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Dark-lines in bifurcation plots of nonlinear dynamic systems*

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Based on the regressive character of chaotic motion in nonlinear dynamic systems, a numerical regression algorithm is developed, which can be used to research the dark-lines passing through chaotic regions in bifurcation plots. The dark-lines of the parabolic mapping are obtained by using the numerical regression algorithm, and compared with those that are accurately acquired through dark-line equations. Thus the validity of this algorithm is proved. Furthermore, for the Brussel oscillation system and the piecewise linear dynamic system of a gear pair, the dark-lines are researched by using the regression algorithm. By researching the dark-lines in the bifurcation plots of nonlinear dynamic systems, the periodic windows embedded in chaotic regions can be ascertained by tangential points of dark-lines, and the turning points of chaotic attractors can be also obtained by intersected points. The results show that this algorithm is helpful to analyse dynamic behaviour of systems and control chaotic motion.

Keywords: nonlinear dynamic system, bifurcation, chaos, dark-lines
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1. Introduction

For nonlinear systems, bifurcation plots can be obtained by numerical iteration. As a representative example, the bifurcation plot of the parabolic mapping is shown in Fig.1. So far, some conclusions,[1−5] such as the Feigenbaum constant, the period-doubling bifurcation sequence, and the self-similarity configuration, etc, have been reached by researching bifurcation and chaos generally existing in nonlinear systems. These conclusions laid the foundation for discovering essentials of nonlinear phenomena.

In Fig.1, many turning points of chaotic attractors, periodic windows embedded in chaotic region, and dark-lines can be found; the dark-lines outline the boundaries of the chaotic region and the profile of the whole bifurcation plot.[1,6,7] By researching the dark-lines of parabolic mapping some conclusions have been obtained. (1) Every tangential point of dark-lines corresponds to a super-stable periodic orbit, and there exists a periodic window in the vicinity of a corresponding parameter value. (2) Every intersected point of dark-lines corresponds to a turning point, where the chaotic attractor collides with an unstable periodic orbit so that it becomes another different attractor.

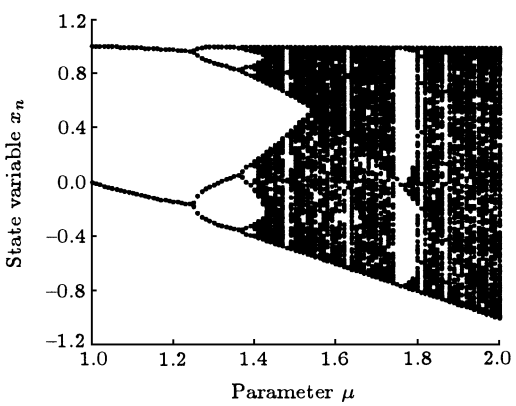


Fig.1. Bifurcation plot of the parabolic mapping.

By researching dark-lines, the corresponding parameter values of periodic windows and the turning points of chaotic attractors can be obtained. Thus some guidance can be provided for studying the nonlinear dynamic behaviour; some help can be given to analysis of the turning mechanism[1] f chaos and control chaotic motion by using the OGY (Ott, Grebogi and Yorke) method.[8,9] Furthermore, the dark-lines represent the singularity in continuous distribution, and have universal significance in some practical

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physical systems, such as defocusing curves of optical models, large-scale structures on the stochastic background of vortex motion,[1] and so on. For this reason, the research on dark-lines of nonlinear dynamic systems is significant in both practice and theory.

At present, the dark-lines are researched only in some discrete mappings with deterministic nonlinear function expression whose dark-line equations can be ascertained. However, most physical systems are continuous and described by nonlinear differential equations. Therefore, an effective algorithm must be developed to research the dark-lines of these systems. The method should be effective in researching the dark-lines of any nonlinear system not depending on dark-line equations. Following the idea of using symbolic dynamics to research nonlinear systems described by differential equations,[10] the regressive mapping can be constructed based on the regressive character of chaotic orbits and related to the single-peak mapping. In this paper, based on the numerical results, the regression algorithm is presented to the research on the dark-lines and proved by some examples, such as the parabolic mapping, the Brussel oscillation system and the piecewise linear gear system. It provides a new view for researching other nonlinear physical systems.

