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Nonexistence of ultra-subharmonic periodic orbits in periodically forced differential equation

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It is proved that if there exists a periodic solution for a class of non-autonomous differential dynamic systems, it can only be subharmonic, ultra-subharmonic periodic solution is impossible. Moreover, the existence of R-type ultra-subharmonic periodic solution defined for a specified planar system is also denied. As an application of the above conclusions, through investigating some typical examples, it is pointed out that the existence of ultra-subharmonic periodic orbits in a planar perturbation system cannot be determined by second-order Melnikov method. An explanation is also provided.

Keywords: nonlinear dynamic system, higher-order Melnikov method, ultra-subharmonic periodic solution, Poincaré map

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1. Introduction

In the study of the dynamic system, many important and interesting problems can be simplified to a nonlinear non-autonomous differential equation in the vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x) + \boldsymbol{g}(x,t), \tag{1}$$

where $\boldsymbol{g}(\boldsymbol{x},t)$ is a time-periodic function. System (1) cannot usually be solved exactly and may exhibit many complicated phenomena such as bifurcation and chaos in some special cases. So from Poincaré's work in the late-nineteenth century, qualitative analysis of geometry became an important approach to study the system. For such a system, the classical and the modern analysis methods as well as some related theories have been outlined in literature.[1] In the planar and multidimensional cases, based on Brouwer's fixed-point theorem and the Lyapunov method and under certain particular conditions, a few theorems about the existence and stability of the periodic solution of the system[2,3] have been proved. Especially in the two-dimensional case, if the second term of the right-hand side of Eq.(1) can be regarded as a small time-periodic perturbation, i.e. $\varepsilon\boldsymbol{g}(\boldsymbol{x},t)$, then the Melnikov method is widely used as a criterion to check whether subharmonic and ultra-subharmonic bifurcation or even chaos will occur. However, in most cases, the first-order Melnikov function can only show the existence of subharmonic periodic orbits $T_{\rm k} = mT$ and all the ultra-subharmonic orbits $T_{\rm k} = mT/n(n \neq 1)$ are denied, here T is the period of $g(x,t)$ and $T_{\rm k}$ is the period of the periodic solution, if exists, in (1). Such results are attributed to the fact that the dissipation and the forcing of the system are of same order of ε, thus the first-level period structure of the Poincaré map of the system can be preserved and the high-level ones cannot exist due to the dissipation. For the reasons mentioned above, the dissipation is set to be of higher order of ε than the forcing, and higher-order Melnikov functions are developed to check whether the ultra-subharmonic bifurcation may occur in such a case, consequently the existence of the ultra-subharmonic periodic solution are justified.[4,5] Reference [6] provides a detailed description of the Melnikov method and its development. Some work in the field of physics and mechanics has also been done using this method.[7−10] In this paper, further analyses are made for system (1). It is proved that if there exists a periodic solution for a class of non-autonomous differential dynamic systems, it can only

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be subharmonic; an ultra-subharmonic periodic solution is impossible. Moreover, the existence of an R-type ultra-subharmonic periodic solution defined for a specified planar system is also denied. As an application of the above conclusions, through investigating some typical examples, it is pointed out that the existence of ultra-subharmonic periodic orbits in a planar perturbation system cannot be determined by the present-day second-order Melnikov method. An explanation is also provided.

2. Nonexistence of ultra-subharmonic periodic solutions

Definition If $\boldsymbol{x}(t)$ is a periodic solution of (1) with period T^*, i.e. $\boldsymbol{x}(t) = \boldsymbol{x}(t+T^*)$ $(\forall\ t \in R)$, T is the smallest positive period of $\boldsymbol{g}(x,t)$, and the ratio of T^*/T is an irrational number, then $\boldsymbol{x}(t)$ is a exceptional periodic solution of (1).

