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Chaotic dynamics and synchronization of fractional-order Genesio–Tesi systems*

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In this paper, we investigate numerically the chaotic behaviours in the fractional-order Genesio–Tesi system. We find that chaos exists in the fractional-order Genesio–Tesi system with order less than 3. The lowest order we find to have chaos is 2.4 in this fractional-order Genesio–Tesi system. We propose a drive-response synchronization method for synchronizing the fractional-order chaotic Genesio–Tesi systems only using a scalar drive signal. This synchronization approach, based on stability theory of fractional-order systems, is simple and theoretically rigorous. It does not require the computation of the conditional Lyapunov exponents. Simulation results are used to visualize and illustrate the effectiveness of the proposed synchronization method.

Keywords: chaos, synchronization, Genesio–Tesi system, fractional-order system, fractional calculus
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1. Introduction

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with the correspondence between Leibniz and L'Hospital in 1695, where the meaning of derivative of order one half was discussed.[1] Although fractional calculus has a 300-year-old history, its applications to physics and engineering are just a recent focus of interest. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems,[2] dielectric polarization,[3] electrode–electrolyte polarization,[4] electromagnetic waves,[5] quantitative finance[6] and quantum evolution of complex systems.[7] More recently, many investigations are devoted to control[8−12] and dynamics[13−20] of fractional-order dynamical systems.

On the other hand, synchronization of chaotic systems has attracted much attention[21] since the seminal paper by Pecora and Carroll.[22]

In this paper, we investigate the chaotic behaviours in the fractional-order Genesio–Tesi system.[23] Like many other studies of dynamics in fractional-order systems, we investigate the dynamics of the fractional-order Genesio–Tesi system through numerical simulations. And a drive-response synchronization method is also presented for synchronizing the fractional-order chaotic Genesio–Tesi systems only using a scalar drive signal. This synchronization approach, based on stability theory of fractional-order systems, is simple and theoretically rigorous. Simulation results are used to visualize and illustrate the effectiveness of the proposed synchronization method.

2. Fractional derivative and its approximation

Fractional calculus is a generalization of integration and differentiation to a non-integer-order integro-differential operator ${}_aD_t^\alpha$ defined by

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} & R(\alpha) > 0, \\ 1 & R(\alpha) = 0, \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-\alpha} & R(\alpha) < 0. \end{cases} \tag{1} $$

Two commonly used definitions for the general fractional differintegral are Grunward–Letnikov definition and Riemann–Liouville definition.[1] The

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Riemann–Liouville definition is given here:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad (2)$$

for $n-1 \le \alpha < n$, where $\Gamma(\cdot)$ is the Gamma function. The geometric and physical interpretation of the fractional derivatives was given in Ref.[24].

The Laplace transform of the Riemann–Liouville fractional derivative (2) has the form[1]

$$\int_0^\infty \mathrm{e}^{-st}{}_0D_t^\alpha f(t)\mathrm{d}t = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k {}_0D_t^{\alpha-k-1} f(t)\,|_{t=0}\,, \quad (3)$$

for $n-1 \le \alpha < n$, where $s \equiv j\omega$ denotes the Laplace operator. Assuming the initial conditions to be zero, Eq.(3) reduces to

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\}. \quad (4)$$

Thus, the fractional integral operator of order "α" can be represented by the transfer function $F(s) = \dfrac{1}{s^\alpha}$ in the frequency domain.

The standard definitions of fractional differintegral do not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer-order operators. In Ref.[25], an effective algorithm is developed to approximate fractional-order transfer functions. Basically, the idea is to approximate the system behaviour in the frequency domain. This approximation method has been adopted in Refs.[13–17] and has sufficient accuracy for time domain hardware implementations. In Table 1 of Ref.[13], approximations for $1/s^\alpha$ with $\alpha = 0.1-0.9$ in steps 0.1 were given with errors of approximately 2dB. We will mainly use these approximations in the following simulations.

