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Painlevé property, symmetries and symmetry reductions of the coupled Burgers system*

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The Painlevé property, inverse recursion operator, infinite number of symmetries and Lie symmetry reductions of the coupled Burgers equation are given explicitly. Three sets of infinitely many symmetries of the considered model are obtained by acting the recursion operator and the inverse recursion operator on the trivial symmetries such as the identity transformation, the space translation and the scaling transformation respectively. These symmetries constitute an infinite dimensional Lie algebra while its finite dimensional Lie point symmetry subalgebra is used to find possible symmetry reductions and then the group invariant solutions.

Keywords: symmetry, recursion operator, symmetry reduction, Painlevé
PACC: 0230, 0340, 0220

1. Introduction

As we know, the symmetry study plays an important role in almost all the scientific fields, especially, in the soliton theory because of the existence of the infinitely many symmetries for integrable systems. In (1+1)-dimensions, to find the infinitely many symmetries of a given integrable model, one of the best way is to construct a recursion operator of the studied model.[1−8] When the recursion operator is found, the infinitely many local symmetries can be obtained by repeatedly acting the operator on some simple known local symmetries such as the space-time translations and scaling transformations. About ten years ago, one of the present authors (Lou) pointed out that the infinitely many nonlocal symmetries can be simply constructed via action of the inverse recursion operator on the identity transformation[10,11] and the conformal transformations.[12,13]

The Buerger's equation (BE)

$$q_t = q_{xx} + 2qq_x, \tag{1}$$

is one of the most important nonlinear systems and a diversity of nonlinear phenomena can be explained by this simple models.[14−16] Recently many kinds of extensions of the model have been proposed.[17−25] Among them, one of the most interesting generalizations is the following symmetric-coupled Burgers equation (SCBE):

$$\begin{cases} u_t = \dfrac{1}{2}u_{xx} + \dfrac{1}{2}v_{xx} + 5uu_x + vu_x + 2vv_x, \\ v_t = \dfrac{1}{2}u_{xx} + \dfrac{1}{2}v_{xx} + 2uu_x + uv_x + 5vv_x, \end{cases} \tag{2}$$

which is first given by Foursov.[25] Obviously, the SCBE (2) will be reduced back to the usual BE (1) when $u = v = q/4$.

The conserved density and the recursion operator of the SCBE have also been given in Ref.[25]. The recursion operator of Eq.(2) reads

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$$\Phi^{uv} = \begin{pmatrix} \frac{1}{2}D + 3u - v + 2u_x D^{-1} & \frac{1}{2}D + 2v + 2u_x D^{-1} \\ \frac{1}{2}D + 2u + 2v_x D^{-1} & \frac{1}{2}D - u + 3v + 2v_x D^{-1} \end{pmatrix}, \tag{3}$$

where $D = \dfrac{\partial}{\partial x}$ and $D^{-1}D = DD^{-1} = 1$.

In this article, one of the main purposes is to study the symmetries of the asymmetric-coupled Burgers equations (ACBE)

$$\begin{cases} w_t = w_{xx} + 4ww_x + 3zz_x, \\ z_t = zw_x + 2wz_x, \end{cases} \tag{4}$$

which is related to the SCBE by

$$w = u + v, \ z = u - v. \tag{5}$$

The Painlevé property, which is linked with many other important integrable properties such as the Bäcklund transformation, bilinear form, symmetries and Lax pairs is one of the most interesting and important properties of integrable systems. So, before studying the symmetries of the ACBE (and then the SCBE), we give a simple discussion on the weak Painlevé property of the model.

2. The weak Painlevé property of the ACBE

A nonlinear system is said possessing Painlevé property if the movable singularities of all the solutions of the model with respect to an arbitrary manifold are poles. To test the Painlevé property, there are some different approaches. For instance, the Ablowitz–Ramani–Segur (ARS) algorithm[26] is a Painlevé test method via testing the Painlevé properties of all the ordinary differential equation (ODE) reductions which may be obtained via nonclassical symmetry reduction approach[27,28] or direct method.[29−34] The Weiss–Tabor–Carnevale (WTC) approach is a generalization of the ODE test, and this method makes the general series solution of an nth order partial differential equation (PDE) include n independent arbitrary functions.[35] The Kruskal simplification is equivalent to the WTC approach if one only uses the method to test the Painlevé property.[36] The Conte method is a conformal group invariant test approach.[37] The Conte invariant approach can be used to obtain possible new exact solutions via nonstandard truncated expansions.[38] The Lou approach is a generalization of Conte's method.[40] Using both the standard and nonstandard truncated expansions in the Lou extended Painlevé test method, one can find more new exact solutions and find some exact solutions for some difficult models which cannot be solved by other methods.[41−43]

