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Effect of noise on trace distance of remote state preparation*

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The influence of noise on the remote preparation of a qubit by an entangled pair is investigated. In Bloch sphere representation, we use the trace distance to describe how close the final state is to the original state to be prepared. Our studies include two cases. Firstly, we consider the sender and the receiver sharing a mixture of two Bell states as quantum channel in remote state preparation (RSP), and calculate the trace distance as a function of mixture ratio and angle of a state to be prepared. Secondly, the system–environment interaction is taken into account by including stochastic fluctuating terms in the system Hamiltonian. Solving the Bloch equations, we obtain the evolution density matrix of the system. We then apply this stochastic model to study the effect of noise on the trace distance of RSP.

Keywords: remote state preparation, trace distance, density matrix
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1. Introduction

Quantum entanglement and the nonlocal property, being important resources in the field of quantum computer and quantum information, have been extensively applied. By these resources, protocols about transmission or preparation of quantum state, such as teleportation[1−4] and remote state preparation (RSP)[5−11] have been proposed. In the standard teleportation protocol,[1] Alice (a sender) can exactly send an unknown quantum state to a remote receiver (called Bob) via dual classical and quantum channels. Compared with teleportation, RSP may be called the teleportation of a known state. In RSP, Alice does not possess a physical instance of the state to be prepared, but is aware of its full classical description. This was first addressed by Pati[5] and Lo.[6] In their protocols, if Alice wants to implement RSP, she performs a measurement on her share of the entangled resource in a basis chosen in accordance with the state to be prepared. Dependent on the outcome of her measurement, the entangled ensemble collapses either into the desired state at Bob's place or can be converted into it by a local unitary operation. Bennett *et al* have shown that the asymptotic classical communication cost of RSP is one bit per qubit and even less when transmitting part of a known entangled state.[7] An experimental scheme of RSP of a single-mode photonic qubit has been proposed in Ref.[8]. An experimental implementation of RSP scheme by nuclear magnetic resonance has been also reported in Ref.[9].

In Pati's RSP protocol, two spin-1/2 particles are studied and a maximally entangled state, e.g. a singlet state, is required for quantum channel. However, the decoherence effects due to the environment make the pure entangled state become a mixture and degrade quantum entanglement in the real world. In this paper, we consider the RSP protocol of Pati in noisy channel. We apply the trace distance[12] to describe how close the replica state is to the original state to be prepared. In Section 2, we assume that the quantum channel of RSP is in statistical mixture of two Bell states due to the influence of noise. We obtain the trace distance depending on the parameters of the state to be prepared and mixture ratio λ of Bell states. If mixture ratio $\lambda = 0$ or 1, the quantum channel is in maximally entangled state. RSP can be successfully realized. If $\lambda = 1/2$, we can realize remote preparation of an arbitrary mixed state. In Section 3, we consider

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that the model of coupling of spins with the environment is in accord with the stochastic model and study the dynamical decoherence of RSP. Our result shows that the average trace distance decays to 1/2 due to effect of noise. A brief conclusion is given in Section 4.

2. RSP via a mixture of two Bell states

In the protocol of RSP of Pati, Alice wants to prepare an arbitrary qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α is a real number and β is a complex number. In the Bloch sphere picture, this state can be expressed as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle$, where θ and φ are the polar and azimuthal angles respectively. The density matrix of the qubit can be written as

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r}\cdot\boldsymbol{\sigma}}{2}. \tag{1}$$

where $\boldsymbol{r}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is known as the Bloch vector for the state ρ, and $\boldsymbol{\sigma}(\sigma_x,\ \sigma_y,\ \sigma_z)$ denotes a vector of Pauli matrices. To realize RSP, Alice and Bob need to share the two-particle maximally entangled state as a quantum channel. Because Alice knows the classical description of the state, she carries out measurement on particle 1 by projecting onto the "qubit basis" $\{|\psi\rangle_1, |\psi_\perp\rangle_1\}$, where $|\psi_\perp\rangle_1$ is orthogonal to the original one, and then sends her outcome of measurement to Bob through the classical channel. According to the result $|\psi_\perp\rangle_1$(or $|\psi\rangle_1$), Bob performs the rotation operation or does nothing on particle 2. Particle 2 will be in the state (1) (or its orthogonal state). Throughout this paper, we do not consider that the state $|\psi_\perp\rangle$ alone can be unitarily converted into $|\psi\rangle$ for the special ensemble, e.g. equatorial states. In this section we assume that the interaction between entangled state and environment induces a kind of single phase decoherence, so the quantum channel is in a mixture of two Bell states:

$$\rho_{12} = \lambda\left|\psi^-\right\rangle_{12,12}\left\langle\psi^-\right| + (1-\lambda)\left|\psi^+\right\rangle_{12,12}\left\langle\psi^+\right|, \tag{2}$$

