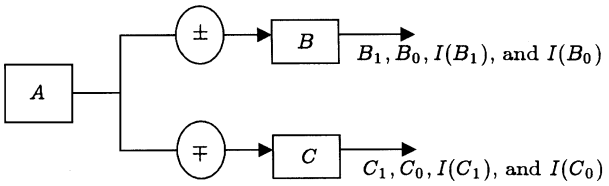


Fig.2. Phase synchronization scheme. System A is a driving system, system B and C are driven systems. B_1, C_1, B_0, and C_0 are the respective complex amplitudes of the subharmonic and fundamental modes of the field. $I(B_1)$, $I(C_1)$, $I(B_0)$, and $I(C_0)$ are the intensity of the subharmonic mode and the fundamental mode. "+" in the circles denotes the positive coupling and "–" denotes negative coupling. B_1 and B_0 will synchronize to C_1 and C_0.

The dynamic equations of system A keep their original form as the Eqs.(1) and (2). The dynamic equations of system B and C can be written in the following expression:
for system B:

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = -(1+\mathrm{i}\Delta_1)(B_1 \pm \varepsilon A_1) + B_1^* B_0, \qquad (3)$$

$$\frac{\mathrm{d}B_0}{\mathrm{d}t} = -(\gamma+\mathrm{i}\Delta_0)B_0 + E_B - B_1^2. \qquad (4)$$

for system C:

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = -(1+\mathrm{i}\Delta_1)(C_1 \pm \varepsilon A_1) + C_1^* C_0, \qquad (5)$$

$$\frac{\mathrm{d}C_0}{\mathrm{d}t} = -(\gamma+\mathrm{i}\Delta_0)C_0 + E_C - C_1^2, \qquad (6)$$

where "+" denotes positive coupling, "–" denotes negative couple. All variables correspond to the Eqs.(1) and (2). System B and C are the same dynamic systems with different chaotic orbits because of different initial conditions. ε is the coupling coefficient from system A to system B and C. Firstly we calculate the MCLE of system B as a function of ε in three cases: (I) three systems are all chaotic; (II) system A is chaotic, system B and C are periodic; (III) system A is periodic , system B and C are chaotic. The parameters in the three systems are the same: $\gamma = \Delta_0$=1.0, and Δ_1=5.0. E_A, E_B, E_C and initial conditions are different in three cases: (I) $E_A = E_B = E_C$=10.8; (II) E_A=10.8, $E_B = E_C$=8.0; (III) E_A=8.0, $E_B = E_C$=10.8. The initial conditions of the systems are A_1^0=0.1+i0.1, A_0^0=0.1+i0.1, B_1^0=0.2+i0.2, B_0^0=0.1+i0.1, C_1^0=0.1+i0.1, and C_0^0=0.2+i0.2. We obtain the MCLEs in the three cases in the range of $0.00 < \varepsilon < 0.30$ as shown in Fig.3.

From Figs.3 (c_1) and (c_2), we know that the MCLE of system B are all positive for case (III), and system B will not synchronize to system C.[15] For case (II), the MCLE are all negative when $\varepsilon > 0.02$. For case (I), the MCLE is negative in the range of $0.18 < \varepsilon < 0.30$. From case (I) and (II) we can find that the MCLE is negative with a suitable value of ε which can be used in a coupling experiment. System B should synchronize to C.

