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Finite symmetry transformation groups and exact solutions of Lax integrable systems*

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In this paper, the finite symmetry transformation groups and then symmetries of Lax integrable nonlinear physical systems, the Davey–Stewartson equation and the (2+1)-dimensional Camassa–Holm equation are investigated by means of a simple direct method.

Keywords: finite transformations, Lie point symmetry, Lax pair, integrable
PACC: 0340K

1. Introduction

To find symmetries, symmetry groups, symmetry reductions and construct group invariant solutions of nonlinear equations for physical systems is one of the important and hot topics in nonlinear science.[1−11] In traditional studies of the symmetry group of a given nonlinear partial differential equation (PDE), one usually only investigates its Lie point symmetry group. In the standard Lie group theory, it is in principle enough to study its infinitesimal form, the Lie algebra, because the related Lie group can be uniquely determined by solving an initial problem related to an ordinary differential equation (ODE) system from the well-known first Lie group theorem. However, for a given nonlinear PDE, once the Lie algebra is obtained, it is still very difficult to solve the related initial problem to obtain the finite transformations and the symmetry group. Nevertheless, if we are restricted to find the symmetry groups of Lax integrable models, we can find a very simple method to directly find the final symmetry group without using the traditional Lie theory.

In Section 2 of this paper, taking the Davey–Stewartson (DS) equation as a simple example, we outline an alternative simple and generalized method to find finite symmetry transformation groups. In Section 3 we give the symmetry transformation groups and the related special Lie symmetries of the (2+1)-dimensional Camassa–Holm equation. The last section is a short summary and discussion.

2. Symmetries of the Davey–Stewartson

The Davey–Stewartson (DS) system is a natural (2+1)-dimensional generalization of the nonlinear Schrödinger (NS) equation. The DS equation can be obtained as the consistency condition of two linear problems

$$L_1\Phi \equiv \left\{2\begin{pmatrix}\partial_x & 0\\ 0 & \partial_y\end{pmatrix}+\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}\right\}\Phi = 0, \tag{1a}$$

$$L_2\Phi \equiv \frac{1}{2}\left\{\mathrm{i}I\partial_t + I\partial_x^2 + I\partial_y^2 + \begin{pmatrix}0 & q_x\\ r_y & 0\end{pmatrix} + A\right\}\Phi = 0, \tag{1b}$$

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where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$A=\begin{pmatrix}u&0\\0&v\end{pmatrix},$$

$$\Phi=\begin{pmatrix}\Phi_1\\\Phi_2\end{pmatrix},$$

$$u\equiv u(x,y,t),\qquad v\equiv v(x,y,t),$$

$$r\equiv r(x,y,t),\qquad q\equiv q(x,y,t).$$

Then the consistent condition $[L_1,L_2]\,\Psi=0$ leads to the nonlinear equation

$$\mathrm{i}Q_t+\sigma_3(Q_{yy}-Q_{xx})+[A,Q]=0, \tag{2}$$

$$\begin{pmatrix}u_x&0\\0&v_y\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}(\det Q)_y&0\\0&(\det Q)_x\end{pmatrix}, \tag{3}$$

where

$$Q=\begin{pmatrix}0&q\\r&0\end{pmatrix}.$$

This coupled set of equations is known as DS III. We can rewrite Eqs.(2) and (3) in explicit form as

$$u_x=-\frac{1}{2}(qr)_y,\qquad \mathrm{i}r_t+r_{xx}-r_{yy}+r(v-u)=0,$$

$$v_y=-\frac{1}{2}(qr)_x,\qquad \mathrm{i}q_t+q_{yy}-q_{xx}+q(u-v)=0. \tag{4}$$

In the same way, Eqs.(1) have the following component form:

$$2\,\Phi_{1x}+q\,\Phi_2=0, \tag{5a}$$

$$2\,\Phi_{2y}+r\,\Phi_1=0, \tag{5b}$$

$$\mathrm{i}\,\Phi_{1t}+\Phi_{1xx}+\Phi_{1yy}+q_x\,\Phi_2+u\,\Phi_1=0, \tag{5c}$$

$$\mathrm{i}\,\Phi_{2t}+\Phi_{2xx}+\Phi_{2yy}+r_y\,\Phi_1+v\,\Phi_2=0. \tag{5d}$$