The two-particle state can be written in terms of Pauli matrices in the following form:

$$\begin{aligned}\rho_{12} =& \frac{\lambda}{4}[\boldsymbol{I}_1\otimes\boldsymbol{I}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2] \\ &+ \frac{1-\lambda}{4}[\boldsymbol{I}_1\otimes\boldsymbol{I}_2 + \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} - \sigma_{1z}\sigma_{2z}],\end{aligned} \tag{3}$$

The corresponding projection operators can be expressed as

$$P_1 = |\psi\rangle_{1,1}\langle\psi| = (\boldsymbol{I}_1 + \boldsymbol{r}_1\cdot\boldsymbol{\sigma}_1)/2, \tag{4a}$$

$$P_{\perp 1} = |\psi_\perp\rangle_{1,1}\langle\psi_\perp| = (\boldsymbol{I}_1 + \boldsymbol{r}_{\perp 1}\cdot\boldsymbol{\sigma}_1)/2. \tag{4b}$$

where Bloch vectors $\boldsymbol{r}_1 = -\boldsymbol{r}_{\perp 1} = \boldsymbol{r}$

Based on Pati's protocol, after Alice performs the measurement of the operator $P_{\perp 1}$, the projected density matrix of the particle 2, is given by

$$\rho_2 = 2\mathrm{Tr}_1[\rho_{12}P_{\perp 1}], \tag{5a}$$

$$= (\boldsymbol{I}_2 + \boldsymbol{r}_2\cdot\boldsymbol{\sigma}_2)/2, \tag{5b}$$

where the factor of 2 assures $\mathrm{Tr}\rho_2 = 1$, and the Bloch vector $\boldsymbol{r}_2$ of particle 2 is expressed as

$$\begin{aligned} r_{2x} &= (2\lambda-1)\sin\theta\cos\varphi, \\ r_{2y} &= (2\lambda-1)\sin\theta\sin\varphi\ , \quad 1/2 \le \lambda \le 1, \\ r_{2z} &= \cos\theta, \end{aligned} \tag{6}$$

when $0 \le \lambda < 1/2$, Bob needs to perform σ_z operation on particle 2, so the Bloch vector $\boldsymbol{r}_2$ is given by

$$\begin{aligned} r_{2x} &= (1-2\lambda)\sin\theta\cos\varphi, \\ r_{2y} &= (1-2\lambda)\sin\theta\sin\varphi, \quad 0 \le \lambda < 1/2, \\ r_{2z} &= \cos\theta. \end{aligned} \tag{7}$$

From Eqs.(6) and Eqs.(7), when $\lambda = 1$ or 0, $\boldsymbol{r}_2 = \boldsymbol{r}$, the state (1) is successfully prepared in Bob's location, which has been discussed in Pati's report.[5] When $\lambda = 1/2$, particle 2 is in a mixed state. In this case, we can realize the RSP of an arbitrary mixed state. For the noisy channel, it is important and nontrivial to know how close the final state (5b) is to the initial state (1). We use the trace distance[12] to describe this question. In the Bloch sphere representation, the trace distance between the states (1) and (5) is defined as

$$D(\rho,\rho_2) = \frac{|\boldsymbol{r} - \boldsymbol{r}_2|}{2}. \tag{8}$$

That is, the distance between two single qubit states is equal to one half the ordinary Euclidean distance between them on the Bloch sphere. For qubits, the trace distance between them has more intuitive geometric picture than the fidelity. According to Eqs.(1), (6), (7) and (8), we can calculate the trace distance as

$$D(\rho,\rho_2) = \begin{cases} \lambda\sin\theta, & 0 \le \lambda < 1/2, \\ (1-\lambda)\sin\theta, & 1/2 \le \lambda \le 1, \end{cases} \tag{9}$$

Here the trace distance depends on the initial state as well as the mixture ratio λ. Since in general a state to

be prepared is arbitrary, it is more useful to calculate the average trace distance. The average trace distance is defined as

$$D_{av} = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} D(\rho,\rho_2)\sin\theta \mathrm{d}\theta, \tag{10}$$

where 4π is the total solid angle of a closed surface along its inner point.

Substituting Eqs.(9) into Eq.(10), we get the average trace distance

$$D_{av} = \begin{cases} \dfrac{\pi}{4}\lambda, & 0 \le \lambda < 1/2, \\ \dfrac{\pi}{4}(1-\lambda), & 1/2 \le \lambda \le 1. \end{cases} \tag{11}$$

When $\lambda = 1/2$, the average trace distance has a maximum value, that is, information about the state (1) mostly lost.

