Vol 14 No 8, August 2005
1009-1963/2005/14(08)/1522-04

Chinese Physics

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Master–slave synchronization of coupled fractional-order chaotic oscillators

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(Received 15 January 2005; revised manuscript received 19 April 2005)

The dynamics of fractional-order systems have attracted increasing attentions in recent years. In this paper, we study the synchronization technique based on the master–slave synchronization scheme and apply it to the synchronization of two coupled nonlinear fractional-order electronic chaotic oscillators. Simulations show that two coupled fractional-order chaotic oscillators can be brought to an exact synchronization with appropriate coupling strength. It is interesting that the synchronization rate of the fractional-order chaotic oscillators is slower than its integer-order counterpart; however, with the increase of system order, the curves of synchronization error can be smoothened, which indicates that the master–slave synchronization of two coupled fractional order oscillators can be smoothened and stabilized.

Keywords: fractional order, coupled chaotic oscillators, master–slave synchronization
PACC: 0545

1. Introduction

Chaotic orbits are known to be sensitive to initial conditions. It is indeed surprising to find that two chaotic orbits which are initiated differently can be brought to synchronize with each other. Owing to its conceptual significance and potential applications in secure communication,[1,2] tremendous efforts have been made recently in the studies of chaotic synchronization. Particularly, many interesting chaotic synchronization schemes have been reported since the seminal paper by Pecora and Carroll,[3] such as adaptive synchronization method,[4,5] pulse synchronization method,[6,7] observer synchronization method,[8,9] variable structure control[10,11] and coupling synchronization method,[12,13] etc.

The applications of fractional calculus to physics and engineering are a recent focus of interest, although fractional calculus is a 300-year-old topic. Many systems are known to display fractional-order dynamics, such as viscoelastic systems,[14] electrode–electrolyte polarization,[15] and electromagnetic waves.[16] More recently, many investigations are devoted to the chaotic behaviours and chaotic control of fractional-order dynamical systems, such as the fractional-order Chua system,[17] fractional-order Chen system[18] and fractional-order Rössler equation.[19] However, not much has been reported on the possibility of chaotic synchronization in fractional-order dynamical systems.

In this paper, we study the master–slave synchronization technique and apply it to the synchronization of two coupled fractional-order chaotic oscillators. Simulations show that the two coupled fractional-order chaotic oscillators can be brought to an exact synchronization with appropriate coupling strength. The synchronization rate of the fractional-order chaotic oscillators is slower than its integer-order counterpart; however, with the increase of system order, the curves of synchronization error can be evidently smoothened, indicating that the master–slave synchronization of two coupled fractional-order chaotic oscillators can be smoothened and stabilized.

2. Fractional derivative and its approximation

Two commonly used definitions for the general fractional differintegral is the Riemann–Liouville (RL)

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definition.[20] The RL definition is given here by

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}}\mathrm{d}\tau, \qquad (1)$$

where $n-1 \le q < n$ and $\Gamma(\bullet)$ is the Euler's gamma function. Upon considering all the initial conditions to be zero, the Laplace transform of the RL fractional derivative is

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\}. \qquad (2)$$

Thus, the fractional integral operator of order "q" can be represented by the transfer function $F(s) = 1/s^q$ in the frequency domain.

The standard definitions of fractional differintegral do not allow direct implementation of the operator in time-domain simulations. An effective method to deal with this problem is to approximate fractional operators by using the standard integer-order operators. In the following simulations, we use the approximation approach proposed in Ref.[21]. In Table 1 of Ref.[17], approximations for $1/s^q$ with q=0.1–0.9 in steps of 0.1 were given with errors of approximately 2dB. We will mainly use these approximations in the following simulations.

3. Synchronization of coupled fractional-order chaotic oscillators

The model studied in this paper is an electronic chaotic oscillator of canonical structure, and one control parameter. It has been reported in Ref.[22], and conjectured as the simplest possible approach to a chaotic oscillator. This oscillator is known to give a double-scroll-like chaotic attractor in the range $1.0 > a > 0.49$, with a being a parameter of the oscillator.

Consider the master–slave synchronization scheme of two fractional-order chaotic oscillators with the master oscillator M and the slave oscillator S.