Now let $\Phi=G\,\Psi(\xi,\ \eta,\ \tau)$, where

$$\Psi=\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix},\quad G=\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix},$$

and $g_{11},\ g_{12},\ g_{21},\ g_{22},\ \xi,\ \eta,\ \tau$ are functions of $x,\ y,\ t$. Substituting Ψ and G into the first two equations in Eqs.(5) and restricting Ψ to satisfy the same equations

$$2\,\Psi_{1\xi}+q_1\,\Psi_2=0, \tag{6a}$$

$$2\,\Psi_{2\eta}+r_1\,\Psi_1=0, \tag{6b}$$

$$\mathrm{i}\,\Psi_{1\tau}+\Psi_{1\xi\xi}+\Psi_{1\eta\eta}+q_{1\xi}\,\Psi_2+u_1\,\Psi_1=0, \tag{6c}$$

$$\mathrm{i}\,\Psi_{2\tau}+\Psi_{2\xi\xi}+\Psi_{2\eta\eta}+r_{1\eta}\,\Psi_1+v_1\,\Psi_2=0, \tag{6d}$$

where

$$u_1\equiv u_1(\xi,\eta,\tau),\quad v_1\equiv v_1(\xi,\eta,\tau),$$

$$r_1\equiv r_1(\xi,\eta,\tau),\quad q_1\equiv q_1(\xi,\eta,\tau),$$

one can obtain

$$\begin{aligned}&(g_{12}(\tau_x\mathrm{i}r_1-r_1\eta_x)+2g_{11x}+qg_{21}+2\mathrm{i}g_{11}\tau_x u_1)\,\Psi_1\\&+(2g_{12x}+g_{11}\tau_x q_{1\xi}\mathrm{i}+qg_{22}\\&+2\mathrm{i}g_{12}\tau_x v_1-g_{11}q_1\xi_x)\,\Psi_2+(2g_{11}\eta_x-g_{12}\tau_x r_1\mathrm{i})\,\Psi_{1\eta}\\&+(2g_{12}\xi_x-g_{11}\tau_x q_1\mathrm{i})\,\Psi_{2\xi}\\&+2g_{12}\mathrm{i}\tau_x\,\Psi_{2\xi\xi}+2g_{11}\mathrm{i}\tau_x\,\Psi_{1\eta\eta}=0,\\&(2\mathrm{i}g_{21}\tau_y u_1+2g_{21y}+rg_{11}+g_{22}(\tau_y r_{1\eta}\mathrm{i}-r_1\eta_y))\,\Psi_1\\&+(2g_{22y}+g_{21}\tau_y q_{1\xi}\mathrm{i}+rg_{12}\\&+2\mathrm{i}g_{22}\tau_y v_1-g_{21}q_1\xi_y)\,\Psi_2+(2g_{21}\eta_y-g_{22}\tau_y r_1\mathrm{i})\,\Psi_{1\eta}\\&+(2g_{22}\xi_y-g_{21}\tau_y q_1\mathrm{i})\,\Psi_{2\xi}\\&+2\mathrm{i}g_{21}\tau_y\,\Psi_{1\eta\eta}+2\mathrm{i}g_{22}\tau_y\,\Psi_{2\xi\xi}=0,\end{aligned}$$

by comparing the different derivatives of Ψ, which means

$$\begin{aligned}&g_{12}(\tau_x\mathrm{i}r_1-r_1\eta_x)+2g_{11x}\\&+qg_{21}+2\mathrm{i}g_{11}\tau_x u_1=0, &(7)\\&2g_{12x}+g_{11}\tau_x q_{1\xi}\mathrm{i}+qg_{22}\\&+2\mathrm{i}g_{12}\tau_x v_1-g_{11}q_1\xi_x=0, &(8)\\&2g_{11}\eta_x-g_{12}\tau_x r_1\mathrm{i}=0,\\&2g_{12}\xi_x-g_{11}\tau_x q_1\mathrm{i}=0, &(9)\\&g_{12}\mathrm{i}\tau_x=0,\quad g_{11}\mathrm{i}\tau_x=0,\\&\mathrm{i}g_{21}\tau_y=0,\quad \mathrm{i}g_{22}\tau_y=0, &(10)\\&2\mathrm{i}g_{21}\tau_y u_1+2g_{21y}+rg_{11}\\&+g_{22}(\tau_y r_{1\eta}\mathrm{i}-r_1\eta_y)=0, &(11)\\&2g_{22y}+g_{21}\tau_y q_{1\xi}\mathrm{i}+rg_{12}\\&+2\mathrm{i}g_{22}\tau_y v_1-g_{21}q_1\xi_y=0, &(12)\\&2g_{21}\eta_y-g_{22}\tau_y r_1\mathrm{i}=0,\\&2g_{22}\xi_y-g_{21}\tau_y q_1\mathrm{i}=0. &(13)\end{aligned}$$