In this section, we use the standard WTC approach to test the Painlevé property of the ACBE. By using the leading order analysis, i.e. substituting

$$w = w_0\phi^{\alpha}, \ z = z_0\phi^{\beta}$$

into Eq.(4), it is straightforward to find that $\alpha = -1$ and $\beta = \dfrac{1}{2}$. The half index for β means that the algebraic branch points with respect to the arbitrary manifold ϕ can enter into the general solution of the ACBE. In other words, the ASBE has no Painlevé property. This kind of situation occurs for other types of integrable systems: say, the Harry–Dym equation. When the algebraic branch points appear, the question that follows is whether the model possesses the weak Painlevé property first introduced by Kruskal. The weak Painlevé property of a model means that if the model can be changed to a new model with the Painlevé property via a suitable variable-dependent transformation, then the original model would possess algebraic branch points with respect to an arbitrary singular manifold.

Fortunately, we can really prove that the ACBE does possess the weak Painlevé property. The transformation we use is

$$z = v^{-1/2}. \tag{6}$$

Substituting Eq.(6) into Eq.(4) results in

$$\begin{cases} 2w_t v^2 = 2w_{xx}v^2 + 8ww_x v^2 - 3v_x, \\ v_t = 2wv_x - 2vw_x. \end{cases} \tag{6}$$

To find the possible Painlevé property of the model equation (7), we expand the fields w, v about an arbitrary singularity manifold $\phi = \phi(x,t) = 0$ as

$$w = \sum_{j=0}^{\infty} w_j \phi^{j+\alpha}, \tag{8}$$

$$v = \sum_{j=0}^{\infty} v_j \phi^{j+\beta}, \tag{9}$$

where $\{w_j,\ v_j\}$ are functions of (x,t) and $u_0 \neq 0, v_0 \neq 0$. If the model possesses the Painlevé property, there should be a primary branch with three arbitrary functions among $w_j,\ v_j$ and ϕ because the system (7) is composed of a second-order partial differential equation and a first-order one. That is to say, there should be two more arbitrary functions in additional to the arbitrary singularity manifold ϕ about Eq.(7).

Substituting

$$w \sim w_0\phi^{\alpha}, \quad v \sim v_0\phi^{\beta} \tag{10}$$

into Eq.(7) and using the leading order analysis, we have

$$\alpha = \beta = -1,\ w_0 = \frac{1}{2}\phi_x,\ v_0 = v_0, \tag{11}$$

where v_0 is an arbitrary function of (x,t).

Now substituting Eq.(9) with Eq.(11) into Eq.(7), we can obtain the recursion relations of the functions w_j and v_j:

$$-2(j+1)(j-2)w_j v_0^2\phi_x^2\phi^{j-5} = f_j, \tag{12}$$

$$\frac{1}{2}j\phi_x(-2w_j v_0 + v_j\phi_x)\phi^{j-3} = g_j, \tag{13}$$

where f_j and g_j are complicatedly dependent on $\{w_0,\ \cdots,\ w_{j-1},\ v_0,\ \cdots,\ v_{j-1}\}$ and the derivatives of ϕ. From Eqs.(12) and (13), we know that the resonance values of j are given by

$$\begin{vmatrix} -2\phi_x(j+1)(j-2)v_0^2 & 0 \\ -jv_0\phi_x & \frac{1}{2}j\phi_x \end{vmatrix}$$

$$= -j(j+1)(j-2)v_0^2\phi_x^4 = 0, \tag{14}$$

i.e.

$$j = -1,\ 0,\ 2. \tag{15}$$

The resonance at $j=-1$ is related to the arbitrariness of the singular manifold, and the arbitrariness of v_0 in the leading order analysis means that the resonance at $j=0$ is satisfied naturally.