The second term of the right-hand side of Eq.(1) can be written in a separated form as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{\psi}(\boldsymbol{x})\,\boldsymbol{h}(t), \tag{2}$$

where $\boldsymbol{x} = \boldsymbol{x}(t) \in R^n$ is a vector-valued function of an independent variable time t and $\boldsymbol{f}(\boldsymbol{x}) : U \to R^n$ is a $C^r(r \geq 1)$ smooth function defined on some subset $U \subseteq R^n$; $\boldsymbol{\psi}(\boldsymbol{x})$ is a invertible matrix on $R^{n\times n}$, $\boldsymbol{h}(t) : R \to R^n$ is a time-periodic $C^r(r \geq 1)$ function with the smallest positive period T. For a degenerate form of (2), i.e. $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{h}(t)$, it has been proved that no exceptional periodic solution exists in such a case.[2] Here we will prove that there is also no ultra-subharmonic periodic solution in Eq.(2).

Proposition 1 If there exists a periodic solution $\boldsymbol{x}(t) = \boldsymbol{x}(t+T^*)$ of period T^* for equation (2), then the solution can only be subharmonic, i.e. $T^* = kT(k \in Z^+)$; the ultra-subharmonic periodic solution of period $T^* = \dfrac{m}{n}T$ (m and n are relatively prime) is impossible.

Proof (reductio ad absurdum approach)

Assume that Eq.(2) has an ultra-subharmonic periodic solution $\boldsymbol{x}^*(t)$ with period $T^* = \dfrac{m}{n}T$. Then $\boldsymbol{x}^*(t)$ satisfies equation (2), i.e.

$$\dot{\boldsymbol{x}}^*(t) = \boldsymbol{f}(\boldsymbol{x}^*(t)) + \boldsymbol{\psi}(\boldsymbol{x}^*(t))\,\boldsymbol{h}(t). \tag{3}$$

Equation (3) is identical with respect to translations in time t, so we have

$$\begin{aligned}\dot{\boldsymbol{x}}^*(t+\frac{m}{n}T) =& \boldsymbol{f}\left(\boldsymbol{x}^*\left(t+\frac{m}{n}T\right)\right)\\ &+\boldsymbol{\psi}\left(\boldsymbol{x}^*\left(t+\frac{m}{n}T\right)\right)\boldsymbol{h}\left(t+\frac{m}{n}T\right).\end{aligned} \tag{4}$$

Since $\boldsymbol{x}^*(t)$ is time-periodic function of period $\dfrac{m}{n}T$, then we have $\boldsymbol{x}^*(t) = \boldsymbol{x}^*\left(t+\dfrac{m}{n}T\right)$ and $\dot{\boldsymbol{x}}^*(t) = \dot{\boldsymbol{x}}^*\left(t+\dfrac{m}{n}T\right)$, comparing (3) with (4), we obtain

$$\boldsymbol{\psi}(\boldsymbol{x}^*(t))\,\boldsymbol{h}(t) = \boldsymbol{\psi}(\boldsymbol{x}^*(t))\,\boldsymbol{h}\left(t+\frac{m}{n}T\right). \tag{5}$$

Since $\boldsymbol{\psi}(\boldsymbol{x}^*(t))$ is invertible, we have

$$\boldsymbol{h}(t) = \boldsymbol{h}\left(t+\frac{m}{n}T\right) \quad (\forall\ t \in R). \tag{6}$$

Since equation (6) is identical to any time t and T is the smallest positive period of $\boldsymbol{h}(t)$, there must exist $k \in Z^+$ which satisfies

$$\frac{m}{n}T = kT. \tag{7}$$

Hence, we have

$$\frac{m}{n} = k. \tag{8}$$

Obviously, (8) conflicts with the presupposition that m and n are relatively prime, so we conclude that if there exists a periodic solution for (2), it can only be subharmonic; an ultra-subharmonic periodic solution is impossible.

In the two-dimensional case, Eq.(1) can represent many single freedom systems under periodic excitation. For later use, let us introduce the definition of an R-type periodic solution.[6]

Definition A solution $(x(t), y(t))$ of Eq.(1) in a planar case is an R-type periodic solution if there exists a positive number $T^* > 0$, which satisfies $(x(t+T^*),\ y(t+T^*)) = (x(t)+2\pi, y(t))$ $(\forall t \in R)$.

Consider the following non-autonomous planar system:

$$\begin{cases}\dot{x} = f_1(x,y) + h_1(t),\\ \dot{y} = f_2(x,y) + h_2(t),\end{cases} \tag{9}$$

where f_i is an x-periodic function, i.e. $f_i(x+2\pi, y) = f_i(x,y)$ $(\forall (x,y) \in R^2, i = 1,2)$, and other requirements are the same as in (2), then we can give the following proposition analogously.