Solving Eqs.(7)–(13), one can find that

$$\tau_x=0,\quad \tau_y=0,\quad \eta_x=0,\quad \xi_y=0. \tag{14}$$

$$g_{12}=0,\quad g_{21}=0,\quad g_{11x}=0,\quad g_{22y}=0, \tag{15}$$

$$q=\frac{g_{11}q_1\xi_x}{g_{22}},\quad r=\frac{g_{22}r_1\eta_y}{g_{11}}. \tag{16}$$

In the same way, after substituting $\Phi = g\Psi(\xi,\ \eta,\ \tau)$, Eqs.(14) and (15) into the last two equations in Eqs.(5) we have

$$
\begin{aligned}
&(g_{11yy} + ug_{11} + g_{11t}\mathrm{i} - g_{11}\tau_x u_1)\,\Psi_1 \\
&+ \frac{1}{2}(2q_x g_{22} - \mathrm{i}g_{11}q_1\xi_t - g_{11}q_1\xi_{xx} \\
&- g_{11}\tau_t q_{1\xi} - g_{11}\xi_x^2 q_{1\xi})\,\Psi_2 \\
&+ (g_{11}\eta_t\mathrm{i} + g_{11}\eta_{yy} + 2g_{11y}\eta_y)\,\Psi_{1\eta} \\
&- g_{11}(\tau_t - \eta_y^2)\,\Psi_{1\eta\eta} + \frac{1}{2}g_{11}q_1(\tau_t - \xi_x^2)\,\Psi_{2\xi} = 0, \\
&(g_{22xx} + g_{22t}\mathrm{i} + vg_{22} - g_{22}\tau_t v_1)\,\Psi_2 \\
&+ \frac{1}{2}g_{22}r_1(\tau_t - \eta_y^2)\,\Psi_{1\eta} + (g_{22}\xi_{xx} \\
&+ 2g_{22x}\xi_x + g_{22}\xi_t\mathrm{i})\,\Psi_{2\xi} - g_{22}(\tau_t - \xi_x^2)\,\Psi_{2\xi\xi} \\
&- \frac{1}{2}(g_{22}r_1\eta_{yy} - 2r_y g_{11} \\
&+ g_{22}\tau_t r_{1\eta} + g_{22}\eta_y^2 r_{1\eta} + g_{22}r_1\eta_t\mathrm{i})\,\Psi_1 = 0.
\end{aligned}
$$

That is

$$
\begin{aligned}
&g_{11yy} + ug_{11} + g_{11t}\mathrm{i} - g_{11}\tau_x u_1 = 0, \quad \tau_t - \eta_y^2 = 0, \\
&2q_x g_{22} - \mathrm{i}g_{11}q_1\xi_t - g_{11}q_1\xi_{xx} \\
&- g_{11}\tau_t q_{1\xi} - g_{11}\xi_x^2 q_{1\xi} = 0, \\
&g_{11}\eta_t\mathrm{i} + g_{11}\eta_{yy} + 2g_{11y}\eta_y = 0, \quad \tau_t - \xi_x^2 = 0, \\
&g_{22xx} + g_{22t}\mathrm{i} + vg_{22} - g_{22}\tau_t v_1 = 0, \\
&g_{22}\xi_{xx} + 2g_{22x}\xi_x + g_{22}\xi_t\mathrm{i} = 0, \\
&g_{22}r_1\eta_{yy} - 2r_y g_{11} + g_{22}\tau_t r_{1\eta} \\
&+ g_{22}\eta_y^2 r_{1\eta} + g_{22}r_1\eta_t\mathrm{i} = 0.
\end{aligned}
$$