To check the resonance conditions at $j=2$, we write down the first few terms of Eq.(7) after the substitution of expansions (8) and (9) with Eq.(11),

$$\frac{v_0^2\phi_x}{\phi^4}(\phi_{xx} - \phi_t + 4\phi_x w_1)$$

$$+ \frac{v_0}{\phi^3}(v_0\partial_x - 2\phi_x)(\phi_{xx} - \phi_t + 4\phi_x w_1) + \cdots = 0, \tag{16}$$

$$-(2w_1 v_0\phi_x - \phi_x^2 v_1 - v_{0x}\phi_x + v_0\phi_{xx} - v_0\phi_t)\phi^{-2}$$

$$+ [\phi_x(v_{1x} + 2v_2\phi_x - 4v_0 w_2)$$

$$+ 2w_1 v_{0x} - 2v_0 w_{1x} - v_{0t} - v_1\phi_{xx}]\phi^{-1} + \cdots = 0. \tag{17}$$

Setting the coefficients of ϕ^{-4} and ϕ^{-2} in Eqs.(16) and (17) to zero respectively, we have

$$w_1 = \frac{\phi_t - \phi_{xx}}{4\phi_x}, \tag{18}$$

$$v_1 = \frac{-2\phi_x v_{0x} + v_0\phi_{xx} - v_0\phi_t}{2\phi_x^2}. \tag{19}$$

To check the compatibility at $j=2$, we write down the determining equations for w_2 and v_2 by setting the coefficients of ϕ^{-3} and ϕ^{-1} in Eqs.(16) and (17) to zero respectively,

$$v_0(v_0\partial_x - 2\phi_x)(\phi_{xx} - \phi_t + 4\phi_x w_1) = 0, \tag{20}$$

$$\phi_x(v_{1x} + 2v_2\phi_x - 4v_0 w_2) + 2w_1 v_{0x}$$

$$- 2v_0 w_{1x} - v_{0t} - v_1\phi_{xx} = 0. \tag{21}$$

It is straightforward to see that Eq.(20) is satisfied naturally because of Eq.(18). From Eq.(20) w_2 or v_2 can be solved simply, while the other remains free. In other words, the resonant condition for $j=2$ is satisfied also. Then we prove that the system (7) possesses the Painlevé property, and then the ACBE and SCBE possess the weak Painlevé property.

3. The symmetries and symmetry algebra of the ACBE

A symmetry of the ACBE (4),

$$\sigma^{wz} \equiv \begin{pmatrix} \sigma^w \\ \sigma^z \end{pmatrix}, \tag{22}$$

is defined as a solution of its linearized system

$$\begin{cases} \sigma_t^w = \sigma_{xx}^w + 4\sigma^w w_x + 4w\sigma_x^w + 3\sigma^z z_x + 3z\sigma_x^z, \\ \sigma_t^z = \sigma^z w_x + z\sigma_x^w + 2\sigma^w z_x + 2w\sigma_x^z, \end{cases} \tag{23}$$

this means Eq.(4) is form invariant under the transformation

$$\begin{pmatrix} w \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} w \\ z \end{pmatrix} + \epsilon\begin{pmatrix} \sigma^w \\ \sigma^z \end{pmatrix}, \tag{24}$$

with ϵ being an infinitesimal parameter.

To give some sets of infinitely many symmetries of the ACBE, one has to find out some seed solutions of Eq.(23) and the recursion operator. Using the standard classical Lie group approach[44,45] or the standard classical Lie symmetry approach,[46] one can find that the only possible Lie point symmetries of the ACBE are the symmetries related to the space–time translations,

$$K_0^{wz} = \begin{pmatrix} w_x \\ z_x \end{pmatrix}, \qquad K_1^{wz} = \begin{pmatrix} w_t \\ z_t \end{pmatrix}, \tag{25}$$

and the scaling transformation

$$\tau_1^{wz} = \begin{pmatrix} tw_t + \frac{1}{2}(xw)_x \\ tz_t + \frac{1}{2}(xz)_x \end{pmatrix}. \tag{26}$$

The recursion operator of the ACBE can be easily obtained from that of the SCBE because of the well-known transformation theorem[47] and the transformation relation Eq.(5). The result reads

$$\Phi \equiv \Phi^{wz} = T\Phi^{uv}T^{-1} = \begin{pmatrix} D + 2w + 2w_xD^{-1} & 3z \\ z + 2z_xD^{-1} & 0 \end{pmatrix}, \qquad T \equiv \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{27}$$