2. Algorithm of dark-lines

2.1. Dark-line equations [1]

The nonlinear function mapping is expressed by

$$x_{n+1} = f(\mu, x_n), \quad x_n, x_{n+1} \in R. \tag{1}$$

Here the variable μ is an external parameter. The dark-lines can be found by performing the following steps:

(1) find the derivative function $f'(\mu, x)$;

(2) solving the equation $f'(\mu, x) = 0$ to obtain the critical values $x_{\rm ac} = c$;

(3) define the initial function $P_0(\mu) = c$;

(4) define recursive functions and express them as

$$P_{n+1}(\mu) = f(\mu, P_n(\mu)), \quad (n = 0, 1, 2, \cdots); \tag{2}$$

(5) If the number of critical points is i for the function $f(\mu, x)$, accordingly define the functions as

$$P_0^{(i)}(\mu) = c_i, P_{n+1}^{(i)}(\mu) = f(\mu, P_n^{(i)}(\mu)). \tag{3}$$

The functions $P_n(\mu)$ $(n = 0, 1, 2, \cdots)$ are the dark-line equations of the nonlinear system.

2.2. Regression algorithm

Based on the regressive character, the chaotic motion of nonlinear dynamic systems is summarized and researched. For both the discrete mapping with deterministic function expression and the dynamic system described by differential equations, the chaotic orbits outline the attractors with clear boundaries. The chaotic attractor may be divided into several segments. With the process of evolution, the orbit returns to the vicinity of the state that once appeared.

(1) For a given parameter, we properly take sampling parameters NS1 and N. Here NS1 is the number of the elementary period for the continuous system, and is the iterative steps for the discrete mapping. And N is the number of the sampling point. For clearly observing the distribution of sampling points, the value of "N" is taken as 200–600 commonly.

(2) If the sampling points are distributed in k regions, we can define a parameter m and make it equal to the value of "k·NS1". Then the chaotic attractor is marked by the expression "mI".

(3) Let the parameter NK equal $i, (i = 1, 2, \cdots, k)$, we can take the ith region from the k regions and research it.

(4) By taking the parameter NS2 and making it equal to m, we take sampling points again and use them to construct the regressive mapping.

(5) The regressive mapping is related to the single-peak mapping, and we can find the critical point of the mapping.

(6) Starting from the critical point we perform n-step numerical iterations or integrals, and the n points are obtained. They correspond to the values of n pieces of dark-lines at the given parameter.

(7) For every parameter value in the whole parameter range, all the steps stated above are performed repeatedly, and the dark-lines can be obtained.

3. Research on dark-lines of nonlinear dynamic systems

3.1. Discrete mapping with deterministic nonlinear function

The parabolic mapping is expressed by

$$x_{n+1} = 1 - \mu x_n^2, \quad \mu \in (0, 2], \quad x \in [-1, 1]. \tag{4}$$

The variable μ is an external parameter. The dark-lines of the parabolic mapping are obtained from the dark-line equations and by using the regression algorithm respectively in the following.

3.1.1. Dark-lines obtained from the dark-line equations

The dark-line equations are expressed as

$$f(\mu, x) = 1 - \mu x^2, \quad f'(\mu, x) = -2\mu x. \tag{5}$$

The critical point is $c = 0$. Thus the dark-line equations are $P_0(\mu) = c = 0$, $P_1(\mu) = f(\mu, P_0(\mu)) = 1$, $P_2(\mu) = f(\mu, P_1(\mu)) = 1 - \mu, \ldots$. On the analogy of these, more dark-line equations may be acquired. By using these equations the dark-lines are obtained and shown in Fig.2.

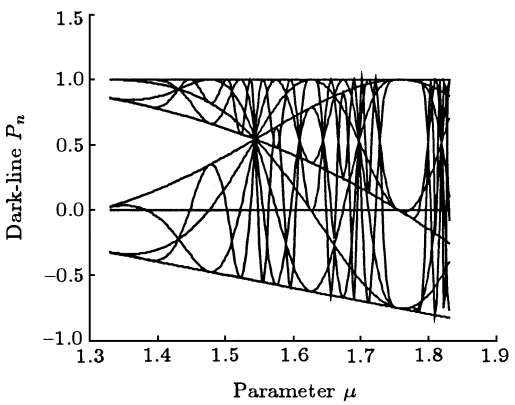


Fig.2. Dark-lines of the parabolic mapping by the dark-line equations.