Finishing the detailed calculations, we have

$$g_{12} = g_{21} = 0, \tag{17a}$$

$$g_{11} = F_1(t)\exp\left[-\frac{\mathrm{i}y}{8\dot{f}}\left(y\ddot{f} + 4\dot{\eta}_1\sqrt{\dot{f}}\right)\right], \tag{17b}$$

$$g_{22} = F_1(t)\exp\left[-\frac{\mathrm{i}x}{8\dot{f}}\left(x\ddot{f} + 4\dot{\xi}_1\sqrt{\dot{f}}\right)\right], \tag{17c}$$

$$\xi = \tau_t^{1/2}x + \xi_{1t}, \quad \eta = \tau_t^{1/2}y + \eta_{1t}, \quad \tau = \tau(t), \tag{18a}$$

$$u = \tau_t u_1(\xi,\ \eta,\ \tau) + \frac{3}{16}\frac{y^2\tau_{tt}^2}{\tau_t^2} - \frac{1}{8}\frac{y^2\tau_{ttt}}{\tau_t} + \frac{1}{2}\frac{\eta_{1t}(t)\tau_{tt}}{\tau_t^{3/2}} - \frac{1}{2}\frac{y\eta_{1tt}(t)}{\tau_t} + \frac{1}{4}\mathrm{i}\frac{\tau_{tt}}{\tau_t} + \frac{1}{4}\frac{\eta_{1t}^2(t)}{\tau_t} + m(t), \tag{18b}$$

$$v = \tau_t v_1(\xi,\ \eta,\ \tau) + \frac{1}{4}\mathrm{i}\frac{\tau_{tt}}{\tau_t} + \frac{3}{16}\frac{x^2\tau_{tt}^2}{\tau_t^2} + \frac{1}{2}\frac{x\xi_{1t}(t)\tau_{tt}}{\tau_t^{3/2}} + \frac{1}{4}\frac{\xi_{1t}^2(t)}{\tau_t} - \frac{1}{8}\frac{x^2\tau_{ttt}}{\tau_t} - \frac{1}{2}\frac{x\xi_{1tt}(t)}{\tau_t} + m(t), \tag{18c}$$

$$r = \tau_t^{1/2}\exp\left\{\mathrm{i}\left(\frac{(y^2 - x^2)\tau_{tt}}{8\tau_t} + \frac{y\eta_1(t) - x\xi_{1t}(t)}{2\tau_t}\right)\right\} r_1(\xi,\ \eta,\ \tau), \tag{18d}$$

$$q = \tau_t^{1/2}\exp\left\{\mathrm{i}\left(\frac{(x^2 - y^2)\tau_{tt}}{8\tau_t} + \frac{x\xi_{1t}(t) - y\eta_1(t)}{2\tau_t}\right)\right\} q_1(\xi,\ \eta,\ \tau). \tag{18e}$$

In other words the following theorem holds.

Theorem 1 If $\{u_1 = u_1(x,\ y,\ t),\ v_1 = v_1(x,\ y,\ t),\ q_1 = q_1(x,\ y,\ t),\ r_1 = r_1(x,\ y,\ t)\}$ are solutions of the DS equations (4), then so is $\{u,\ v,\ q,\ r\}$ expressed by Eqs.(18).

Furthermore, the Lie point symmetries, the generators of the symmetry group, of the DS equation have the following forms:

$$
\begin{aligned}
\sigma \equiv\ & \sigma_1(f(t))+\sigma_2(g(t))+\sigma_3(h(t))+\sigma_4(n(t)) \\
=\ & \begin{pmatrix}
f(t)u_t+\frac{1}{2}\dot{f}(xu_x+yu_y+2u)+\frac{\mathrm{i}}{4}\ddot{f}(t)-\frac{1}{8}y^2\overset{\cdots}{f}(t) \\
f(t)v_t+\frac{1}{2}\dot{f}(xv_x+yv_y+2v)+\frac{\mathrm{i}}{4}\ddot{f}(t)-\frac{1}{8}x^2\overset{\cdots}{f}(t) \\
f(t)q_t+\frac{1}{2}\dot{f}(xq_x+yq_y+q)+\frac{\mathrm{i}}{8}\ddot{f}(t)(x^2-y^2) \\
f(t)r_t+\frac{1}{2}\dot{f}(xr_x+yr_y+r)-\frac{\mathrm{i}}{8}\ddot{f}(t)(x^2-y^2)
\end{pmatrix} \\
& +\begin{pmatrix} g(t)u_x \\ g(t)v_x \\ g(t)q_x+\frac{\mathrm{i}}{2}\dot{g}xq \\ g(t)r_x-\frac{\mathrm{i}}{2}\dot{g}xr \end{pmatrix}+\begin{pmatrix} h(t)u_y \\ h(t)v_y \\ h(t)q_y-\frac{\mathrm{i}}{2}\dot{h}yq \\ h(t)r_y+\frac{\mathrm{i}}{2}\dot{h}yr \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ n(t) \\ n(t) \end{pmatrix},
\end{aligned}
$$