Applying the recursion operator Φ^{wz} to the seeds K_0^{wz} and τ_1^{wz}, we obtain two sets of infinitely many general symmetries (positive K symmetries and τ symmetries),

$$K_n^{wz} = \Phi^n K_0^{wz}, \qquad (n = 0,\ 1,\ 2,\ \ldots), \tag{28}$$

and

$$\tau_n^{wz} = \Phi^{n-1}\tau_1^{wz}, \qquad (n = 1,\ 2,\ \ldots). \tag{29}$$

In addition to K_0^{wz}, K_1^{wz} and τ_1^{wz}, some further simple examples of K_n^{wz} and τ_n^{wz} read:

$$K_2^{wz} = \begin{pmatrix} (w_{xx} + 6ww_x + 3zz_x + 4w^3 + 6wz^2)_x \\ z(w_x + 2w^2)_x + 2z_x(w_x + 2w^2 + 3z^2) \end{pmatrix}, \tag{30}$$

$$K_3^{wz} = \begin{pmatrix} [w_{xxx} + 8ww_{xx} + 6w_x^2 + (9z^2w + 8w^3 + 3zz_x)_x + 18z^2w^2 + \frac{9}{2}z^4 + 8w^4]_x \\ z^{-1}[z^2(w_{xx} + 3zz_x + 6ww_x + 4w^3 + 6wz^2)]_x \end{pmatrix} \tag{31}$$

and

$$\tau_2^{wz} = tK_2^{wz} + \begin{pmatrix} \frac{x}{4}(2w_x + 4w^2 + 3z^2)_x + w_x + w^2 + \frac{3}{2}z^2 \\ \frac{1}{2}(zw_x + 2wz_x)x + \frac{1}{2}zw \end{pmatrix}. \tag{32}$$

To find more general symmetries, one has to use the inverse recursion operator and new seed symmetries.

Suppose the inverse of the recursion operator Φ to be in the form

$$\Phi^{-1} = \begin{pmatrix} A & C \\ B & E \end{pmatrix}, \tag{33}$$

where the operators A, B, C and D are determined by

$$\Phi^{-1}\Phi = \Phi\Phi^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

The left inverse condition $\Phi^{-1}\Phi = I$ leads to

$$A\left(D + 2w + 2w_xD^{-1}\right) + B\left(z + 2z_xD^{-1}\right) = 1, \tag{35}$$

$$3Az = 0, \tag{36}$$

$$C\left(D + 2w + 2w_xD^{-1}\right) + E\left(z + 2z_xD^{-1}\right) = 0, \tag{37}$$

$$3Cz = 1. \tag{38}$$

From Eq.(36) we have

$$A = 0. \tag{39}$$

Substituting Eq.(39) into Eq.(35) leads to

$$B = Dz^{-2}D^{-1}z. \tag{40}$$

Employing Eq.(38) with $(3z)^{-1}$ from the right yields

$$C = \frac{1}{3}z^{-1}. \tag{41}$$

Substituting Eq.(41) into Eq.(37), we find

$$E = -\frac{1}{3}z^{-1}\left(D^2 + 2wD + 2w_x\right)z^{-2}D^{-1}z. \tag{42}$$

Finally, the left inverse of recursion operator Φ reads

$$\Phi^{-1} = \begin{pmatrix} 0 & Dz^{-2}D^{-1}z \\ \frac{1}{3}z^{-1} & -\frac{1}{3}z^{-1}D(D + 2w)z^{-2}D^{-1}z \end{pmatrix}. \tag{43}$$

It is straightforward to verify that Φ^{-1} given by Eq.(43) is also the right inverse of Φ.

As discussed in Refs.[10] and [11], for the KdV equation, the modified KdV equation, the Sawada–Kortera equation. etc, employing the inverse recursion operators on the trivial symmetry $\sigma = 0$ (related to identity transformation) and taking $D^{-1}0 = C = const$, one may obtain new seed symmetries and then the new set(s) of general symmetries. For the ACBE, taking $C = -1/2$, we have

$$\begin{aligned}\bar{K}_1^{wz} &= \Phi^{-1}0 \\ &= \begin{pmatrix} 0 & Dz^{-2}D^{-1}z \\ \frac{1}{3}z^{-1} & -\frac{1}{3}z^{-1}D(D+2w)z^{-2}D^{-1}z \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} z^{-3}z_x \\ \frac{1}{6}z^{-1}D(D+2w)z^{-2} \end{pmatrix}. \end{aligned} \tag{44}$$

Direct substitution of $\sigma^{wz} = \bar{K}_1^{wz}$ into the symmetry definition Eqs.(23) shows that $\bar{K}_1^{wz}$ is really a symmetry of the ACBE.