3. RSP in stochastic model

In this section, the system–environment interactions are taken into account by including stochastic fluctuating terms in the system Hamiltonian. In RSP,[5] the entangled state of two spin-1/2 particles is taken to be any of the four Bell states. Due to the interaction of the system with the environment, the two-particle system will evolve with time. Here we consider isotropical decoherence, so the Bell state degenerates into a Werner state. The most general form for the two-particle density matrix has the form

$$\rho_{12}(t) = \frac{1}{4n}\sum_{m=1}^{n}[\boldsymbol{I}_1 + \boldsymbol{P}_1(t,m)\cdot\boldsymbol{\sigma}_1] \otimes [\boldsymbol{I}_2 + \boldsymbol{P}_2(t,m)\cdot\boldsymbol{\sigma}_2] \tag{12}$$

where $\boldsymbol{P}_i(t,m)$ is the usual polarization vector corresponding to an initial condition labelled by index m. In this section, we treat the spins of the entangled pair to be independent. This is not strictly true, as the environment does induce some interaction between the particles. Here we assume that the particles in the entangled pair are sufficiently well separated as one expects in the situation of RSP, so this interaction should be too small to have a qualitative effect and our treatments should be valid. Because of neglecting interaction between particles, each $\boldsymbol{P}_i(t,m)$ is determined by its independent dynamical evolution due to its initial condition $\boldsymbol{P}_i(0,m)$ and environmental interaction. At the initial time $t=0$, we assume that the entangled state of two-particle is in the spin singlet, so the density matrix of the system can be written as

$$\rho_{12}(0) = \frac{1}{4}[\boldsymbol{I}_1\otimes\boldsymbol{I}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2], \tag{13}$$

This density matrix can be written in terms of sums over six initial conditions $\{\boldsymbol{P_i}(0,m)\}$ in the following way:[13]

$$\begin{aligned} &\boldsymbol{P}_1(0,1) = -\boldsymbol{P}_1(0,2) = -\boldsymbol{P}_2(0,1) = \boldsymbol{P}_2(0,2) = \sqrt{3}\hat{x}, \\ &\boldsymbol{P}_1(0,3) = -\boldsymbol{P}_1(0,4) = -\boldsymbol{P}_2(0,3) = \boldsymbol{P}_2(0,4) = \sqrt{3}\hat{y}, \\ &\boldsymbol{P}_1(0,5) = -\boldsymbol{P}_1(0,6) = -\boldsymbol{P}_2(0,5) = \boldsymbol{P}_2(0,6) = \sqrt{3}\hat{z}, \end{aligned} \tag{14}$$

where $\hat{\alpha}(\alpha = x,y,z)$ denotes the unit vector along the α axis. For the convenience of calculation, we assume that $\boldsymbol{P}_i(t,\alpha)$ denotes the solution for the polarization vector $\boldsymbol{P}_i(t)$ corresponding to the initial condition $\boldsymbol{P}_i(0) = \hat{\alpha}$. The solution for $\rho_{12}(t)$ can be expressed in terms of quantities $\boldsymbol{P}_i(t,\alpha)$

$$\rho_{12}(t) = \frac{1}{4}[\boldsymbol{I}_1\otimes\boldsymbol{I}_2 - \sum_{\alpha}[\boldsymbol{\sigma}_1\cdot\boldsymbol{P}_1(t,\alpha)] \otimes[\boldsymbol{\sigma}_2\cdot\boldsymbol{P}_2(t,\alpha)]. \tag{15}$$

To obtain the evolution density matrix of the system, we only need to make explicit evaluations of the polarization vectors $\boldsymbol{P}_i(t,\alpha)$. Here we consider that the model of coupling of spins to the environment is in the stochastic model and treat the effect of the environment as a set of classical fluctuating field acting on the system. This type of coupling is given by the Hamiltonian[14−16]

$$H = -\frac{1}{2}[\boldsymbol{\sigma}_1\cdot\boldsymbol{B}_1(t) + \boldsymbol{\sigma}_2\cdot\boldsymbol{B}_2(t)] \tag{16}$$

where $\boldsymbol{B}_i(t)$ describes the fluctuating fields due to the coupling to the environment, whose probability distributions are taken to be Gaussian such that

$$\begin{aligned} &\langle B_{i\alpha}(t)\rangle = 0, \quad \alpha = x,y, \\ &\langle B_{iz}(t)\rangle = \omega_i, \\ &\left\langle B_{i\alpha}(t)B_{j\beta}(t')\right\rangle = \frac{1}{2}k_{i\alpha}\delta_{\alpha\beta}\delta_{ij}\delta(t-t'). \end{aligned} \tag{17}$$