which can be obtained from Theorem 1 by setting

$$\tau(t)=t+\epsilon f(t),\ \xi_1(t)=\epsilon g(t),\ \eta_1(t)=\epsilon h(t),\ m(t)=\epsilon n(t).$$

where ϵ is an infinitesimal parameter.

The nonzero commutators among $\sigma_1(f(t))$, $\sigma_2(g(t))$, $\sigma_3(h(t))$ and $\sigma_4(n(t))$ are

$$
\begin{aligned}
&[\sigma_1(f_1),\ \sigma_1(f_2)]=\sigma_1(f_1\dot{f}_2-\dot{f}_1f_2), \\
&[\sigma_1(f),\ \sigma_2(g)]=\sigma_2(f\dot{g}), \\
&[\sigma_1(f),\ \sigma_3(h)]=\sigma_3(f\dot{h}), \\
&[\sigma_1(f),\ \sigma_4(n)]=\sigma_3(f\dot{n}),
\end{aligned}
$$

where the commutator $[A,\ B]$ for the arbitrary vectors

$$A\equiv\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix},\ B\equiv\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}$$

is defined as

$$
[A,\ B]=\begin{pmatrix} A'_{1u} & A'_{1v} & A'_{1q} & A'_{1r} \\ A'_{2u} & A'_{2v} & A'_{2q} & A'_{2r} \\ A'_{3u} & A'_{3v} & A'_{3q} & A'_{3r} \\ A'_{4u} & A'_{4v} & A'_{4q} & A'_{4r} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}-\begin{pmatrix} B'_{1u} & B'_{1v} & B'_{1q} & B'_{1r} \\ B'_{2u} & B'_{2v} & B'_{2q} & B'_{2r} \\ B'_{3u} & B'_{3v} & B'_{3q} & B'_{3r} \\ B'_{4u} & B'_{4v} & B'_{4q} & B'_{4r} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}
$$

and $A'_{1u},\ \ldots,\ B'_{4r}$ are partial Gateaux derivatives, that means the definition of $A'_{1u}f$ is

$$A'_{1u}f\equiv\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}A_1(u+\epsilon f,\ v,\ q,\ r),$$

for arbitrary f.

3. Symmetries of the Camassa–Holm equation

The (2+1)-dimensional integrable generalization of the Camassa–Holm (CH) equation[12]

$$v_t - uv_x + 2vu_x = 0, \tag{19a}$$

$$u_y + vu_x - vu_{xx} - 2vw_x = 0, \tag{19b}$$

$$w_y + v_x w + vw_x + v_x w_x + vw_{xx} = 0. \tag{19c}$$

has been derived from the system of (2+1)-dimensional shallow water equations by using a multi-scale decomposition. The system (19) is reduced to the (1+1)-dimensional CH equation[13]

$$m_t = m_x u + 2mu_x,$$
$$m = \frac{1}{2}(u - u_{xx} + k), k = \text{const.}$$

which is called "a shallow water model equation" or "Camassa–Holm–Fokas–Fuchssteiner (CHFF) equation"[14] under the reduction $\partial_y = 0$. The spectral problem has been found for Eqs.(19):[12]

$$\Phi_{xt} + u\Phi_{xx} - (u+w)\Phi_x - \Phi_t = 0, \tag{20a}$$

$$\Phi_y + v\Phi_{xx} - v\Phi_x = 0, \tag{20b}$$

which is called the Lax pair of the Camassa–Holm equation.

In Ref.[15], by using the standard one-parameter Lie group of infinitesimal transformations,[7] the vector of the Camassa–Holm equation had been written as

$$V = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial u} + \phi_2\frac{\partial}{\partial v} + \phi_3\frac{\partial}{\partial w}, \tag{21}$$

with

$$\xi_1 = f(t), \quad \xi_2 = g(y), \quad \xi_3 = h(t),$$
$$\phi_1 = -\dot{h}(t)u - \dot{f}(t), \quad \phi_2 = -g_y(y),$$
$$\phi_3 = -\dot{h}(t)w. \tag{22}$$

Here, we begin with the Lax pair of the CH equation.