Now using the inverse recursion operators on the seed symmetries $\bar{K}_1^{wz}$ and τ_1^{wz} yields two new sets of symmetries (negative $\bar{K}$ symmetries and τ symmetries):

$$\bar{K}_n^{wz} = \Phi^{-n+1}K_1^{wz}, \qquad (n = 1,\ 2,\ ...), \tag{45}$$

and

$$\tau_{-n}^{wz} = \Phi^{-n-1}\tau_1^{wz}, \qquad (n = 0,\ 1,\ 2,\ ...). \tag{46}$$

Three of the further explicit simple examples of K_{-n}^{wz} and τ_{-n}^{wz} read

$$\bar{K}_2^{wz} = \begin{pmatrix} \frac{1}{3}(z^{-4}w - z^{-5}z_x)_x \\ z^{-1}\left(-\frac{1}{6z^2} - \frac{w_x + 2w^2}{9z^4} + \frac{z_{xx} + 6wz_x}{9z^5} - \frac{5z_x^2}{9z^6}\right)_x \end{pmatrix}, \tag{47}$$

$$\bar{K}_3^{wz} = \begin{pmatrix} \left(-\frac{1}{6z^4} - \frac{w_x + 2w^2}{9z^6} + \frac{z_{xx} + 6wz_x}{9z^7} - \frac{5z_x^2}{9z^8}\right)_x \\ \frac{1}{9z}\left[\frac{zw - z_x}{z^5} + (D+2w)\left(-\frac{1}{6z^4} - \frac{w_x + 2w^2}{9z^6} + \frac{z_{xx} + 6wz_x}{9z^7} - \frac{5z_x^2}{9z^8}\right)\right]_x \end{pmatrix}, \tag{48}$$

$$\tau_0^{wz} = \begin{pmatrix} tw_x + \tau_{01}^w \\ tz_x + \tau_{01}^z \end{pmatrix}, \tag{49}$$

$$\tau_{-1}^{wz} = \begin{pmatrix} z^{-1}\tau_{01}^z - 2z^{-3}z_xD^{-1}z\tau_{01}^z \\ \frac{\tau_{01}^w}{3z} - \frac{2w\tau_{01}^z + \tau_{01x}^z}{3z^4} + \frac{\tau_{01}^z z_x}{z^5} - \frac{2(z^2w_x - zz_{xx} - 2wzz_x + 3z_x^2)}{3z^5}D^{-1}z\tau_{01}^z \end{pmatrix}, \tag{50}$$

where

$$\tau_{01}^w \equiv \frac{1}{2}(1 - z^{-3}z_xD^{-1}z^2), \tag{51}$$

$$\tau_{01}^z \equiv -\frac{wz - z_x}{6z^2} - \frac{z^2w_x + 3z_x^2 - zz_{xx} - 2zwz_x}{6z^5}D^{-1}z^2. \tag{52}$$

From Eqs.(45)–(52), it is noticed that τ_n^{wz} $(n \le 0)$ are all nonlocal symmetries and the negative $\bar{K}_n^{wz}$ $(n \le 1)$ symmetries are all local symmetries while for the KdV and other known S integrable models, all the negative symmetries are nonlocal symmetries.[10,11]

Because of the symmetry K_0^{wz} is the kernel of Φ^{-1}, i.e.

$$\Phi^{-1}K_0^{wz} = 0,$$

all the negative τ symmetries τ^{wz}_{-n}, $(n \geq 1)$ are not explicitly time-dependent.

For the SCBE Eq.(2), the related symmetries can be obtained simply by means of the transformation relation Eq.(5), the results can be expressed as

$$\sigma^{wz} = \begin{pmatrix} \sigma^w \\ \sigma^z \end{pmatrix} = \begin{pmatrix} \sigma^u + \sigma^v \\ \sigma^u - \sigma^v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} \equiv T\sigma^{uv}, \tag{53}$$

i.e.

$$\sigma^{uv} = T^{-1}\sigma^{wz}, \tag{54}$$

where σ^{wz} and σ^{uv} are symmetries of the ACBE and SCBE respectively.

