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Poisson structure and Casimir functions for a noncentral dynamical system in four-dimensional phase space

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In this paper, we express the differential equations of a noncentral dynamical system in Ermakov formalism to obtain the Ermakov invariant. In term of Hamiltonian theories and using the Ermakov invariant as the Hamiltonian, the Poisson structure of a noncentral dynamical system in four-dimensional phase space are constructed. The result indicates that the Poisson structure is degenerate and the noncentral dynamical system possesses four invariants: the Hamiltonian, the Ermakov invariant and two Casimir functions.

Keywords: noncentral dynamical system, Ermakov invariant, Poisson structure, Casimir functions
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1. Introduction

Research on the invariant has already become an important and active field of modern mathematical, mechanical and physical sciences. The modern methods to find conserved quantities are mainly the Noether method, the Lie method and the form invariance and a series of important results have been obtained in the past 20 years.[1−6]

Ermakov systems have attracted attention in the last decades due to their important physical applications and mathematical properties as well.[7−12] A central mathematical property of Ermakov systems is the existence of the Ermakov invariant. Hamiltonian theories have been widely used in almost all areas of physics, constructing Poisson structure in term of Hamiltonian theories have become an important topic in mathematical physics.[12−16] Noncentral dynamical system is one of the fundamental systems in mechanics and quantum mechanics.[12,17] In this paper, we express the differential equations of a noncentral dynamical system in Ermakov formalism to obtain the Ermakov invariant. In terms of Hamiltonian theories and using the Ermakov invariant as the Hamiltonian, the Poisson structure of noncentral dynamical system in four-dimensional (4D) phase space are constructed. The result indicates that the Poisson structure is degenerate and the noncentral dynamical system possesses four invariants, the Hamiltonian, the Ermakov invariant and two Casimir functions.

2. Ermakov formalism and Ermakov invariant of a noncentral dynamical system

The Hamiltonian of a noncentral dynamical system in polar coordinates is[9]

$$\begin{aligned} H =& \frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{\sigma}{r}+\frac{1}{r^2\sin^2\theta}(g_1+g_2\cos\theta) \\ =& \frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)+\bar{V}(r)+\frac{1}{r^2}\Phi(\theta), \end{aligned} \tag{1}$$

where σ, g_1 and g_2 are positive constants, and

$$\bar{V}(r)=-\frac{\sigma}{r}, \tag{2}$$

$$\Phi(\theta)=\frac{1}{\sin^2\theta}(g_1+g_2\cos\theta). \tag{3}$$

The Hamiltonian function (1) generates the following

equations for the motion of the system:

$$\dot{r} = p_r, \tag{4}$$
$$\dot{\theta} = \frac{p_\theta}{r^2}, \tag{5}$$
$$\dot{p}_r = \frac{p_\theta^2}{r^3} - \frac{\sigma}{r^2} + \frac{2\Phi(\theta)}{r^3}, \tag{6}$$
$$\dot{p}_\theta = -\frac{1}{r^2}\frac{\partial \Phi(\theta)}{\partial \theta}. \tag{7}$$

We can easily obtain the second-order equations from the first-order equations:

$$\ddot{r} - r\dot{\theta}^2 + \frac{\sigma}{r^2} = \frac{2\Phi(\theta)}{r^3}, \tag{8}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{1}{r^3}\frac{\partial \Phi(\theta)}{\partial \theta}. \tag{9}$$

According to Ref.[12], Eqs.(8) and (9) are Ermakov systems with $\omega^2 = \frac{\sigma}{r^3}$, $F(\theta) = 2\Phi(\theta)$ and $G(\theta) = \frac{\partial \Phi(\theta)}{\partial \theta}$. The essential property of Ermakov systems is that they possess a constant of motion, the Ermakov invariant,

$$I = \frac{p_\theta^2}{2} + \Phi(\theta). \tag{10}$$

3. Poisson structure and two Casimir functions for noncentral dynamical system in four-dimensional phase space

In 4D phase space, we choose

$$(x_1, x_2, x_3, x_4) = (r, \theta, p_r, p_\theta). \tag{11}$$

The noncentral dynamical system is said to be a generalized Hamiltonian system when it can be cast in the following form:

$$\dot{x}_\mu = J^{\mu v}\partial_v H, \qquad \mu = 1, 2, 3, 4, \tag{12}$$

where sum convention is assumed and $\partial_v = \frac{\partial}{\partial x_v}$. Here, H is the Hamiltonian function and $J^{\mu v}$ is the Poisson matrix for the system. The Poisson matrix must be skew-symmetric, $J^{\mu v} = -J^{v\mu}$, and should satisfy the following system of partial differential equations:[12]

$$J^{\mu 1}\partial_\mu J^{23} + J^{\mu 2}\partial_\mu J^{31} + J^{\mu 3}\partial_\mu J^{12} = 0, \tag{13}$$
$$J^{\mu 1}\partial_\mu J^{24} + J^{\mu 2}\partial_\mu J^{41} + J^{\mu 4}\partial_\mu J^{12} = 0, \tag{14}$$
$$J^{\mu 1}\partial_\mu J^{34} + J^{\mu 3}\partial_\mu J^{41} + J^{\mu 4}\partial_\mu J^{13} = 0, \tag{15}$$
$$J^{\mu 2}\partial_\mu J^{34} + J^{\mu 3}\partial_\mu J^{42} + J^{\mu 4}\partial_\mu J^{23} = 0. \tag{16}$$