M:

$$\begin{aligned}
\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= y_1,\\
\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= z_1,\\
\frac{\mathrm{d}^q z_1}{\mathrm{d}t^q} &= -a[x_1 + y_1 + z_1 - f(x_1)];
\end{aligned} \qquad (3)$$

S:

$$\begin{aligned}
\frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= y_2 + c(x_1 - x_2),\\
\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= z_2,\\
\frac{\mathrm{d}^q z_2}{\mathrm{d}t^q} &= -a[x_2 + y_2 + z_2 - f(x_2)],
\end{aligned} \qquad (4)$$

where

$$f(x) = \mathrm{sgn}(x) = \left\{\begin{array}{ll} 1, & x \ge 0\\ -1, & x < 0 \end{array}\right\},$$

$q > 0$ is the fractional order, $c > 0$ is the coupling strength. For the sake of simplicity, only the first variable x is used for coupling the two fractional-order chaotic oscillators. Define the error signal as $e = x - y$, the aim of the synchronization scheme is to design the coupling strength such that $||e(t)|| \to 0$ as $t \to \infty$. This scheme is similar to the master–slave synchronization of classical integer-order chaotic systems.

When q= 0.9 and a=0.21, the fractional-order oscillator is chaotic. The phase plot of x and z is shown in Fig.1.

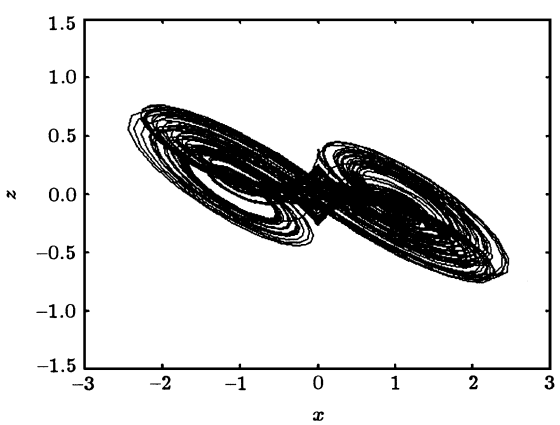


Fig.1. Phase plot of the fractional-order oscillator with q=0.9.

Next, we study numerically the synchronization of fractional-order chaotic oscillators. To obtain a critical value of c to make the two oscillators synchronized, we continuously increase the coupling strength c. When $c < 0.5$, no synchronous phenomenon is observed. When c=0.5, the curve of the synchronization error $J(t) = \log(||e(t)||)$ is shown in Fig.2(a), which indicates that the master–slave synchronization is achieved. In Fig.2(b), we show the curve of the synchronization error when c=2, in which the synchronization effect is better than that of c=0.5.

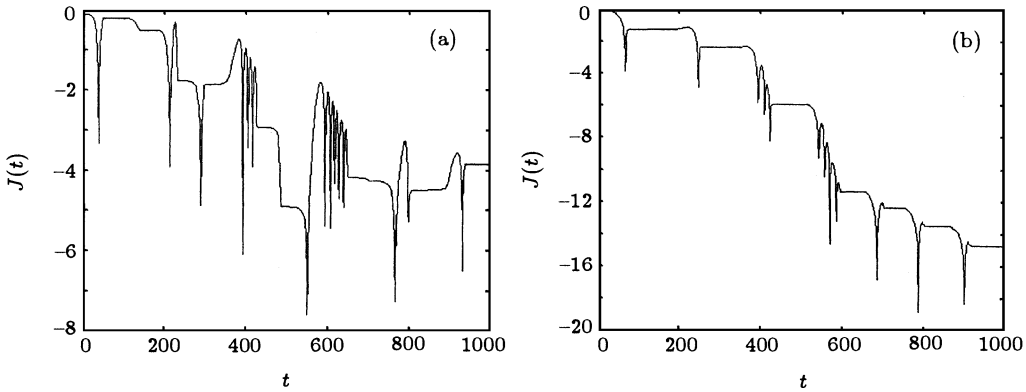


Fig.2. Synchronization error of the fractional-order chaotic oscillators with q=0.9. (a) c=0.5, (b) c=2.

In Figs.3(a) and (b), we show the synchronization of two nonlinear chaotic oscillators under different initial conditions, the control is applied at time t=80.

For the purpose of comparison, we also plot the curves of synchronization error of the integer order (a=0.8) and fractional order (q=3.2, a=1) chaotic oscillators in Figs.4 and 5. Comparing Fig.2 with Figs.4 and 5, we can find that the synchronization rate of the fractional-order chaotic oscillators is slower than its integer-order counterpart. However, with the increase of system order q and coupling strength c, the curves of synchronization error can be smoothened, which indicates that the master–slave synchronization of two coupled fractional-order oscillators can be smoothened and stabilized.