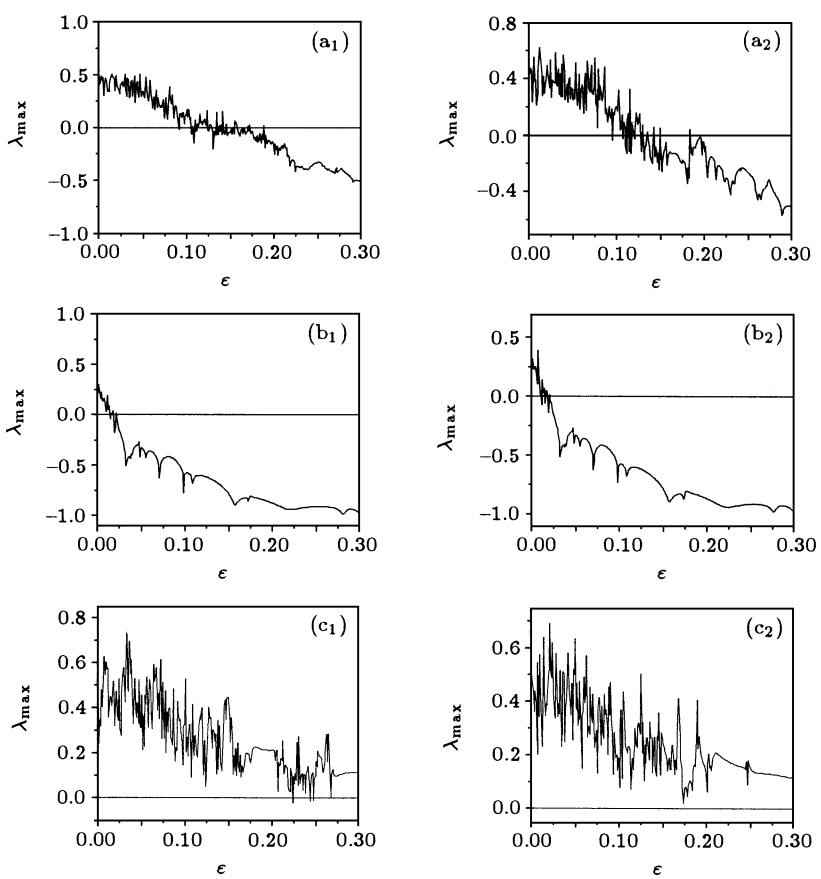


Fig.3. MCLE as a function of unidirectional coupling coefficient ε in three cases, where (a_1), (b_1), (c_1) correspond to positive coupling, (a_2), (b_2), (c_2) correspond to negative coupling.

In the calculation, we take ε=0.2 for case (I) and (II), and the other parameters are the same as those which correspond to the respective cases. Figure 4 shows case (I); though system B and C are chaotic originally, the signals from system B and C driven by system A are different from the original signal and is determined by system A. Case (II) is shown in Fig.5: system B and C become chaotic when it is driven by system A. The chaotic outputs from system B and C are the result of being driven by system A. From Fig.4(b) and Fig.5(b), we cannot find a regular relation between system A and system B. From Fig.4(c) and Fig.5(c) we can find $\mathrm{Re}(B_1)$=$\mathrm{Re}(C_1)$ which is always true after the transient process dies out, the difference of the intensity of the subharmonic mode $I(B_1)$–$I(C_1)$ becomes zero as shown in Fig.4(d) and Fig.5(d), other corresponding variables are also the same. The states

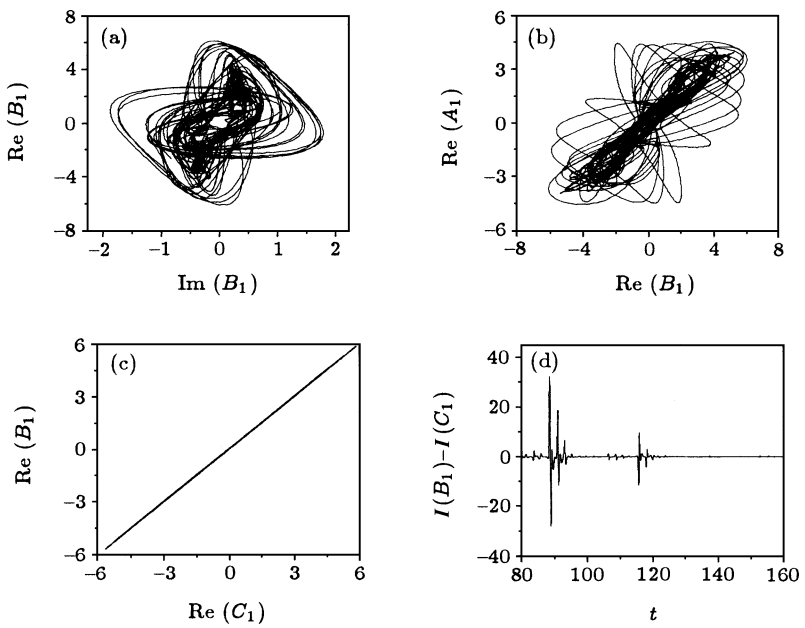


Fig.4. Phase synchronization of system B and C for case (I): (a) the phase portrait $\mathrm{Re}(B_1)$ versus $\mathrm{Im}(B_1)$; (b) the relation between $\mathrm{Re}(A_1)$ and $\mathrm{Re}(B_1)$; (c) the relation between $\mathrm{Re}(B_1)$ and $\mathrm{Re}(C_1)$; (d) the difference of the intensity of the subharmonic mode $I(B_1)$–$I(C_1)$ versus time t.