3.1.2. Dark-lines obtained by using the regression algorithm

For $\mu = 1.50$, when taking NS1=1, the sampling result is shown in Fig.3. The Poincaré section of the chaotic attractor is distinctly distributed in two bands. So the chaotic attractor is marked with "2I". By taking NK=1, NS2=2 and sampling points again, the regressive mapping is constructed by using the sampling points and shown in Fig.4. The critical point $x = 0.81620$ can be found from Fig.4. By performing n-step numerical integrals starting from the initial value $P_0(\mu = 1.50) = 0.81620$, the values of $P_n(\mu = 1.50)$ $(n = 1, \cdots, 11)$ are obtained and listed in Table 1.

Table 1. Values of $P_n(\mu = 1.50)$, $(n = 0, 1, \cdots, 11)$.

n	$P_n(\mu = 1.50)$	n	$P_n(\mu = 1.50)$
0	0.81620	6	0.74284
1	0.00071	7	0.17229
2	1.00000	8	0.95547
3	–0.50000	9	–0.36939
4	0.62500	10	0.79532
5	0.41406	11	0.05120

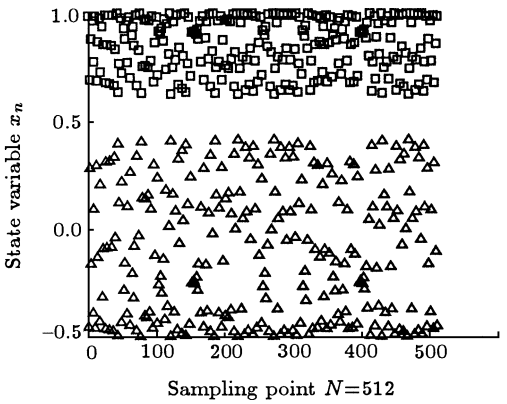


Fig.3. Sampling result of the chaotic attractor (μ=1.50).

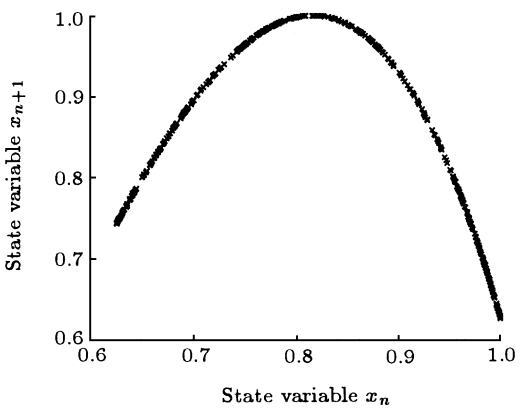


Fig.4. Regression mapping of the sampling points (μ=1.50).

If the parameter interval [1.33,1.83] is given and divided into 251 discrete parameter values by taking $\Delta\mu = 0.002$, the regressive mapping is constructed for every parameter value. Thus the critical points $P_0(\mu_i)$ $(i = 1, 2, \cdots, 251)$ are obtained. Then starting from every critical point we perform n-step numerical iterations, and the values of n pieces of dark-lines $P_n(\mu_i)$ are obtained. The dark-lines in the parameter

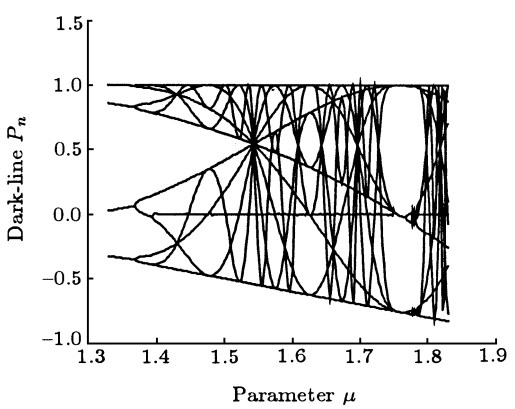


Fig.5. Dark-lines of the parabolic mapping by the regression algorithm.

interval are obtained and shown in Fig.5. By comparing Fig.5 with Fig.2, the dark-lines are found to be approximately consistent with each other. The result shows that the numerical regression algorithm can be used to research the dark-lines of any nonlinear dynamic system not depending on dark-line equations.