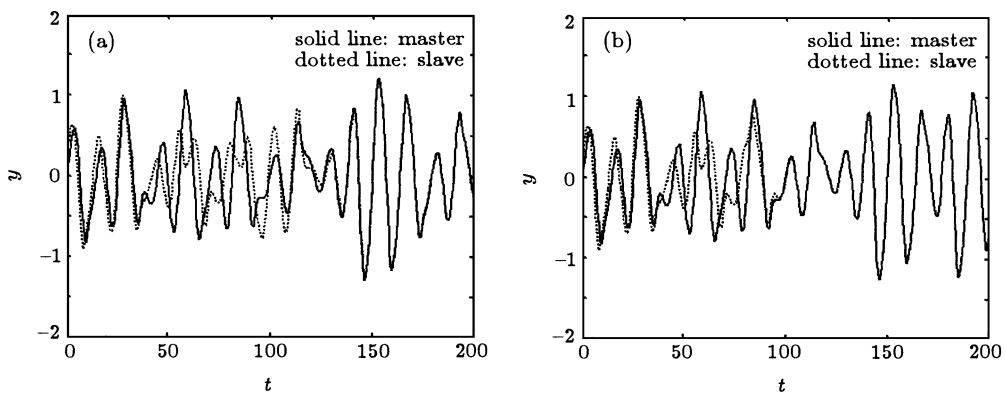


Fig.3. Synchronization of the fractional-order chaotic oscillators with q= 0.9. (a) c=0.5, (b) c=2.

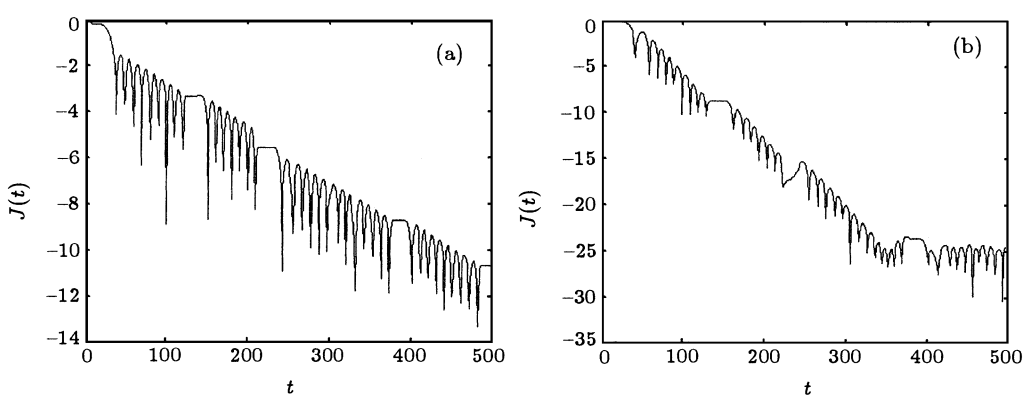


Fig.4. Synchronization error of the integer-order chaotic oscillators. (a) c=0.5, (b) c=2.

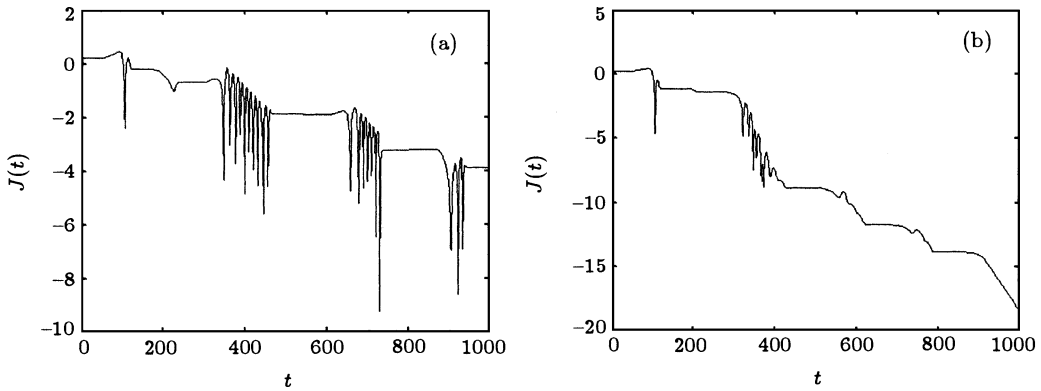


Fig.5. Synchronization error of the fractional-order chaotic oscillators with q=3.2. (a) c=0.5, (b) c=2.

4. Conclusion

We have studied the master–slave synchronization of coupled fractional-order chaotic oscillators. We find that two fractional-order chaotic oscillators can be brought to an exact synchronization with appropriate coupling strength. With the increase of system order, the process of synchronization of two coupled fractional-order oscillators can be smoothened and stabilized. Chaotic synchronization in fractional-order systems is intricate. Future work regarding this topic includes the investigation of other synchronization phenomena of chaotic dynamical systems, such as phase synchronization,[23,24] generalized synchronization,[25,26] and projective synchronization[27] etc.

Acknowledgement

The authors are grateful to Professor Li Chun-Guang and Dr Zhang Jian for discussions on the chaotic synchronization schemes.

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