Furthermore, in Eq.(16), we neglect the interaction between particles. This model of coupling has been extensively studied in nuclear magnetic resonance. The decoherence time is approximately given by $1/k_{i\alpha}$. Here we consider that the fluctuating field has a time-averaged nonzero value in one direction(taken to be z axis) for the entangled pair. For simplicity, we take $\omega_1 = \omega_2 = \omega$ and $k_{i\alpha} = k$. For the Gaussian stochastic field specified in Eq.(17), the average polarization

vectors obey the following Bloch equations:[17]

$$\frac{\partial P_{ix}}{\partial t} = \omega P_{iy} - kP_{ix},$$
$$\frac{\partial P_{iy}}{\partial t} = -\omega P_{ix} - kP_{iy},$$
$$\frac{\partial P_{iz}}{\partial t} = -kP_{iz}, \tag{18}$$

where $P_{i\alpha} = \langle P_{i\alpha}(t)\rangle$. According to the initial conditions, we can get solutions of Eqs.(18)

$$P_{ix}(t,x) = P_{iy}(t,y) = \mathrm{e}^{-kt}\cos(\omega t),$$
$$P_{ix}(t,y) = -P_{iy}(t,x) = \mathrm{e}^{-kt}\sin(\omega t),$$
$$P_{iz}(t,z) = -\mathrm{e}^{-kt}. \tag{19}$$

Substituting Eqs.(19) into Eq.(15), we can obtain the evolution density matrix of the two-particle system. After implementing RSP, from Eqs.(5) and Eq.(15) we obtain the following projected density matrix for particle 2

$$\rho_2(t) = \begin{bmatrix} \frac{1}{2}(1+\mathrm{e}^{-2kt}\cos\theta) & \frac{1}{2}\mathrm{e}^{-2kt}\sin\theta\mathrm{e}^{-i\varphi} \\ \frac{1}{2}\mathrm{e}^{-2kt}\sin\theta\mathrm{e}^{\mathrm{i}\varphi} & \frac{1}{2}(1-\mathrm{e}^{-2kt}\cos\theta) \end{bmatrix}, \tag{20}$$

In the Bloch sphere representation, the Bloch vector $\boldsymbol{r}_2$ of the state $\rho_2(t)$ can be expressed as

$$r_{2x} = \mathrm{e}^{-2kt}\sin\theta\cos\varphi,$$
$$r_{2y} = \mathrm{e}^{-2kt}\sin\theta\sin\varphi,$$
$$r_{2z} = \mathrm{e}^{-2kt}\cos\theta. \tag{21}$$

The trace distance and the average trace distance of RSP are given by

$$D_{av} = D(\rho,\rho_2) = \frac{1}{2}(1-\mathrm{e}^{-2kt}). \tag{22}$$

If $2kt \ll 1$, $D_{av}(\rho,\rho_2) \approx kt$. In the limit of $2kt \gg 1$, the average trace distance decays to 1/2, which means that the result of RSP is a half-half of $|0\rangle$ and $|1\rangle$ states. i.e., information about the state ρ is totally lost. The plot of Eq.(22) is shown in Fig.1.

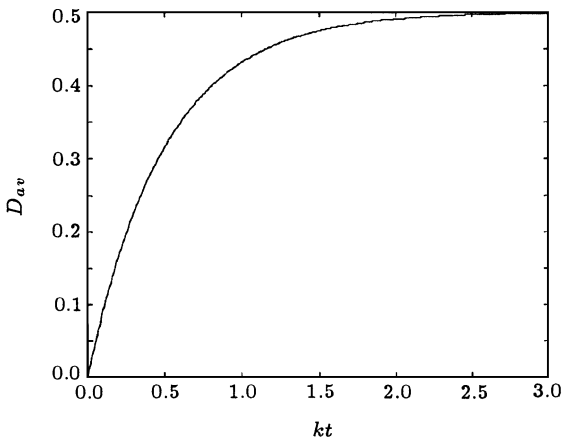


Fig.1. The average trace distance as a function of kt.

4. Conclusion

In conclusion, we calculated the trace distance and the average trace distance of RSP via noisy channels. In two different cases, we considered the effect of noise on the RSP. In the first case, two parties of RSP share the mixture of two Bell states as quantum channels. We found the dependence of the trace distance on the mixture ratio λ and angles of the state to be prepared. In the second case, we considered that the model of coupling of spins to the environment is in the stochastic model. The density matrix of the system is expressed in terms of a set of polarization vectors. We calculated the explicit time dependences of polarization vector and also calculated the trace distance as a function of decoherence rates. In this paper, the decoherence effects due to the environment make the pure entangled state becoming mixed state, and make the trace distance between the initial state and the final state becoming large. Increasing the maximum entangled fraction by applying local operation[18,19] or using entanglement purification[20,21] can decrease the trace distance.

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