3.2. Continuous nonlinear systems described by nonlinear differential equations

3.2.1. Brussel oscillation system

The Brussel oscillation system with periodic excitation is expressed by

$$\begin{cases} \dot{x}_1 = A - (B+1)x_1 + x_1^2 x_2 + \alpha\cos(\omega t), \\ \dot{x}_2 = Bx_1 - x_1^2 x_2. \end{cases} \tag{6}$$

The model describes a problem of chemical dynamics with reaction among three kinds of molecules. In Eq.(6), A and B are the concentrations of two kinds of ingredients, which are invariable in the process of reaction. x_1 and x_2 are the concentrations of mid-products. α is the external parameter, and ω is the exciting frequency. The parameters $A = 0.4$, $B = 1.2$, $\omega = 0.8$, and $\alpha \in [0.050, 0.080]$ are taken. Based on some conclusions in Ref.[3], the dark-lines of this system are researched by using the regression algorithm as follows.

At the given parameter $\alpha = 0.08$, when taking NS1=2 the sampling result is shown in Fig.6, so the chaotic attractor is marked with "4I". By taking NK=1 and NS2=4, the regressive mapping is constructed and shown in Fig.7. The critical point $x_1 = 0.33751$ can be found. By performing n-step integrals from the initial value $P_0(\alpha = 0.08) = 0.33751$, the values of $P_n(\alpha = 0.08)$ $(n = 1, \cdots, 19)$ are obtained and listed in Table 2.

Table 2. Values of $P_n(\alpha = 0.08)$, $(n = 0, 1, \cdots, 19)$.

n	$P_n(\alpha = 0.08)$	n	$P_n(\alpha = 0.08)$
0	0.33751	10	0.28283
1	0.43309	11	0.38062
2	0.27911	12	0.35147
3	0.37557	13	0.43342
4	0.36705	14	0.28060
5	0.42261	15	0.37797
6	0.29087	16	0.35966
7	0.39280	17	0.42939
8	0.31718	18	0.28392
9	0.42049	19	0.38294

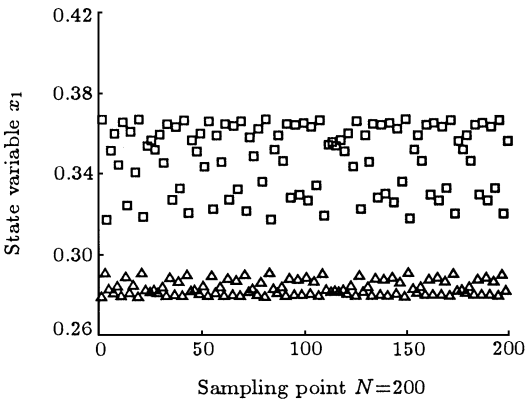


Fig.6. Sampling result of the chaotic attractor (α=0.08).

For the given parameter interval, the dark-lines are obtained and shown in Fig.8. Obviously there are many intersected points and tangential points in the figure. The intersected points correspond to turning points of chaotic attractors, and the tangential points correspond to periodic windows embedded in the chaotic region.

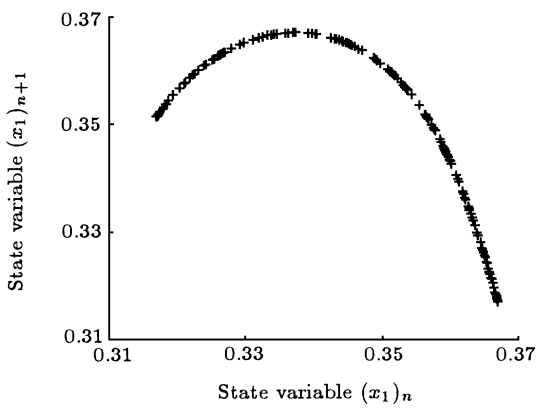


Fig.7. Regression mapping of the sampling points (α=0.08).

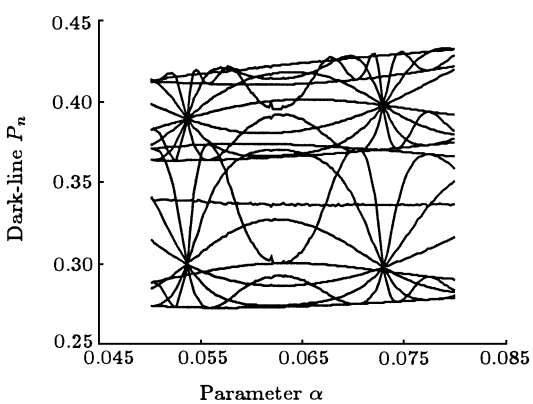


Fig.8. Dark-lines of Brussel oscillation system.