3. Chaotic dynamics of the fractional-order Genesio–Tesi system

The Genesio–Tesi system[23] is described by

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_3,\\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^2,\end{aligned} \quad (5)$$

where β_1, β_2, β_3 and β_4 are system parameters. When $\beta_1 = 1.0$, $\beta_2 = 1.1$, $\beta_3 = 0.44$ and $\beta_4 = 1$, in the system (5) there exists a chaotic attractor as shown in Fig.1.

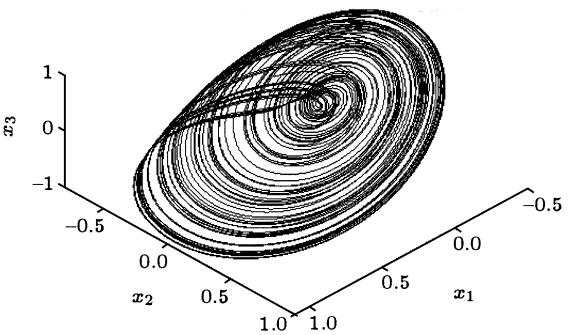


Fig.1. Chaotic attractor of the integer-order Genesio–Tesi system (5).

Now, we consider the fractional-order Genesio–Tesi system

$$\begin{aligned}\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= x_2,\\ \frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= x_3,\\ \frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^2.\end{aligned} \quad (6)$$

where $0 < q \le 1$. In the following simulations in this section, we only vary derivative order q and the system parameter β_3, the other system parameters are fixed.

Simulations are performed for $q = 0.9$, 0.8, 0.7, 0.6, 0.5. The simulation results demonstrate that chaos indeed exists in the fractional-order Genesio–Tesi system (6) with order less than 3. When $q = 0.9$, 0.8, chaotic attractors are found and the phase portraits are shown in Figs.2, and 3, respectively. The largest Lyapunov exponents of the fractional-order Genesio–Tesi systems (6) are obtained as follows from the simulation time series:

1) when $q = 0.9$ and $\beta_3 = 0.15$, the largest Lyapunov exponent is 2.6;

2) when $q = 0.8$ and $\beta_3 = -0.232$, the largest Lyapunov exponent is 2.0.

When q=0.7, 0.6, 0.5, no chaotic behaviour is found, indicating that in this case the lowest limit of the fractional order q for the system (6) to be chaotic is q=0.7–0.8. Thus, the lowest order we found for this fractional-order Genesio–Tesi system (6) to yield chaos is 2.4.

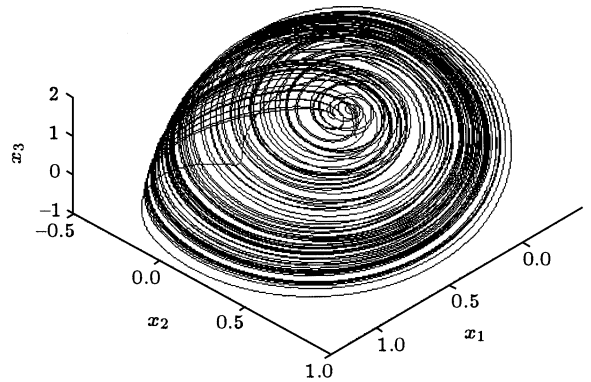


Fig.2. Chaotic attractor of the fractional-order Genesio–Tesi system (6) with $q = 0.9$ and $\beta_3 = 0.15$.

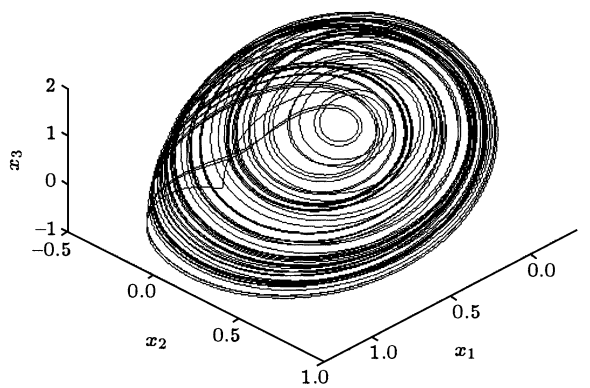


Fig.3. Chaotic attractor of the fractional-order Genesio–Tesi system (6) with $q = 0.8$ and $\beta_3 = -0.232$.