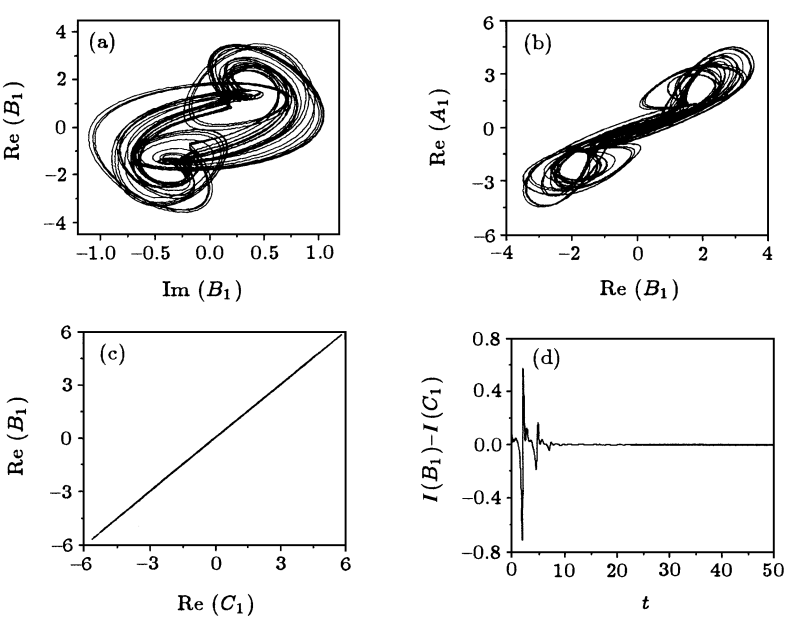


Fig.5. Phase synchronization of system B and C for case (II): (a) the phase portrait $\mathrm{Re}(B_1)$ versus $\mathrm{Im}(B_1)$; (b) the relation between $\mathrm{Re}(A_1)$ and $\mathrm{Re}(B_1)$; (c) the relation between $\mathrm{Re}(B_1)$ and $\mathrm{Re}(C_1)$; (d) the difference of the intensity of the subharmonic mode $I(B_1)$–$I(C_1)$ versus time t.

of system B and C are the same after the transient process dies out even though the initial values are different. In other words, their initial conditions are irrelevant, there is a definite relationship between the two systems. If there is a positive Lyapunov exponent at least, the relationship between driven and driving systems is not definite, though the driving signal also affect the driven system, but they could not reach phase synchronization. The case (III) is a good example. We have gotten the similar result with negative coupling.

4. A unidirectional coupling scheme of anti-phase synchronization

If system A drives system B via positive coupling and drives C via negative coupling or vice versa, anti-phase synchronization of chaos in DOPOs can be realized. we put forward the anti-phase synchronization scheme shown in Fig.6.

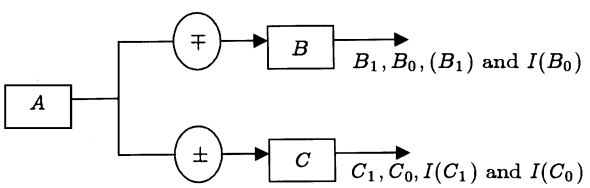


Fig.6. Anti-phase synchronization scheme. A is driving system, B and C are driven systems. All variables and signs are the same as in Fig.(2). B_1 and C_1 will realize anti-phase synchronization.