Similar to the usual Burgers equation and the Liouville equation, though there are infinitely many symmetries for the coupled Burgers systems, there exist only finite conservation laws in the form $\rho_t = J_x$. We can find only two conserved densities,

$$\rho_1 \equiv u + v = w, \tag{55}$$

$$\rho_2 \equiv u^2 + v^2 - 2uv = z^2. \tag{56}$$

The first conserved density is given by Foursov.[25]

Finally, we can prove that the symmetries K^{wz}_n, $n = 0,\ 1,\ ...$, $\bar{K}^{wz}_n$, $n = 1,\ 2,\ ...$ and τ^{wz}_n, $n = 0,\ \pm 1,\ \pm 2,\ ...$ constitute the following infinite-dimensional closed symmetry algebra:

$$[K^{wz}_m,\ K^{wz}_n] = 0, \quad (m = 0,\ 1,\ 2,\ ...,\ n = 0,\ 1,\ 2,\ ...), \tag{57}$$

$$[\bar{K}^{wz}_m,\ \bar{K}^{wz}_n] = 0, \quad (m = 1,\ 2,\ ...,\ n = 1,\ 2,\ ...), \tag{58}$$

$$[K^{wz}_m,\ \bar{K}^{wz}_n] = 0, \quad (m = 0,\ 1,\ 2,\ ...,\ n = 1,\ 2,\ ...), \tag{59}$$

$$[K^{wz}_m,\ \tau^{wz}_n] = \frac{m+1}{2} K^{wz}_{m+n-1}, \quad (m = 0,\ 1,\ 2,\ ...,\ n = 0,\ \pm 1, \pm 2,\ ...), \tag{60}$$

$$[\bar{K}^{wz}_m,\ \tau^{wz}_n] = -\frac{m+1}{2} \bar{K}^{wz}_{m-n+1}, \quad (m = 1,\ 2,\ ...,\ n = 0,\ \pm 1, \pm 2,\ ...), \tag{61}$$

$$[\tau^{wz}_m, \tau^{wz}_n] = \frac{m-n}{2} \tau^{wz}_{m+n-1},\ (m = 0,\ \pm 1,\ \pm 2,\ ...,\ n = 0,\ \pm 1,\ \pm 2,\ ...), \tag{62}$$

where $K^{wz}_m = 0,\ \forall\ m \leq -1$ and $\bar{K}^{wz}_m = 0,\ \forall\ m \geq 0$.

4. Similarity reductions of the ACBE

As we know, starting from any one symmetry σ of a partial differential equation system, one can obtain the corresponding group invariant solutions simply by solving the symmetry construction equation

$$\sigma = 0 \tag{63}$$

and the original evolution equation. However, it is quite difficult to find all the group invariant solutions for a completely integrable model because there exist infinitely many symmetries for an integrable equation system just as shown in the last section for the ACBE. Fortunately, for some special cases one can obtain the group invariant solutions easily by solving the special characteristic equations. In this section, we just make symmetry reductions for the ACBE by using the Lie point symmetries $S = \{K^{wz}_0, K^{wz}_1, \tau^{wz}_1\}$ of the ACBE given in the last section.

To obtain Lie symmetry group invariant solutions of the ACBE, we take the symmetry constraint (63) in the form

$$\sigma = aK^{wz}_0 + bK^{wz}_1 + c\tau^{wz}_1 = 0, \tag{64}$$

with three arbitrary constants. To find the solutions of equation system (4), two different cases should be discussed for $c = 0$ and $c \neq 0$.

Case 1 $c = 0$.

In this case the symmetry constraint (64) reduces to

$$\sigma = aK^{wz}_0 + bK^{wz}_1 = \begin{pmatrix} aw_x + bw_t \\ az_x + bw_t \end{pmatrix} = 0. \tag{65}$$

Obviously, it is just the travelling wave reduction.

Solving Eq.(65), we obtain

$$\begin{aligned} w &= W(\frac{at-bx}{a}) = W(\xi), \\ z &= Z(\frac{at-bx}{a}) = Z(\xi). \end{aligned} \tag{66}$$

Substituting Eq.(66) into Eq.(4) yields the ordinary differential equations

$$\begin{aligned} &aW_\xi - b^2 W_{\xi\xi} + 4bWW_\xi + 3bZZ_\xi = 0, \\ &aZ_\xi + bZW_\xi + 2bWZ_\xi = 0. \end{aligned} \tag{67}$$