3.2.2. Piecewise linear dynamic system of single gear pair

The dimensionless dynamic equations[11] of the piecewise linear gear system are expressed by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2\zeta x_2 - [1 + \varepsilon\cos(\Omega t)]f(x_1) \\ \qquad +P_{\mathrm{m}} + P_{\mathrm{a}}\cos(\Omega t). \end{cases} \tag{7}$$

$$f(x_1) = \begin{cases} x_1 - 1, & x_1 > 1, \\ 0, & -1 \le x_1 \le 1, \\ x_1 + 1, & x_1 < -1. \end{cases} \tag{8}$$

The model describes a nonlinear gear system with clearance and time-varying mesh stiffness. The piecewise linear function $f(x_1)$ represents the nonlinear factor of clearance. The state variables x_1 and x_2 are the dimensionless displacement and velocity, respectively. The variable ζ is the damping ratio, Ω is the exciting frequency and ε is the fluctuation factor of the time-varying mesh stiffness. The variables P_{m} and P_{a} are the average value of dimensionless force and the amplitude of dimensionless periodic exciting force respectively.

According to our studies on the gear system, when the parameters $\varepsilon = 0.1$, $\Omega = 1.0$, $P_{\mathrm{m}} = 0.1$ and $P_{\mathrm{a}} = 0.2$ are given and the parameter ζ is changed from 0.090 to 0.050, some conclusions have been reached. The bifurcation plot is shown in Fig.9. In the parameter interval [0.057, 0.058], the "8I" chaotic attractor turns to the "4I" one. In addition, in the vicinity of $\zeta = 0.0579$, there is a periodic window embedded in the "8I" chaotic region, and in the vicinity of $\zeta = 0.0574$, there is a periodic window embedded in the "4I" region.

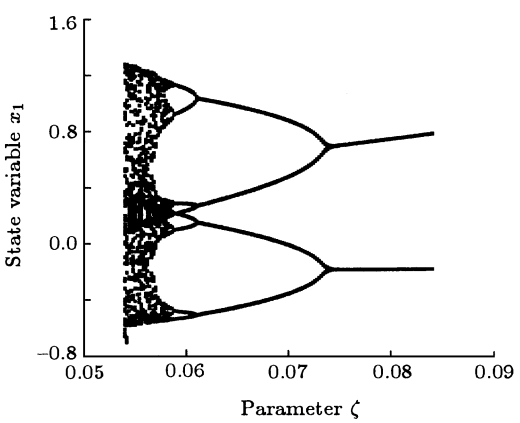


Fig.9. Bifurcation plot of the gear system.

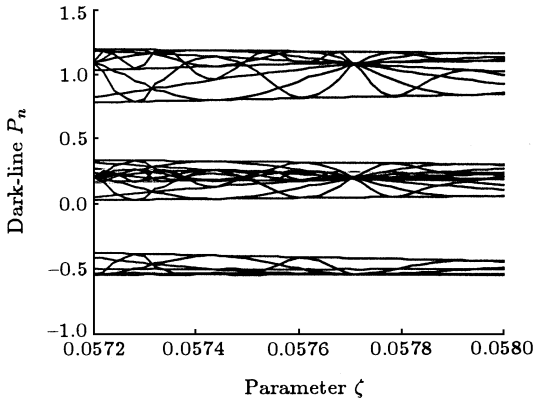


Fig.10. Dark-lines of the gear system.

By using the regression algorithm, the dark-lines in the interval [0.057, 0.058] are researched and shown in Fig.10. A one-quarter embranchment on the top of the dark-lines is taken and shown in Fig.11. Thus we can more clearly observe and analyze the intersection and tangent of these dark-lines. Then the turning points of chaotic attractors and the positions of periodic windows can be obtained.