The dynamic equations of system B and C are expressed as follows:

for system B:

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = -(1+\mathrm{i}\Delta_1)(B_1 \mp \varepsilon A_1) + B_1^* B_0, \tag{7}$$

$$\frac{\mathrm{d}B_0}{\mathrm{d}t} = -(\gamma+\mathrm{i}\Delta_0)B_0 + E_B - B_1^2, \tag{8}$$

for system C:

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = -(1+\mathrm{i}\Delta_1)(C_1 \pm \varepsilon A_1) + C_1^* C_0, \tag{9}$$

$$\frac{\mathrm{d}C_0}{\mathrm{d}t} = -(\gamma+\mathrm{i}\Delta_0)C_0 + E_C - C_1^2. \tag{10}$$

In the calculation, we also take ε=0.2, the other parameters are the same as those which correspond to respective cases. Figure 7 and Fig.8 show the anti-phase synchronization of the subharmonic mode and phase synchronization of the fundamental mode. The fundamental mode is not affected directly, its phase doesn't change so it reaches phase synchronization. The subharmonic mode is coupled in two opposite orientations, the output of the subharmonic mode changes its original orientation, thus anti-phase synchronization of the subharmonic mode appears. The generalized synchronization of chaos was put forward by Henry *et al*;[15] the relation between generalized synchronization and phase synchronization was investigated by Zheng and Hu,[17] who pointed out that if there is a definite function relationship between the two chaotic systems denoted $x(t)$ and $y(t)$, $y(t)$=$\phi(x(t))$ between two different chaotic orbits with different initial point, the chaotic system $x(t)$ and $y(t)$ can be thought of as generalized synchronization. If the two chaotic systems $x(t)$ and $y(t)$ are equal in number and opposite in direction $\lim_{t\to\infty} y(t) = -x(t)$, we can expect them to reach anti-phase synchronization. So anti-phase synchronization is also a generalized synchronization.

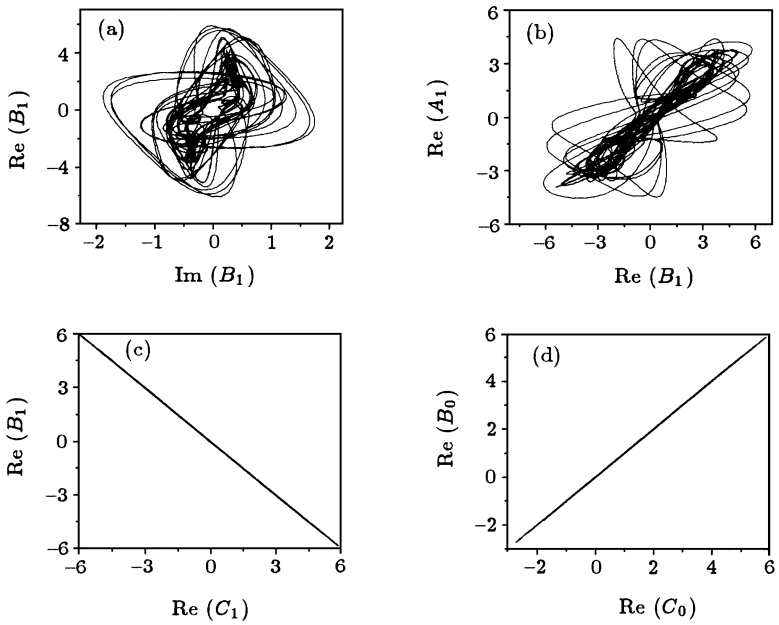


Fig.7. Anti-phase synchronization of B_1 and C_1 for case (I): (a) the phase portrait Re(B_1) versus Im(B_1); (b) the relation between Re(A_1) and Re(B_1); (c) the relation between Re(B_1) and Re(C_1); (d) the relation between Re(B_0) and Re(C_0).