4. Synchronization of the fractional-order chaotic Genesio–Tesi systems

It is worth noting that the adoption of a scalar drive signal is a suitable and expected feature for chaos synchronization and secure communication in practical application. For this reason, in this section, we propose a drive-response synchronization method for synchronizing the fractional-order chaotic Genesio–Tesi systems only using a scalar drive signal. This synchronization approach, based on control theory of fractional-order systems, is simple and theoretically rigorous.

In the proposed drive-response synchronization method, the Genesio–Tesi system (6) is the drive system, the linear scalar drive signal is chosen to be

$$s(t) = x_1, \tag{7}$$

and the driven (response) system is constructed as

$$\begin{aligned}
\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= y_2 + k_1(x_1 - y_1),\\
\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= y_3 + k_2(x_1 - y_1),\\
\frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= -\beta_1 y_1 - \beta_2 y_2 - \beta_3 y_3 + \beta_4 x_1^2 + k_3(x_1 - y_1),
\end{aligned} \tag{8}$$

where $K = [k_1,\ k_2,\ k_3]^{\mathrm{T}}$ are design parameters to be chosen such that the synchronization error $\|x(t) - y(t)\|_2 \to 0$ as $t \to \infty$. Defining the synchronization error $e(t) = x(t) - y(t)$, the fractional-order synchronization error system can be written as follows:

$$\begin{aligned}
\frac{\mathrm{d}^q e_1}{\mathrm{d}t^q} &= e_2 - k_1 e_1,\\
\frac{\mathrm{d}^q e_2}{\mathrm{d}t^q} &= e_3 - k_2 e_1,\\
\frac{\mathrm{d}^q e_3}{\mathrm{d}t^q} &= -\beta_1 e_1 - \beta_2 e_2 - \beta_3 e_3 - k_3 e_1.
\end{aligned} \tag{9}$$

Equation (9) can be written as

$$\frac{\mathrm{d}^q e}{\mathrm{d}t^q} = (A - KC)e, \tag{10}$$

where $e = [e_1,\ e_2,\ e_3]^{\mathrm{T}}$, $C = [1,\ 0,\ 0]$ and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\beta_1 & -\beta_2 & -\beta_3 \end{bmatrix}.$$

From stability theory of fractional-order linear systems,[26] we obtain the following theorem.

Theorem 1 For $0.8 \le q \le 1$, the global asymptotic chaos synchronization between the fractional-order Genesio–Tesi system (6) and the response system (8) can be obtained by choosing K such that $|\arg(\lambda_i(A - KC))| > 0.5\pi\, q, i = 1,\ 2,\ 3$, where $\arg(\lambda_i(A - KC))$ denotes the argument of the eigenvalue λ_i of $(A - KC)$ in Eq.(10).

Since the observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

is of full rank, from linear system theory,[27] it follows that the eigenvalues of $A - KC$ in the system (10) can be placed anywhere by a proper choice of gains k_i, $i = 1, 2, 3$ and then there surely exist k_i, $i = 1, 2, 3$ such that $|\arg(\lambda_i(A - KC))| > 0.5\pi\, q$, $i = 1, 2, 3$. The solution to the problem of determining K such that the arguments of all the eigenvalues (or poles) of the matrix $A - KC$ in Eq.(10) satisfy

$|\arg(\lambda_i(A-KC))| > 0.5\pi q$, i.e., the fractional-order Genesio–Tesi system (6) and the response system (8) are in a state of global asymptotic synchronization that can be easily obtained by the "pole placement technique" from control systems theory such as Ackermann's method.[27] In the following simulations, we take the fractional-order Genesio–Tesi systems with $q = 0.8,\ 0.9$ as examples.