Solving Eq.(67), we have

$$Z = \frac{z_0}{\sqrt{a+2bW}}, \tag{68}$$

where z_0 is an arbitrary constant, and W satisfies the following ordinary differential equation:

$$W_\xi = \frac{3}{2}\frac{z_0^2}{a+2bW} + C_1 + \frac{a+2bW}{b^2}W, \tag{69}$$

where C_1 is an arbitrary integral constant. It is clear that the general solution of Eq.(69) can be implicitly expressed by means of an integration. If we write the constants $\{a,\ C_1,\ z_0\}$ as

$$\begin{aligned} a &= -(z_1+z_2+z_3)b, \\ C_1 &= \frac{(z_3-z_2)^2 - z_1(2z_2+2z_3-z_1)}{2b}, \\ z_0^2 &= \frac{1}{3}b(z_3+z_2-z_1)(z_2-z_3+z_1)(z_3+z_1-z_2), \end{aligned} \tag{70}$$

then the general solution of Eq.(69) is given by a transcendental function, which is determined by

$$\begin{aligned} W =& \frac{z_3-z_2}{2}\tanh\left\{\frac{(z_2-z_1)(z_3-z_1)(\xi-\xi_0) + b(z_3+z_2-z_1)\ln\dfrac{(W-z_1)^2}{(W-z_3)(W-z_2)}}{2b[z_1(z_3+z_2-2z_1)+(z_2-z_3)^2](z_3-z_2)^{-1}}\right\} \\ &+ \frac{z_3+z_2}{2}, \end{aligned} \tag{71}$$

where $z_1,\ z_2,\ z_3,\ \xi_0$ and b are all arbitrary constants. From Eq.(71), it is readily seen that when $z_1 = z_3+z_2$ ($z_0 = 0$ and $z = 0$), Eq.(71) is just the usual kink solution of the Burgers equation.

Case 2 $c \neq 0$.

In this case the symmetry constraint reads

$$\sigma = \begin{pmatrix} aw_x + bw_t + c(tw_t + \frac{1}{2}(xw)_x) \\ az_x + bz_t + c(tz_t + \frac{1}{2}(xz)_x) \end{pmatrix}. \tag{72}$$

To solve the symmetry constraint condition, Eq.(72), is equivalent to find the solutions of the following characteristic equation:

$$\frac{\mathrm{d}x}{a+\frac{1}{2}cx} = \frac{\mathrm{d}t}{b+ct} = \frac{\mathrm{d}w}{\frac{1}{2}cw} = \frac{\mathrm{d}z}{\frac{1}{2}cz}. \tag{73}$$

From Eq.(73) we obtain

$$\frac{b+ct}{c(4a^2+4acx+c^2x^2)} = \xi, \tag{74}$$

$$w = \frac{1}{2a+cx}W(\xi), \quad z = \frac{1}{2a+cx}Z(\xi). \tag{75}$$

The functions of $W(\xi)$ and $Z(\xi)$ are determined by the following equations which are obtained by substituting Eq.(75) into the ACBE (4):

$$\begin{aligned} &(2c\xi - \xi^2 - 8W\xi)W_\xi - 4c\xi^2 W_{\xi\xi} \\ &- 2cW + 3Z^2 + 4W^2 - 6\xi ZZ_\xi = 0, \end{aligned} \tag{76}$$

$$3ZW - 2\xi ZW_\xi - (\xi^2 + 4\xi W)Z_\xi = 0. \tag{77}$$

From Eq.(77), it is easy to find

$$Z_\xi = \frac{Z(-2\xi W_\xi + 3W)}{\xi(\xi+4W)}. \tag{78}$$

Substituting Eq.(78) into Eq.(76), the solution of Z is obtained as follows:

$$\begin{aligned} Z =& \frac{1}{2}\{[3\xi W_\xi(\xi^2 - 2c - 8cW + 12\xi W + 32W^2) + 12c\xi^3 W_{\xi\xi} \\ &+ 6c\xi W(1+8\xi W_{\xi\xi}) + 2\xi W^2(2c-1) - 48W^3]/(2W - \xi - 4\xi W_\xi)\}^{1/2}, \end{aligned} \tag{79}$$

where W satisfies the equation

$$\begin{aligned}&64(c-6W-\xi)\xi^3W_\xi^3-2\xi^2[64c\xi^2W_{\xi\xi}+40W(2c+\xi)-288W^2-10c\xi\\&+13\xi^2]W_\xi^2+[8c\xi^3(8W-9\xi)W_{\xi\xi}-16c\xi^4(\xi+4W)W_{\xi\xi\xi}+2\xi(70\xi W^2-144W^3\\&+60cW^2+4\xi^2W-22c\xi W-\xi^3)]W_\xi+16c\xi^4(\xi+4W)W_{\xi\xi}^2+[8cW^2(4c-3\xi)\\&+44c\xi W-10\xi^2-\xi^3-6c\xi^2]\xi^2W_{\xi\xi}+4c\xi^3(\xi+4W)(2W-\xi)W_{\xi\xi\xi}\\&-24(W-\xi)(c-2W)W^2=0.\end{aligned}\tag{80}$$