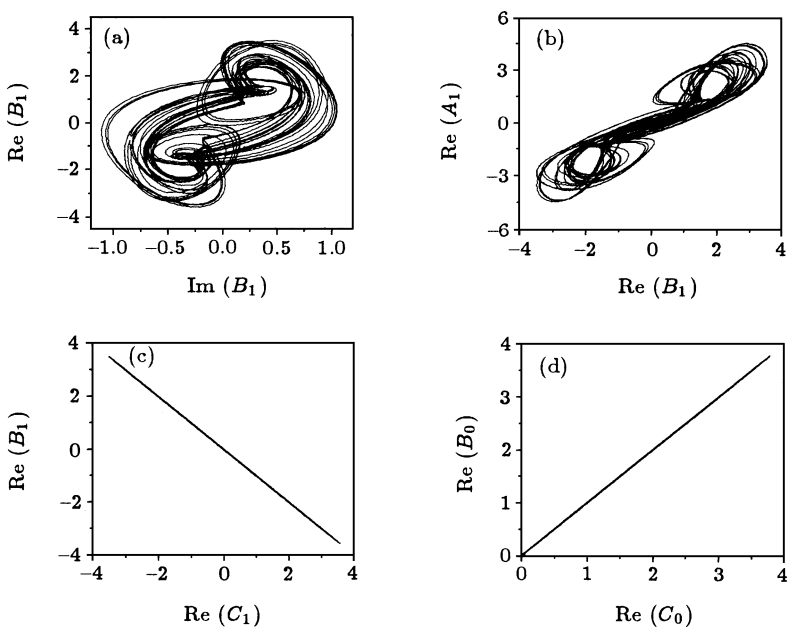


Fig.8. Anti-phase synchronization of B_1 and C_1 for case (II): (a) the phase portrait Re(B_1) versus Im(B_1); (b) the relation between Re(A_1) and Re(B_1); (c) the relation between Re(B_1) and Re(C_1); (d) the relation between Re(B_0) and Re(C_0).

5. Conclusion

Two chaotic systems of the DOPOs with different initial conditions can realize generalized synchronization by unidirectional coupling only if the MCLE of the driven system is negative. This can be realized by adjusting the unidirectional coupling coefficients. The phase synchronization and anti-phase synchronization can be realized in chaotic systems of the DOPOs. The phase synchronization of the subharmonic mode occurs when the two DOPOs are both driven by either positive coupling or negative coupling and the anti-

phase synchronization occurs when one is driven by positive coupling and the other is driven by negative coupling. The fundamental mode of the DOPOs is phase synchronization no matter what coupling way is used. Numerical results have verified these conclusions.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[2] Zhang J S 2000 *Chin. Phys.* **9** 337
[3] Nan M, Wong C N, Tsang K F and Shi X Q 2000 *Phys. Lett.* A **2** 6861
[4] Sugawara T *et al* 1994 Phys. *Rev. Lett.* **72** 3502
[5] Li Z and Han C Z 2002 *Chin. Phys.* **11** 9
[6] Terry J R *et al* 1999 *Phys. Rev.* E **59** 4036
[7] Zhang H, Ma X K, Yang Y and Xu C D 2005 *Chin. Phys.* **1** 67
[8] Lu J G and Xi Y G 2005 *Chin. Phys.* **2** 274
[9] Sivaprakasam S *et al* 2001 *Phys. Rev.* A **64** 01805-1
[10] Wang R and Shen K 2001 *Chin. Phys.* **10** 711
[11] Zhang S H and Shen K 2002 *Chin. Phys.* **9** 894
[12] Winful H G and Rahman L 1990 *Phys. Rev. Lett.* **65** 1575
[13] Newell T C, Alsing P M, Gavrielides A and Kovanis V 1994 *Phys. Rev. Lett.* **72** 1647
[14] Carlos L and Pando L 1998 *Phys. Rev.* E **57** 2725
[15] Henry D I A, Nikolai F R and Mikhai M S 1996 *Phys. Rev.* E **53** 4528
[16] Zhu S Q, Luo X Q and Fang J X 2001 *Chin. Phys.* **9** 337
[17] Zhang Z G and Hu G 2000 *Phys. Rev.* E **62** 7882
[18] Pettiaux N P, Li R D and Mandel P 1998 *Opt. Commun.* **72** 256
[19] Oppo G L, Bramblla and Lugiato L A 1994 *Phys. Rev.* A **49** 2028
[20] Oppo G L, Bramblla, Gamesasca D G and Lugiato L A 1994 *J. Mod. Opt.* **41** 1151
[21] Valcarcel D E, Staliunas G J, Roloav E and Morcillo V S 1996 *Phys. Rev.* A **54** 1609
[22] Trillo S, Haelterman M and Shepparo A 1997 *Opt. Lett.* **22** 970
[23] Sialiunas K and Morcillo S V 1998 *Phys. Rev.* A **57** BR1454
[24] Sialiunas K and Morciuo V J 1999 *Opt. Commun.* **139** 360