Let $\Phi = G\Psi(\xi,\ \eta,\ \tau)$, where $G,\ \xi,\ \eta,\ \tau$ are functions of $x,\ y,\ t$. Substituting Ψ and G into the first equation of Eqs.(20) and restricting Ψ to satisfy the following equations:

$$\Psi_{\xi\tau} + u_1\Psi_{\xi\xi} - (u_1 + w_1)\Psi_\xi - \Psi_\tau = 0,$$

$$\Psi_\eta + v_1\Psi_{\xi\xi} - v_1\Psi_\xi = 0,$$

one can have

$$v_1^2 G\eta_x(u\eta_x + \eta_t)\Psi_{\xi\xi\xi\xi} + G\tau_x(u\tau_x + \tau_t)\Psi_{\tau\tau} + F(x, y, t, \Psi, \Psi_\xi, \Psi_\tau, ...) = 0, \tag{23}$$

where $u_1 \equiv u_1(\xi,\ \eta,\ \tau)$, $v_1 \equiv v_1(\xi,\ \eta,\ \tau)$, $w_1 \equiv w_1(\xi,\ \eta,\ \tau)$, and F is not depend on $\Psi_{\xi\xi\xi\xi}$ and $\Psi_{\tau\tau}$.

Comparing the different derivatives of Ψ in Eq.(23) we find that

$$\eta_x = 0, \quad \tau_x = 0. \tag{24}$$

Substituting Eq.(24) into Eq.(23) we have another reduction. Step by step, one can find that η is a function of y and t, and τ is a function of t. In the same way, substitute $\Phi = G\Psi(\xi,\ \eta,\ \tau)$ into Eq.(20b). Through direct calculations we have

$$\xi = x + \xi_1(t), \quad \eta = \eta(y), \quad \tau = \tau(t), \quad G = G(t), \tag{25a}$$

$$u = \tau_t u_1(\xi,\ \eta,\ \tau) - \xi_{1t}(t),$$
$$w = \tau_t w_1(\xi,\ \eta,\ \tau),$$
$$v = \eta_y(y)v_1(\xi,\ \eta,\ \tau), \tag{25b}$$

where $\xi_1(t),\ \eta(y),\ \tau = \tau(t)$ and $G = G(t)$ are arbitrary functions of the indicated variables.

In other words, the following theorem is correct.

Theorem 2 If $\{u_1 = u_1(x,\ y,\ t),\ v_1 = v_1(x,\ y,\ t),\ w_1 = w_1(x,\ y,\ t)\}$ are solutions of the Camassa–Holm equations (19), then so is $\{u,\ v,\ w\}$ given by Eqs.(25).

The equivalent symmetry vector form Eq.(21) can be read off from Theorem 2 when we take

$$\xi_1 = \epsilon f(t),\ \eta = \epsilon g(y),\ \tau = t + \epsilon h(t).$$

4. Discussion

In summary, starting from the Lax pair expressions of integrable models, one can readily obtain the symmetry groups by means of a simple combination of the gauge transformation and the space-time transformations no matter how complicated the models are. Various detailed explicit symmetry groups of many famous expressions such as the Kadomtsev–Petviashvili equation, the Nizhnik–Novikov–Vesselov (NNV) equation, the asymmetric NNV equation, the Davey–Stewartson equation, the dispersive long-wave equation, the (2+1)-dimensional Camassa–Holm equation and the extended self-dual Yang–Mills equations are listed in this paper.

When the finite symmetry transformation group of a given system is obtained, the related Lie point symmetries can be obtained simply by restricting the arbitrary functions in infinitesimal forms. For the high-dimensional integrable models discussed in this paper, the Lie point symmetries all have a Kac–Moody–Virasoro structure.

Although the symmetry groups of the Lax integrable models can also be obtained by means of the traditional methods, the final expressions are much more complicated and difficult to be used to find concrete explicit exact solution forms for nonmathematical scientists. Nevertheless, the traditional Lie approach(es) can be used for any systems no matter whether the models are integrable or not. In Ref. [16], a simple direct method had also been established for both integrable and nonintegrable models. It is worthy of further study on the simple method proposed here and extending the method such that the nonlinear systems without Lax pairs can also be treated.

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