Proposition 2 If there exists an R-type periodic solution of period T^* for (9), then the solution

can only be subharmonic, i.e. $T^* = kT$ ($k \in Z^+$); the R-type ultra-subharmonic periodic solution of period $T^* = \frac{m}{n}T$ (m and n are relatively prime) is impossible.

Proof Assume that (9) has an R-type ultra-subharmonic periodic solution $(\boldsymbol{x}^*(t), y^*(t))$ with period $T^* = \frac{m}{n}T$. Then $(\boldsymbol{x}^*(t), y^*(t))$ satisfies (9), i.e.

$$\begin{cases} \dot{\boldsymbol{x}}^*(t) = f_1(\boldsymbol{x}^*(t), y^*(t)) + h_1(t), \\ \dot{y}^*(t) = f_2(\boldsymbol{x}^*(t), y^*(t)) + h_2(t). \end{cases} \tag{10}$$

Equation (10) is identical with respect to translations in time t, so we have

$$\begin{cases} \dot{\boldsymbol{x}}^*(t + mT/n) = f_1(\boldsymbol{x}^*(t + mT/n), y^*(t + mT/n)) + h_1(t + mT/n), \\ \dot{y}^*(t + mT/n) = f_2(\boldsymbol{x}^*(t + mT/n), y^*(t + mT/n)) + h_2(t + mT/n). \end{cases} \tag{11}$$

Since $(\boldsymbol{x}^*(t), y^*(t))$ is an R-type periodic solution of period $\frac{m}{n}T$, then obviously we have

$$\boldsymbol{x}^*(t + \frac{m}{n}T) = \boldsymbol{x}^*(t) + 2\pi,$$

$$\dot{\boldsymbol{x}}^*(t + \frac{m}{n}T) = \dot{\boldsymbol{x}}^*(t),$$

$$y^*(t + \frac{m}{n}T) = y^*(t)$$

and

$$\dot{y}^*(t + \frac{m}{n}T) = \dot{y}^*(t).$$

Comparing (10) with (11) and under the consideration of the periodicity of $f_i(x, y)$ ($i = 1, 2$) with respect to variable x, we obtain

$$h_i(t) = h_i\left(t + \frac{m}{n}T\right), \quad (\forall\, t \in R, i = 1, 2). \tag{12}$$

Since (12) is identical to any time t and T is the smallest positive period of $\boldsymbol{h}(t)$, there must exist $k \in Z^+$ which satisfies

$$\frac{m}{n}T = kT, \quad \text{i.e.,} \quad \frac{m}{n} = k. \tag{13}$$

Obviously, Eq.(13) conflicts with the presupposition that m and n are relatively prime, so we conclude that if there exists an R-type periodic solution for (9), it can only be subharmonic; an R-type ultra-subharmonic periodic solution is impossible.

3. Higher-order Melnikov method

Let us recall the higher-order Melnikov method. Consider a planar system of the form[4–6]

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}, t) + \varepsilon^2 \boldsymbol{h}(\boldsymbol{x}, t), \tag{14}$$

where $\boldsymbol{x} = \boldsymbol{x}(t) = \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$; $\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix}$, $\boldsymbol{g}(\boldsymbol{x}, t) = \begin{pmatrix} g_1(\boldsymbol{x}, t) \\ g_2(\boldsymbol{x}, t) \end{pmatrix}$ and $\boldsymbol{h}(\boldsymbol{x}, t) = \begin{pmatrix} h_1(\boldsymbol{x}, t) \\ h_2(\boldsymbol{x}, t) \end{pmatrix}$ are sufficiently smooth ($C^r, r \geq 2$) and bounded on bounded set, and $\boldsymbol{g}(\boldsymbol{x}, t)$ is T-periodic in t. Specific assumptions on system (14) are:

1) For $\varepsilon = 0$, (14) is a Hamiltonian system; i.e. there is a function $H : R^2 \to R$ such that $f_1 = \partial H/\partial v$, $f_2 = -\partial H/\partial u$.