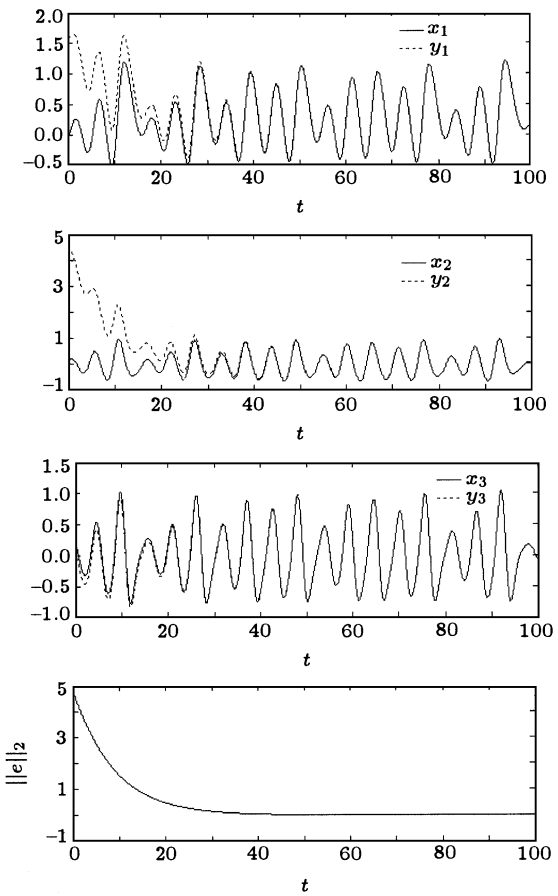


Fig.4. The simulation results with $q = 0.9$ and $\beta_3 = 0.15$.

When $q = 0.9$ and $\beta_3 = 0.15$, the attractor of the fractional-order Genesio–Tesi system (6) is shown in Fig.2. From Theorem 1, we know that the fractional-order Genesio–Tesi system (6) and the response system (8) with $q = 0.9$ are in a state of global asymptotic synchronization for $|\arg(\lambda_i(A-KC))| > 0.45\pi q$, $i = 1, 2, 3$. For example, we choose $k_1 = 2.85$, $k_2 = 1.4725$, $k_3 = -3.3559$ such that the eigenvalues of $A-KC$ in the system (10) are those specified in vector $[-1, -1, -1]$. The simulation results are shown in Fig.4, which indicates that the chaos synchronization between systems (6) and (8) is achieved.

When $q = 0.8$ and $\beta_3 = -0.232$, the attractor of the fractional-order Genesio–Tesi system (6) is shown in Fig.3. From Theorem 1, we know that the fractional-order Genesio–Tesi system (6) and the response system (8) with $q = 0.8$ are in a state of global asymptotic synchronization for $|\arg(\lambda_i(A-KC))| > 0.4\pi q$, $i = 1, 2, 3$. For example, we choose $k_1 = 3.232$, $k_2 = 2.6498$, $k_3 = -2.9404$ such that the eigenvalues of $A-KC$ in the system (10) are those specified in vector $[-1, -1, -1]$. The simulation result is shown in Fig.5, which indicates that the chaos synchronization between systems (6) and (8) is achieved.

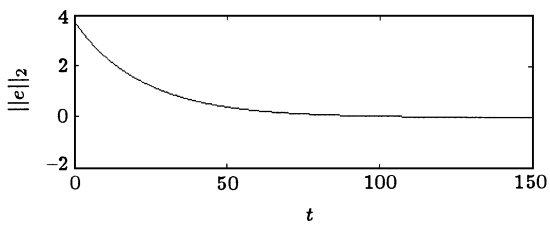


Fig.5. The simulation result with $q = 0.8$ and $\beta_3 = -0.232$.

5. Conclusion

In this paper, we have studied numerically the chaotic behaviours in the fractional-order Genesio–Tesi system. The lowest order we find to have chaos is 2.4 in this fractional-order Genesio–Tesi system. Our results have been validated by the existence of a positive Lyapunov exponent. We have also proposed a drive-response synchronization method for synchronizing the fractional-order chaotic Genesio–Tesi systems via a scalar drive signal. This synchronization approach, based on the stability theory for fractional-order systems, is simple and theoretically rigorous. It does not require the computation of the conditional Lyapunov exponents. Simulation results have illustrated the effectiveness of the proposed synchronization method.

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