Using the Ermakov invariant as the Hamiltonian,

$$H = \frac{x_4^2}{2} + \Phi(x_2), \tag{17}$$

Eqs.(4)–(7) can be written as

$$\dot{x}_1 = x_3 = J^{12}G(x_2) + J^{14}x_4, \tag{18}$$
$$\dot{x}_2 = \frac{x_4}{x_1^2} = J^{24}x_4, \tag{19}$$
$$\dot{x}_3 = \frac{x_4^2}{x_1^3} - \frac{\sigma}{x_1^2} + \frac{2\Phi(x_2)}{x_1^3} = J^{32}G(x_2) + J^{34}x_4, \tag{20}$$
$$\dot{x}_4 = -\frac{G(x_2)}{x_1^2} = J^{42}G(x_2). \tag{21}$$

Setting

$$J^{12} = 0, \tag{22}$$

we obtain

$$J^{14} = \frac{x_3}{x_4}, \tag{23}$$
$$J^{24} = \frac{1}{x_1^2}. \tag{24}$$

Substituting Eqs.(22)–(24) into Eq.(14), we obtain

$$J^{23} = \frac{x_3}{x_1^2 x_4}. \tag{25}$$

Inserting Eq.(25) into Eq.(20), we obtain

$$J^{34} = \frac{x_4}{x_1^3} - \frac{\sigma}{x_4 x_1^2} + \frac{2\Phi(x_2)}{x_4 x_1^3} + \frac{x_3}{x_1^2 x_4^2}G(x_2). \tag{26}$$

Substituting Eqs.(22)–(26) into Eq.(16), we obtain

$$J^{13} = \left(\frac{x_3}{x_4}\right)^2 + \frac{\sigma}{x_1 x_4^2} - \frac{2\Phi(x_2)}{x_1^2 x_4^2} - \frac{x_4}{x_1 x_4^3}G(x_2). \tag{27}$$

Inserting Eqs.(22)–(27) into Eq.(13), we obtain

$$\frac{\sigma}{x_1 x_4^2} - \frac{2\Phi(x_2)}{x_1^2 x_4^2} - \frac{x_4}{x_1 x_4^3}G(x_2) = 0. \tag{28}$$

So

$$J^{34} = \frac{x_4}{x_1^3}, \tag{29}$$
$$J^{13} = \left(\frac{x_3}{x_4}\right)^2. \tag{30}$$

From the results of the above discussion, we obtain the following Poisson matrix:

$$J^{\mu v} = \begin{pmatrix} 0 & 0 & \left(\frac{x_3}{x_4}\right)^2 & \frac{x_3}{x_4} \\ 0 & 0 & \frac{x_3}{x_1^2 x_4} & \frac{1}{x_1^2} \\ -\left(\frac{x_3}{x_4}\right)^2 & -\frac{x_3}{x_1^2 x_4} & 0 & \frac{x_4}{x_1^3} \\ -\frac{x_3}{x_4} & -\frac{1}{x_1^2} & -\frac{x_4}{x_1^3} & 0 \end{pmatrix}. \tag{31}$$

The Poisson matrix is degenerate, because of

$$\det J^{\mu v} = 0. \tag{32}$$

We can obtain Casimir functions from the defining equations

$$J^{\mu v}\partial_v C = 0. \tag{33}$$

Using the Poisson matrix, we obtain following equations for Casimirs:

$$x_3\frac{\partial C}{\partial x_3} + x_4\frac{\partial C}{\partial x_4} = 0, \tag{34}$$

$$-\left(\frac{x_3}{x_4}\right)^2\frac{\partial C}{\partial x_1} - \frac{x_3}{x_4 x_1^2}\frac{\partial C}{\partial x_2} + \frac{x_4}{x_1^3}\frac{\partial C}{\partial x_4} = 0. \tag{35}$$

Equation (34) shows that the Casimirs depend on the variables $\left(\frac{x_3}{x_4}, x_1, x_2, t\right)$. Setting

$$\frac{x_3}{x_4} = \alpha, \tag{36}$$

we can transform Eq.(35) into

$$\alpha\frac{\partial C}{\partial x_1} + \frac{1}{x_1^2}\frac{\partial C}{\partial x_2} + \frac{1}{x_1^3}\frac{\partial C}{\partial \alpha} = 0. \tag{37}$$

By changing the coordinates according to

$$\bar{x}_1 = \frac{1}{x_1}, \quad \bar{x}_2 = x_2, \quad \bar{\alpha} = -\alpha, \tag{38}$$

Eq.(37) becomes

$$\bar{\alpha}\frac{\partial C}{\partial \bar{x}_1} + \frac{\partial C}{\partial \bar{x}_2} - \bar{x}_1\frac{\partial C}{\partial \bar{\alpha}} = 0. \tag{39}$$

The characteristic equations for Eq.(39) are

$$\frac{\mathrm{d}\bar{x}_1}{\bar{\alpha}} = \frac{\mathrm{d}\bar{x}_2}{1} = \frac{\mathrm{d}\bar{\alpha}}{-\bar{x}_1}. \tag{40}$$

From Eq.(40), the two Casimir functions are

$$C_1 = \frac{1}{2}\left(\frac{\mathrm{d}\bar{x}_1}{\mathrm{d}\bar{x}_2}\right)^2 + \frac{1}{2}\bar{x}_1^2, \tag{41}$$

$$C_2 = \bar{x}_2 - \int^{\bar{x}_1}\frac{\mathrm{d}\bar{x}_1}{(2C_1 - \bar{x}_1^2)^{1/2}}. \tag{42}$$

C_1 is the energy and C_2 is the integration constant. In term of the original coordinates, the two Casimir functions are

$$C_1 = \frac{1}{2}\left(\frac{x_3}{x_4}\right)^2 + \frac{1}{2x_1^2},$$

$$C_2 = x_2 - \int^{x_1}\frac{\mathrm{d}x_1}{(1 - 2C_1 x_1^2)^{1/2}}.$$

Thus the noncentral dynamical system possess four invariants, namely Hamiltonian, the Ermakov invariant and two Casimir functions.

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