2) There exists an open set $U \in R^2$ such that the unperturbed Hamiltonian system has a continuous family of periodic orbits $\boldsymbol{q}^\alpha(t)$, $\alpha \in R$, contained in U.

3) Let $h_\alpha = H(\boldsymbol{q}^\alpha(t))$ and T_α be the period of $\boldsymbol{q}^\alpha(t)$. Then T_α is a differentiable function of h_α and $\mathrm{d}T_\alpha/\mathrm{d}h_\alpha > 0$ (or $\mathrm{d}T_\alpha/\mathrm{d}h_\alpha < 0$).

Under the above assumptions, for a periodic orbit $\boldsymbol{q}^\alpha(t)$ with period $T_\alpha = mT/n$ (m and n are relatively prime), first-order and second-order Melnikov functions are defined by

$$M_1^{m/n}(t_0) = \int_0^{mT} \boldsymbol{f}(\boldsymbol{q}^\alpha(t)) \wedge \boldsymbol{g}(\boldsymbol{q}^\alpha(t), t + t_0)\,\mathrm{d}t, \tag{15}$$

$$M_2^{m/n}(t_0) = \int_0^{mT} \boldsymbol{f}(\boldsymbol{q}^\alpha(t)) \wedge \left[\frac{1}{2}\boldsymbol{D}^2\boldsymbol{f}(\boldsymbol{q}^\alpha(t))(\boldsymbol{q}_1^\alpha(t + t_0, t_0))^2 + \boldsymbol{D}\boldsymbol{g}(\boldsymbol{q}^\alpha(t), t + t_0)(\boldsymbol{q}_1^\alpha(t + t_0, t_0)) + \boldsymbol{h}(\boldsymbol{q}^\alpha(t), t + t_0)\right]\mathrm{d}t, \tag{16}$$

where $\boldsymbol{q}_1^{\alpha}(t,t_0)$ is the solution of the variational equation

$$\dot{\boldsymbol{q}}_1^{\alpha}(t,t_0)=D\boldsymbol{f}(\boldsymbol{q}^{\alpha}(t-t_0))\boldsymbol{q}_1^{\alpha}(t,t_0)+\boldsymbol{g}(\boldsymbol{q}^{\alpha}(t-t_0),t). \tag{17}$$

Then the following conclusions are drawn:[4−6]

(1) If $M_1^{m/n}(t_0)\not\equiv 0$ and $\forall t_0\in[0,\ T]$, $M_1^{m/n}(t_0)$ has simple zeros independent of ε, then for ε sufficiently small, system (14) has a subharmonic orbit with period $mT/n(n=1)$ or ultra-subharmonic orbit with period $mT/n(n>1)$.

(2) If $M_1^{m/n}(t_0)\equiv 0$ and $\forall t_0\in[0,\ T]$ $M_2^{m/n}(t_0)$ has simple zeros independent of ε, then for ε sufficiently small, system (14) has a subharmonic orbit with period $mT/n(n=1)$ or ultra-subharmonic orbit with period $mT/n(n>1)$.

Obviously, conclusions (1) and (2) provide a criterion for the existence of ultra-subharmonic period orbit.

Counterexamples:

Consider the Josephson Junction model equation

$$\begin{aligned}&\dot{\boldsymbol{x}}=y,\\&\dot{y}=-\sin x-\tfrac{1}{\sqrt{\beta}}(1+\mu\cos x)y+\rho+\alpha\sin\omega t,\end{aligned} \tag{18}$$

where α, β, ρ, μ, Ω are nondimensional parameters. In order to analyse using the higher-order Melnikov method, a small parameter ε is introduced, and we let $\sqrt{\beta^{-1}}=\varepsilon^2\delta$, $\rho=\varepsilon^2 b\delta$, $\alpha=\varepsilon c$, where δ, b, c is of $O(1)$; thus (18) has a substituted form

$$\begin{aligned}&\dot{\boldsymbol{x}}=y,\\&\dot{y}=-\sin x-\varepsilon^2\delta[b-(1+\mu\cos x)y]+\varepsilon c\sin\omega t.\end{aligned} \tag{19}$$