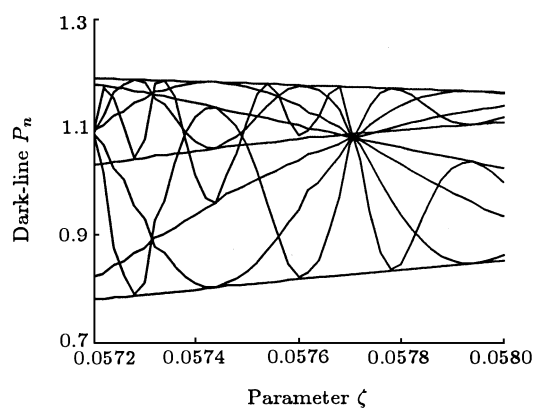


Fig.11. A one-quarter embranchment on the top of the dark-lines of the gear system.

The result shows that the dark-lines intersect at the point of $\zeta = 0.057710$ where the "8I" chaotic attractor turns to the "4I" one. In addition, among the tangential points, there are two points that correspond to the parameter values $\zeta = 0.057420$ and $\zeta = 0.057940$, where there exist periodic windows.

For example, at $\zeta = 0.058$ the attractor is marked with "8I". The sampling result is shown in Fig.12, and the regressive mapping is shown in Fig.13.

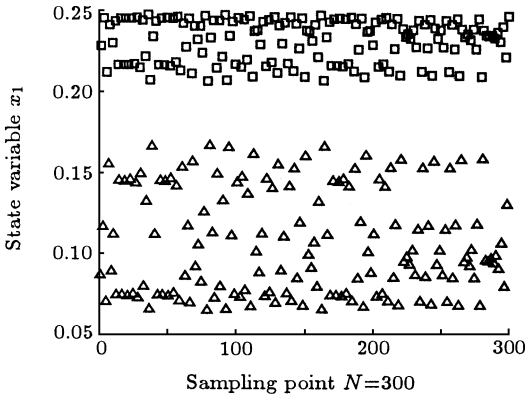


Fig.12. Sampling result of the chaotic attractor (ζ=0.058).

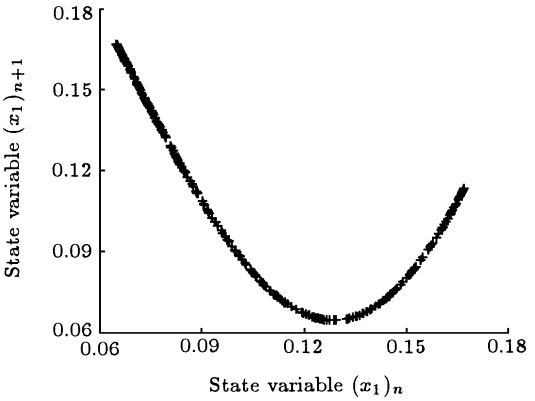


Fig.13. Regression mapping of the sampling points (ζ=0.058).

4. Conclusions

The research results show that the regression algorithm based on numerical calculation of chaotic orbits does not depend on the dark-line equations, and can be effectively used to research the dark-lines of practical nonlinear physical systems. By researching the dark-lines, we find the turning points where the mutation of chaotic attractors occurs. Furthermore, the parameter ranges corresponding to the periodic windows embedded in the chaotic regions are clearly determined. So for nonlinear system, researching on dark-lines can provide guidance for analysing the dynamic behaviour, selecting design parameters, and controlling chaos.

References

[1] Hao B L 1995 *Starting With Parabolas: An Introduction to Chaotic Dynamics* (Shanghai: Shanghai Scientific and Technological Education Publishing House) p33 (in Chinese)

[2] Hao B L 1983 *Prog. Phys.* **3** 336 (in Chinese)

[3] Hao B L and Zhang S Y 1983 *Acta Phys. Sin.* **32** 198 (in Chinese)

[4] Hua C C and Liu Y Z 2002 *Chin. Phys.* **11** 547

[5] Tang J S and Fu W B 2002 *Chin. Phys.* **11** 1004

[6] Zeng W Z, Hao B L, Wang G R and Chen S G 1984 *Commun. Theor. Phys.* **3** 283

[7] Jensen R and Myers C R 1985 *Phys. Rev.* A **32** 1222

[8] Grebogi C, Ott E and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196

[9] Zhu W Y, Wang D J and Yu H 2000 *Journal of Northeastern University* (Natural Science) **21** 465

[10] Zheng W M and Hao B L 1994 *Applied Symbolic Dynamics* (Shanghai: Shanghai Scientific and Technological Education Publishing House) p165 (in Chinese)

[11] Gao Z Y 2004 *Chin. J. Mech. Eng.* **40**(5) 17