The ordinary differential equation (80) is not easy to solve. Here, we just investigate the Painlevé property of this equation. To study the Painlevé property of Eq.(80), we expand W as

$$W=\sum_{j=0}^{\infty}W_j\psi^{j+\gamma},\ \psi=\xi-\xi_0,\tag{81}$$

where ξ_0 is an arbitrary constants and the expansion coefficients W_j should be determined later, and W can be expressed by three arbitrary constants (ξ_0 and two of W_j) and $W_0\neq 0$. Using leading order analysis, i.e. replacing W by $W_0\psi^\gamma$, we have

$$\gamma=-1,\qquad W_0=c\xi_0.\tag{82}$$

Now substituting Eq.(81) with Eq.(82) into Eq.(80), we can obtain the recursion relations of the function W_j:

$$64c^3\xi_0^6(j+1)(j-2)(j-3)W_j\psi^{j-7}=F_j,\tag{83}$$

where we have omitted the details on F_j (which is only dependent on $W_0,\ \cdots,\ W_{j-1}$ and ξ_0) because of its complexity.

From the left-hand side of Eq.(83), we know that the three required resonance conditions are

$$j=-1,\ 2,\ 3.\tag{84}$$

Similarly, the resonance for $j=-1$ corresponds to the arbitrariness of ξ_0. If Eq.(80) possesses Painlevé property, the other two resonance conditions, $F_2=0$ and $F_3=0$ must be satisfied simultaneously. To check the resonant conditions $j=2$ and $j=3$, we write $F_1,\ F_2$ and F_3 here explicitly,

$$F_1=32c^3\xi_0^6(6c-\xi_0),\tag{85}$$

$$F_2=160c^3\xi_0^5(8W_1+\xi_0-6c),\tag{86}$$

$$F_3=2c^2\xi_0^4(8W_1+\xi_0-6c)\times(142c-11\xi_0-48\xi_0W_2+24W_1).\tag{87}$$

So from $j=1$ of Eq.(83), we have

$$W_1=\frac{1}{8}(6c-\xi_0).\tag{88}$$

Now it is straightforward to see that the resonance conditions $F_2=0$ and $F_3=0$ are satisfied naturally. In other words, the reduction equation (80) possesses the Painlevé property and has the general Laurent series solution (81) with Eq.(82) and three arbitrary integral constants $\xi_0,\ W_2$ and W_3. It should be pointed out that the weak Painlevé property is already implied in the relation (79) due to the entrance of the square root. It should be also pointed out that the series solution (81) is valid only around the movable singular point, i.e. $\xi_0\neq 0$. Around the fixed singular point some kinds of the branch points may occur. For instance, it is straightforward to prove that the reduction equation (80) possesses a special solution

$$W=-\frac{\xi}{2}+C\sqrt{\xi},$$

which contains an algebraic branch point around the fixed singular point $\xi=0$.

5. Summary

In summary, the coupled Burgers equations (the SCBE (2) and the ACBE (4) are proved to possess weak Painlevé property. Starting from the known recursion operator of the SCBE, the explicit form of the inverse recursion operator of the ACBE is obtained. Acting the recursion operator and its inverse on the trivial symmetries, we have obtained three sets of infinitely many symmetries, positive local $K_n^{wz}(n=0,\ 1,\ 2,\ \ldots)$ symmetries, negative local $\bar{K}_m^{wz}(m=1,\ 2,\ \ldots)$ symmetries and full (both positive local and negative nonlocal) $\tau_n^{wz}(n=0,\ \pm1,\ \pm2,\ \ldots)$ symmetries. These three sets of symmetries construct an infinite dimensional Lie algebra. Though the ACBE (and then SCBE) possesses infinitely many symmetries, only two conservation laws are found. Furthermore, the Lie point symmetries of the model

are used to find possible group invariant solutions via classical Lie group approach.

The travelling wave reduction of the model can be solved by direct integrations. By expressing the integral constants appropriately, the kink solution is expressed by a transcendental function which will naturally tend to the usual kink solution expressed by a hyperbolic tangent function for the standard Burgers equation when the solution of the coupling field tends to zero.

The scaling symmetry related similarity reduction equation (80) is proved to possess the Painlevé property while the weak Painlevé property of the original model is reflected by the relation equation (79). The reduction equation (80) provides an example where the algebraic branch point around a fix point may occur for a Painlevé integrable model.

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