For $\varepsilon=0$, (19) is a Hamiltonian system with a centre at (x,y)=(0,0) and Hamiltonian $H(x,y)=y^2/2-\cos x=h$. If $h>1$, there are two families of R-type periodic orbits

$$\begin{cases}x_{\pm}(t,k)=\pm 2\arcsin\left(\mathrm{sn}\dfrac{t}{k}\right),\\ y_{\pm}(t,k)=\pm\dfrac{2}{k}\mathrm{dn}\left(\dfrac{t}{k}\right),\end{cases} \tag{20}$$

where sn and dn are the Jacobi elliptic functions, k is elliptic modulus, $0<k<1$, and $h(k)=2/k^2-1>1$. For orbits (20), Ref.[17] has given a detailed calculation, and the results showed that a first-order Melnikov function is always equal to zero when $n\neq 1$, and that a second-order Melnikov function has simple zeros at $t_0\in[0,2\pi/\omega]$ if certain parameter conditions are satisfied. So by the conclusion (2) above, it is thought in Ref.[7] that system (19) will have R-type ultra-subharmonic periodic orbits with period $mT/n=2m\pi/n\omega$.

Another analogous example is a simple pendulum with weak periodic perturbation

$$\begin{aligned}&\dot{\boldsymbol{x}}=y,\\&\dot{y}=-\sin x+\varepsilon r\cos\omega t-\varepsilon^2\delta y.\end{aligned} \tag{21}$$

Equation (21) also has been analysed using a high-order Melnikov method in Ref.[5] and similar results were obtained, that R-type ultra-subharmonic periodic orbits will exist in (21) under required parameter conditions.

Obviously, both (19) and (21) are consistent with (9). Based on the conclusion of proposition 2, no ultra-subharmonic periodic orbits of the period $T=2m\pi/n\omega$ exist in these equations.

From the above results, we think that the second-order Melnikov method may give an improper result on the existence of ultra-subharmonic periodic orbits in a planar perturbation system.

4. Geometric explanation

Here we give a simple explanation of why the second-order Melnikov method yields an improper result. Consider a suspended autonomous system of (14)

$$\left.\begin{aligned}&\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\varepsilon g(\boldsymbol{x},\theta)+\varepsilon^2\boldsymbol{h}(x,\theta)\\&\dot{\theta}=1\end{aligned}\right\};(\boldsymbol{x},\theta)\in R^2\times S^1, \tag{22}$$

where the circular component $S^1=R(\mathrm{mod}\ T)$ reflects the periodicity of the vector field $\boldsymbol{f}$ and $\boldsymbol{g}$ in θ, thus we can define a global cross section (Poincaré section)

$$\Sigma^{t_0}=\left\{(\boldsymbol{x},\theta)\in R^2\times S^1\,|\theta=t_0\in[0,T]\right\}. \tag{23}$$

Then Poincaré map $P_{\varepsilon}^{t_0}:\ \Sigma^{t_0}\to\Sigma^{t_0}$ is given by

$$\begin{aligned}P_{\varepsilon}^{t_0}(\boldsymbol{q}_{\varepsilon}(t_0,t_0))=&\pi(q_{\varepsilon}(t_0+T,t_0),\theta(T))\\=&\boldsymbol{q}_{\varepsilon}(t_0+T,t_0),\end{aligned} \tag{24}$$

where $\boldsymbol{q}_{\varepsilon}(t_0,t_0)$ is a solution of (14), π denotes projection onto the first factor. For a periodic orbit $\boldsymbol{q}^{\alpha}(t)$ of period $T_{\alpha}=mT/n$ in the unperturbed system ($\varepsilon=0$), its Melnikov function is used to measure the distance between two mapping points $\boldsymbol{q}_{\varepsilon}^{\alpha}(t_0,t_0)$ and $\boldsymbol{q}_{\varepsilon}^{\alpha}(t_0+mT,t_0)$, see details in Ref.[1]. $M_1^{m/n}(t_0)$ and $M_2^{m/n}(t_0)$ is the first term and the second term coefficient of the asymptotic expansion of the distance

function, respectively. If $M_1^{m/n}(t_0)$ has simple zeros, or $M_1^{m/n}(t_0) \equiv 0$ and $M_2^{m/n}(t_0)$ has simple zeros, then it implies that the map $(P_\varepsilon^{t_0})^m$ has a fixed point near $\boldsymbol{q}^\alpha(0)$, i.e. Poincaré map (24) has m period points. So both the first-order Melnikov method and the second-order one can only make a justification whether a Poincaré map has periodic points and the conclusions (1) and (2) in Section 2 are taken as a straightforward result from the facts above. But a question arises in the last step. Let us make a careful investigation in two different cases:

In the case of $n = 1$, if Poincaré map (24) has m period points, then during the time interval of mT, the perturbed orbit will finish one circle and remain a closed orbit. Compared with the unperturbed periodic orbit of period $T_\alpha = mT$, the perturbed orbit may be distorted under the perturbation but not broken (see Fig 1(a)). So the conclusions are uniformly valid in such a case.

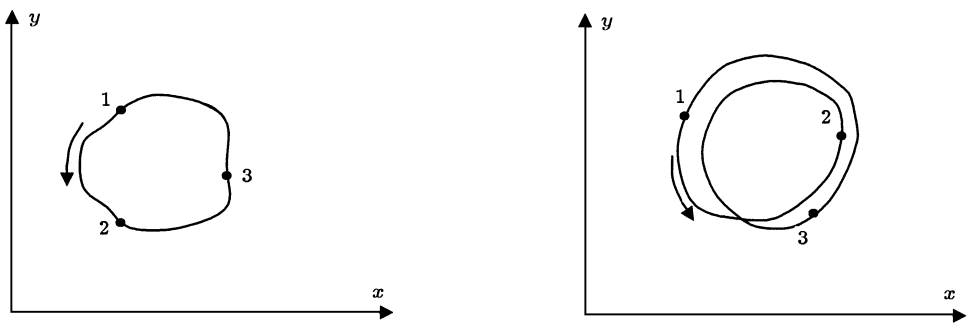


Fig.1. Poincaré section.

In the case of $n > 1$, if Poincaré map (24) has m period points, the perturbed orbit will start at time t_0 and make n circles in the neighbourhood of the original unperturbed orbit of period $T_\alpha = mT/n$, then get closed at the time $t_0 + mT$. However, whether each one of the n circles is a closed orbit is unknown; a possible case is that the original orbit will break under the perturbation and reform a new closed orbit after finishing n circles. Now, the period of the new closed orbit is mT but not mT/n, i.e. system (14) has no ultra-subharmonic periodic orbit of period $mT/n (n > 1)$ in such a case (See Fig.1(b)).

From the analysis above, we know that even if Poincaré map (24) has m period points in the case of $n > 1$, the existence of ultra-subharmonic orbit is not a necessary result, so the conclusions in Section 2 is not appropriate.

5. Conclusions

In this paper, the non-existence of ultra-subharmonic solutions for a class of non-autonomous differential dynamic systems is proved. Since this system can stand for many important and interesting physical models, the conclusions presented here are universally useful. As an application of above conclusions, through investigating some typical examples, it is pointed out that the existence of ultra-subharmonic periodic orbits in a planar perturbation system cannot be determined by a second-order Melnikov method. A geometric explanation is also provided. Further research need to be done for such a problem.

References

[1] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (New York: Springer) p166
[2] Sansone G and Conti R 1964 *Nonlinear Differential Equations* (New York: Pergamon) p28
[3] Li L 1997 *The Qualitative Theories and Quantitative Methods of Strong Nonlinear Oscillation System* (Beijing: Scientific Publishing Press) p85 (in Chinese)
[4] Liu Z R and Gu G Q 1990 *Phys. Lett.* A **143** 213
[5] Guo Y Z and Liu Z R 1991 *Applied Mathematics and Mechanics* **12** 19 (in Chinese)
[6] Liu Z R 1994 *Perturbation Criteria For Chaos* (Shanghai: Shanghai Scientific and Technological Education Publishing House) p111 (in Chinese)
[7] Liu Z R, Jiang X M, Han Z B and Gu G Q 1990 *Acta Phys. Sin.* **39** 823 (in Chinese)
[8] Han Q, Zhang N M and Yang G T 1999 *Appl. Math. Mech.* **20** 960
[9] Zhang N M, Han Q and Yang G T 1999 *Appl. Math. Mech.* **20** 1319
[10] Zhang N M, Yang G T and Xu B Y 2000 *Appl. Math. Mech.* **21** 1008